

A CHARACTERIZATION OF BICYCLIC BUT NOT TWO-CYCLIC GRAPHS EACH WITH A NON-CUT VERTEX AS SELF VERTEX SWITCHING

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Abstract: A vertex $v \in V(G)$ is said to be a self vertex switching of G if G is isomorphic to G^v , where G^v is the graph obtained from G by deleting all edges of G incident to v and adding all edges incident to v which are not in G . A graph G is called a bicyclic but not two-cyclic graph if it is a connected $(p, p+1)$ graph with exactly three cycles. Trees, forests, both connected and disconnected unicyclic graphs and both connected and disconnected two-cyclic graphs are characterized, each with a self vertex switching. In this paper, we characterize bicyclic but not two-cyclic graphs each with a non-cut vertex as a self vertex switching.

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Key Words: switching, self vertex switching, unicyclic, two-cyclic, bicyclic but not two-cyclic, $SS_1(G)$, $ss_1(G)$

1. Introduction

For a finite undirected simple graph $G(V, E)$ with $|V(G)| = p$ and a set $\sigma \subseteq V$, the switching of G by σ is defined as the graph $G^\sigma(V, E')$, which is obtained

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from G by removing all edges between σ and its complement $V - \sigma$ and adding all non-edges as edges between σ and $V - \sigma$. As switching has been defined by Seidel [6] it is referred to as Seidel switching. When $\sigma = \{v\} \subset V$, we call the corresponding switching $G^{\{v\}}$ as vertex switching and denote it as G^v [1, 8]. A subset σ of $V(G)$ is a self switching of G if $G \cong G^\sigma$. The set of all self switching of G with cardinality k is denoted by $SS_k(G)$ and its cardinality by $ss_k(G)$. When $k = 1$, then we call the corresponding self switching as self vertex switching [1, 9].

A branch at v in G is a maximal connected subgraph B of G such that the intersection of B with the vertex v is v and $B - \{v\}$ is connected [9]. A branch B at v in G is said to be self switching branch if $B \cong B^v$. In G , two branches B_1 and B_2 at v are said to be complementary switching branches, if there exist isomorphisms f_1 between B_1 and B_2^v and f_2 between B_2 and B_1^v such that $f_1(v) = f_2(v)$ [9]. A simple graph in which each pair of distinct vertices joined by an edge is called a complete graph. A complete graph on n vertices is denoted by K_n . A walk in a graph is a finite non-null sequence whose terms are alternatively vertices and edges. A path is a walk in which all the vertices are distinct. A path with n vertices is denoted by P_n . Paths with unknown number of vertices are denoted by $P_1, P_2, \dots$. A closed path is called a cycle. A cycle with n vertices is denoted by C_n . Cycles with unknown number of vertices are denoted by $C_1, C_2, \dots$. The line joining the two non-adjacent vertices in a cycle is called a chord. We define the joining of two non-adjacent vertices in a cycle by a path is called a path chord. Let $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ be any two graphs. The union of the graphs G_1 and G_2 is a graph denoted by $G_1 \cup G_2$ whose vertex set is $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2$. We denote $G \cup G \cup \dots \cup G$ (with m copies of G) by mG . The join of disjoint graphs G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2 . Two vertices u and v are said to be interchange similar if there is an automorphism f of G such that $f(u) = v$ and $f(v) = u$. A graph with p vertices and q edges is denoted by (p, q) graph. A connected $(p, p+1)$ graph is called a bicyclic graph. A two-cyclic graph is a $(p, p+1)$ graph with exactly two cycles. We define a graph as bicyclic but not two-cyclic if it is a connected $(p, p+1)$ graph with exactly three cycles. The graph $K_4 - \{e\}$ is a $(4, 5)$ graph with three cycles, where e is an edge in K_4 and hence it is a bicyclic but not two-cyclic graph.

In [10], a characterization of interchange similar vertices to be self vertex switching was given. A graph is said to be a self centered graph if its every vertex is a central vertex. A connected graph G is said to be highly irregular, if each of its vertices is adjacent only to vertices with distinct degrees. In [7], it is proved that, there is no highly irregular graph with a self vertex switching

and for every $m \geq 2$, there exists a self centered graph of order $2m + 1$ with a self vertex switching. Also, it is proved that for every $n \geq 1$, there exists a graph G of order $4n + 1$ with a self vertex switching v such that $G - \{v\}$ is self complementary and every graph is an induced subgraph of a graph H with a self vertex switching v such that $H - \{v\}$ is self complementary. In [2], trees and forests are characterized, each with a self vertex switching. In [3], connected unicyclic graphs, each with a self vertex switching is characterized. In [4], disconnected unicyclic graphs, each with a self vertex switching is characterized. In [5], connected two-cyclic graphs, each with a self vertex switching is characterized. In Section 2 of this paper, we characterize for any non-cut vertex v of G , G^v is bicyclic but not two-cyclic. In Section 3, we characterize bicyclic but not two-cyclic graphs, each with a non-cut vertex as a self vertex switching. To characterize G^v as a bicyclic but not two-cyclic graph, we have to consider both connected and disconnected graphs. Here we are dealing with simple graphs only. We denote the degree of the vertex v in G by $d_G(v)$. Now consider the following results, which are required in the subsequent sections.

Theorem 1.1. [1] *If v is a self vertex switching of a graph G of order p , then p is odd and $d_G(v) = \frac{(p-1)}{2}$.*

Theorem 1.2. [9] *Let v be any vertex of a connected graph G such that G^v is connected. Then B is a branch at v in G if and only if B^v is a branch at v in G^v .*

Theorem 1.3. [2] *Let $v \in V(G)$ and $|V(G)| \geq 2$. Then G^v is connected if and only if $d_G(v) = 0$ or $d_G(v) < |V(B)| - 1$ for every branch B at v in G .*

Theorem 1.4. [2] *Let v be a vertex of a non-trivial connected graph G . Then G^v is a tree if and only if $G - \{v\}$ is acyclic and $d_B(v) = |V(B)| - 2$ for every branch B at v in G .*

Theorem 1.5. [2] *Let v be a self vertex switching of a connected graph G and let B be a branch at v in G . Then $|V(B)| \geq 3$.*

Theorem 1.6. [9] *Let B_i be either a branch at v in G or the union of v and a component of G not containing v , $i = 1, 2, \dots, k(G - \{v\})$. Then $G = \bigcup_{i=1}^k B_i$ and $G^v = \bigcup_{i=1}^k B_i^v$ where $k = k(G - \{v\})$ is the number of components of $G - \{v\}$.*

Theorem 1.7. [3] *Let $v \in V(G)$ be a non-cut vertex of G of order $p \geq 3$. Let $D \neq K_1, K_2$ be a component of G containing v . Then G^v is connected and unicyclic if and only if one of the following statements is true:*

- (a) $G = K_2 \cup (p-2)K_1$ and v is one of the K_1 's .
- (b) G is connected, $G - \{v\}$ is acyclic and $d_G(v) = |V(G)| - 3$.
- (c) G is connected, $G - \{v\}$ is unicyclic and $d_G(v) = |V(G)| - 2$.
- (d) $G = D \cup (p - |V(D)|)K_1$, $G - \{v\}$ is unicyclic and $d_G(v) = |V(D)| - 2$.
- (e) $G = D \cup (p - |V(D)|)K_1$, $G - \{v\}$ is acyclic and $d_G(v) = |V(D)| - 3$.
- (f) $G = D \cup K_2 \cup (p-2-|V(D)|)K_1$, $G - \{v\}$ is acyclic and $d_G(v) = |V(D)| - 2$.

2. Characterization of a Non-Cut Vertex v of G Such That G^v to be Bicyclic but Not Two-Cyclic Graph

Let v be the vertex of a graph G and G^v be the switching of G by v . Theorem 1.3 gives conditions on the vertex v of G such that G^v is connected. In [3], we characterized vertex v of G such that G^v is connected and unicyclic. In [5], we characterized vertex v of G such that G^v is connected and two-cyclic. In this section, we characterize a non-cut vertex v of G such that G^v is a connected bicyclic but not two-cyclic graph.

Theorem 2.1. *Let v be a non-cut vertex of a graph G of order $p \geq 4$. Then G^v is bicyclic but not two-cyclic if and only if one of the following statements is true:*

- (a) $G = P_3 \cup (p-3)K_1$ and v is one of the K_1 's.
- (b) G is connected, $G - \{v\}$ is acyclic, and $d_G(v) = |V(G)| - 4$.
- (c) G is connected, $G - \{v\}$ is unicyclic, $d_G(v) = |V(G)| - 3$, and either
 - (i) the two non-adjacent vertices of v do not lie on the cycle of $G - \{v\}$ and at least two vertices of the cycle lie on the two paths connecting the two non-adjacent vertices in $G - \{v\}$ or
 - (ii) the two non-adjacent vertices lie on the cycle of $G - \{v\}$ or
 - (iii) one of the two non-adjacent vertices lies on the cycle of $G - \{v\}$ and at least one vertex of the cycle lies on the two paths connecting the two non-adjacent vertices in G .
- (d) G is connected, $G - \{v\}$ is bicyclic but not two-cyclic, and $d_G(v) = |V(G)| - 2$.

- (e) $G = D \cup P_3 \cup (p-3-|V(D)|)K_1$, $G - \{v\}$ is acyclic, and $d_G(v) = |V(D)| - 2$.
- (f) $G = D \cup (p - |V(D)|)K_1$, $G - \{v\}$ is acyclic, and $d_D(v) = |V(D)| - 4$.
- (g) $G = D \cup (p - |V(D)|)K_1$, $G - \{v\}$ is unicyclic, $d_D(v) = |V(D)| - 3$, and either
- (i) the two non-adjacent vertices of v does not lie on the cycle of $G - \{v\}$ and at least two vertices of the cycle lie on the two paths connecting the two non-adjacent vertices in $G - \{v\}$ or
 - (ii) the two non-adjacent vertices lie on the cycle of $G - \{v\}$ or
 - (iii) one of the two non-adjacent vertices lies on the cycle of $G - \{v\}$ and at least one vertex of the cycle lies on the two paths connecting the two non-adjacent vertices in G .
- (h) $G = D \cup (p - |V(D)|)K_1$, $G - \{v\}$ is bicyclic, and $d_G(v) = |V(D)| - 2$.

Proof. Let G^v be a bicyclic but not two-cyclic graph. Then G^v is a connected $(p, p+1)$ graph with exactly three cycles. Now $G - \{v\}$ is either acyclic or unicyclic or bicyclic but not two-cyclic. Since G^v is connected, using Theorem 1.3, $d_G(v) = 0$ or $d_B(v) < |V(B)| - 1$ for every branch B at v in G .

If $d_G(v) = 0$, then G is disconnected. Since G^v is bicyclic but not two-cyclic, only one branch at v in G^v is bicyclic. Clearly, the other branches at v in G^v are K_2 's. Let B be the bicyclic branch at v in G^v . Then $|V(B)| \geq 4$. Also, $B - \{v\}$ is a component of G . If $|V(B)| > 4$, then the component $B - \{v\}$ has at least 4 vertices and hence B has at least 4 cycles which is not possible. Therefore, $|V(B)| = 4$ and hence the component $B - \{v\}$ in G has only 3 vertices which is either P_3 or K_3 . If $B - \{v\} = K_3$, then B is not bicyclic. Therefore, $B - \{v\} = P_3$. This implies that $G = P_3 \cup (p-3)K_1$ and v is one of the K_1 's. Hence (a) is proved.

Now consider the case $d_G(v) \neq 0$ and $d_B(v) < |V(B)| - 1$. That is, when $d_B(v) \leq |V(B)| - k, k \geq 2$. Since v is not a cut vertex of G , B is either G or a component of G containing v according as G is connected or disconnected. We consider the following 6 cases with respect to G and $G - \{v\}$.

Case 1. G is connected and $G - \{v\}$ is acyclic.

Here $B = G$. Since v is not a cut vertex of G , $G - \{v\}$ is connected. If $d_G(v) = |V(G)| - 2$, then G^v is a tree, using Theorem 1.4. If $d_G(v) = |V(G)| - 3$, then there exist exactly two vertices, say u and w , in G such that they are non-adjacent to v . Since $G - \{v\}$ is connected and acyclic, there exists a unique u - w path in $G - \{v\}$ and hence in G^v also. Hence the edge vu , the u - w path and

the edge wv form a unique cycle in G^v , which is a contradiction. If $d_G(v) = |V(G)| - 4$, then there exist exactly three vertices, say u_1, u_2 and u_3 , in G such that they are non-adjacent to v . Since $G - \{v\}$ is connected and acyclic, there exist unique u_1-u_2 , u_2-u_3 and u_1-u_3 paths in $G - \{v\}$ and hence in G^v also. Let $P1$ be the u_1-u_2 path, $P2$ be the u_2-u_3 path and $P3$ be the u_1-u_3 path. Let x be the last vertex common to both $P1$ and $P3$. If $u_1 = x$, then $P1 \cup P3$ is the u_2-u_3 path $P2$. In this case vu_1 is the chord of the cycle C which is formed by taking the edge vu_2 , the path $P2$ and the edge u_3v . If $u_1 \neq x$, then let $P1^*$ be the $x-u_2$ path and $P3^*$ be the $x-u_3$ path. Clearly, $P1^* \cup P3^*$ is the path $P2$. Now the edge vu_2 , the path $P2$ and the edge u_3v form a cycle C in G^v in which the edge vu_1 and the u_1-x path together forms a path chord in C . Hence G^v is bicyclic but not two-cyclic. If $d_G(v) < |V(G)| - 4$, then there exist at least four vertices which are non-adjacent to v in G and hence G^v has at least $4C_2 = 6$ cycles since there are 6 possible paths between the four vertices, which is a contradiction. Hence (b) is proved.

Case 2. G is connected and $G - \{v\}$ is unicyclic.

Let C be the unique cycle in $G - \{v\}$. Then C is also a cycle of G^v not containing v . Since G^v is connected, using Theorem 1.3, we have $d_G(v) \leq |V(G)| - 2$. If $d_G(v) = |V(G)| - 2$, then there exists exactly one vertex say u , which is non-adjacent to v in G . Thus in G^v , v is adjacent only to u . Hence G^v has a unique cycle C , which is a contradiction. If $d_G(v) < |V(G)| - 3$, then v is non-adjacent to at least three vertices, say u_1, u_2 and u_3 in G . Since $G - \{v\}$ is connected, there exist u_1-u_2 , u_2-u_3 and u_1-u_3 paths in $G - \{v\}$ and hence in G^v also. Now the paths together with the edges u_1v, u_2v and u_3v form at least three cycles in G^v and each cycle contains v and hence G^v has at least four cycles, which is a contradiction. If $d_G(v) = |V(G)| - 3$, then v is non-adjacent to exactly two vertices, say u and w . We consider the following three subcases.

Subcase 2.1. u and w do not lie on the cycle C .

Since $G - \{v\}$ is connected, there exist either one $u-w$ path or two $u-w$ paths in $G - \{v\}$ according as the $u-w$ path has at most one vertex of C or at least two vertices of C and hence in G^v also.

Subcase 2.1.a. One $u-w$ path in $G - \{v\}$ i.e., $u-w$ path has at most one vertex of C .

Let $P1$ be the unique $u-w$ path in $G - \{v\}$. Then the edge vu , the $u-w$ path $P1$ and the edge wv form another cycle $C1$ in G^v , which contains the vertex v . Therefore, G^v is two-cyclic.

Subcase 2.1.b. Two u - w paths in $G - \{v\}$ i.e., u - w path has at least two vertices of C .

Let $P1$ and $P2$ be the two u - w paths in $G - \{v\}$. Let $P1$ be $uu_1u_2 \dots u_nw$ and $P2$ be $uv_1v_2 \dots v_mw$. Let u_i be the first vertex in $P1$ such that $u_i \notin V(P2)$. Therefore, $u_x = v_x$ for $1 \leq x \leq i-1$. Let u_j be the last vertex in $P1$ such that $u_j \notin V(P2)$. Then $u_{j+1} = v_r$ for some r such that $m-r+1 = n-j$ and hence $u_{j+1} = v_r, u_{j+2} = v_{r+1}, \dots, v_n = v_m$. Now the edge vu , the path $P1$ and the edge wv form a cycle $C1$ and the edge vu , the path $P2$ and the edge wv form another cycle $C2$ in G^v . Hence G^v has three cycles and the cycle $C2$ has $v_{i-1} = u_{i-1}u_iu_{i+1} \dots u_{j+1} = v_r$ as the path chord. Hence G^v is bicyclic but not two-cyclic. Thus c(i) is proved.

Subcase 2.2. u and w lie on the cycle C .

Then there exist exactly two u - w paths, say $P1$ and $P2$, in C and they have no common vertex other than u and w and hence in G^v also. Now the edge vu , the path $P1$ and the edge wv form a cycle, say $C1$, in G^v and the edge vu , the path $P2$ and the edge wv form another cycle, say $C2$, in G^v . Thus G^v has three cycle in which uvw is a path chord of the cycle C . Hence G^v is bicyclic. Thus c(ii) is proved.

Subcase 2.3. Either u or w lies on the cycle C .

Let us assume that u lies on C . Since $G - \{v\}$ is connected, there exists either one u - w path or two u - w paths according as either the u - w path has no vertex of C or a vertex of C other than u and hence, there arise two subcases.

Subcase 2.3.a. One u - w path in $G - \{v\}$ i.e., u - w path has no vertex of C other than u .

The u - w path contains no vertex of the cycle C implies that the edge vu , the u - w path and the edge wv form a cycle different from C . Hence G^v has two cycles and these two cycles have no common edges. Therefore, G^v is two-cyclic.

Subcase 2.3.b. Two u - w paths in $G - \{v\}$ i.e., u - w path has a vertex of C other than u .

Let $uu_1u_2 \dots u_nu$ be the cycle C . Let $P1$ and $P2$ be the two w - u paths. Let u_i be the vertex of the cycle lies on both $P1$ and $P2$. Let P^* be the w - u_i path. Let $P1^*$ be the path $u_iu_{i+1} \dots u_nu$ and $P2^*$ be the path $u_iu_{i-1} \dots u_1u$. Then $P1 = P^* \cup P1^*$ and $P2 = P^* \cup P2^*$. Now the edge vw , the path $P1$ and the edge wv form a cycle $C1$ and the edge vw , the path $P2$ and the edge wv form another cycle $C2$. Hence G^v has 3 cycles and the cycle $C2$ has $P1^*$ as the path chord. Hence G^v is bicyclic. Thus c(iii) is proved.

Case 3. G is connected and $G - \{v\}$ is bicyclic but not two-cyclic.

Let $C1, C2$ and $C3$ be the three cycles in $G - \{v\}$. Clearly $C1, C2$ and $C3$ are the cycles in G^v . Since G^v is bicyclic, v does not lie on all the three cycles. Since G^v is connected, using Theorem 1.3, $d_G(v) \leq |V(G)| - 2$. If $d_G(v) = |V(G)| - 2$, then there exists exactly one vertex, say u , which is non-adjacent to v in G . Hence $G^v = G - \{v\} + uv$, and therefore G^v is a bicyclic but not a two-cyclic graph. If $d_G(v) < |V(G)| - 2$, then there exist at least two vertices, say u_1 and u_2 , which are non-adjacent to v . Since $G - \{v\}$ is connected, there exists u_1 - u_2 path in $G - \{v\}$ and hence in G^v also. Now the edge vu_1 , the path u_1 - u_2 and the edge u_2v together form another cycle which contains v , which is a contradiction. Thus the only possibility is $d_G(v) = |V(G)| - 2$. Hence (d) is proved.

Let G be a disconnected graph. Let v be in a component D of G . Let $D_1, D_2, \dots, D_m$ be the other components of G . Then $G = D \cup D_1 \cup D_2 \cup \dots \cup D_m = D \cup \left(\bigcup_{i=1}^m D_i \right)$. By Theorem 1.6, $G^v = D^v \cup \left(\bigcup_{i=1}^m D_i^v \right)$. Since $d_G(v) \neq 0, |V(D)| \geq 2$. If $D_i = K_1$, then $D_i^v = K_2$. If $D_i = K_2$, then $D_i^v = C_3$. If $D_i \neq K_1$ and K_2 , then D_i^v has at least 3 cycles in G^v .

Case 4. G is disconnected and $G - \{v\}$ is acyclic.

If $d_D(v) = |V(D)| - 2$, then by Theorem 1.4, D^v is a tree. Since G^v is bicyclic but not two-cyclic, one D_i is P_3 and others K_1 's. Therefore, $G = D \cup P_3 \cup (p - 3 - |V(D)|)K_1$ and $d_G(v) = |V(D)| - 2$. Thus (e) is proved.

If $d_D(v) = |V(D)| - 3$, then by Theorem 1.7, D^v is unicyclic. $G^v = D^v \cup \left(\bigcup_{i=1}^m D_i^v \right)$ implies that G^v is either two-cyclic or not bicyclic.

If $d_D(v) = |V(D)| - 4$, then by Case 1, D^v is bicyclic. Since G^v is bicyclic, each component of G is K_1 . Thus $G = D \cup (p - |V(D)|)K_1$ and $d_D(v) = |V(D)| - 4$. Thus (f) is proved.

If $d_D(v) > |V(D)| - 4$, then G^v is not bicyclic.

Case 5. G is disconnected and $G - \{v\}$ is unicyclic.

$G - \{v\}$ is unicyclic implies that $D - \{v\}$ is unicyclic or D_i is unicyclic. If D_i is unicyclic, then D_i^v has at least four cycles in G^v and hence G^v is not bicyclic. Therefore, $D - \{v\}$ is unicyclic. If $D_i \neq K_1$, then D_i^v has at least one cycle and hence G^v is either two-cyclic or not bicyclic. Therefore, each component of G other than D is K_1 and hence $G = D \cup (p - |V(D)|)K_1$. Clearly, $G^v = D^v \cup (p - |V(D)|)(K_1 + v)$. Since G^v is a connected and bicyclic but not a two-cyclic graph, D^v is also a connected and bicyclic but not a two-cyclic graph. Apply Case 2 to D , we get $d_D(v) = |V(D)| - 3$ and either the two non-adjacent vertices of v do not lie on the cycle of $G - \{v\}$ and at least

two vertices of the cycle lie on the two paths connecting the two non-adjacent vertices in $G - \{v\}$ or the two non-adjacent vertices lie on the cycle of $G - \{v\}$ or one of the two non-adjacent vertices lies on the cycle of $G - \{v\}$ and at least one vertex of the cycle lies on the two paths connecting the two non-adjacent vertices in G . Hence (g) is proved.

Case 6. G is disconnected and $G - \{v\}$ is bicyclic but not two-cyclic.

Clearly D_i 's are trivial graphs. This implies that $G = D \cup (p - |V(D)|)K_1$ and hence $G^v = D^v \cup (p - |V(D)|)(K_1 + v)$. Since G^v is a connected bicyclic but not a two-cyclic graph, D^v is also a connected bicyclic but not a two-cyclic graph. Apply Case 3 to D , we get $d_D(v) = |V(D)| - 2$. Thus $G = D \cup (p - |V(D)|)K_1$ and $d_G(v) = |V(D)| - 2$ and hence (h) is proved.

Conversely, let either (a), (b), (c), (d), (e), (f) or (g) hold. From (a) to (g), we see either $d_G(v) = 0$ or $d_G(v) < |V(G)| - 1$. Using Theorem 1.3, G^v is connected. Also each case implies that G^v is bicyclic but not two-cyclic. Hence the theorem is proved. $\square$

3. Characterizing Bicyclic but Not Two-Cyclic Graphs Each With a Non-Cut Vertex as Self Vertex Switching

First we introduce the notation for representing a class of graphs.

Notation 3.1. Let $C_{n(v_1),r,x,n-x}$ denote a cycle graph of length n oriented clockwise starting at v_1 with a chord of length r connecting the vertices v_1 and v_x of the cycle. We use the ordered list $C_{n(v_1),r,x,n-x}(B_1, B_2, \dots, B_n)$ to represent the graph formed by adding a branch B_i to vertex v_i in $C_{n(v_1),r,x,n-x}$. For example the graphs $C_{5(v),1,2,3}(0, 0, 0, 0, P_3)$, $C_{6(v),1,2,4}(0, P_2, 0, 0, 0, 0)$, $C_{4(v),2,2,2}(0, 2P_2, 0, 0)$ and $C_{6(v),1,3,3}(0, P_2, 0, 0, 0, 0)$ are illustrated in Figure 3.1.

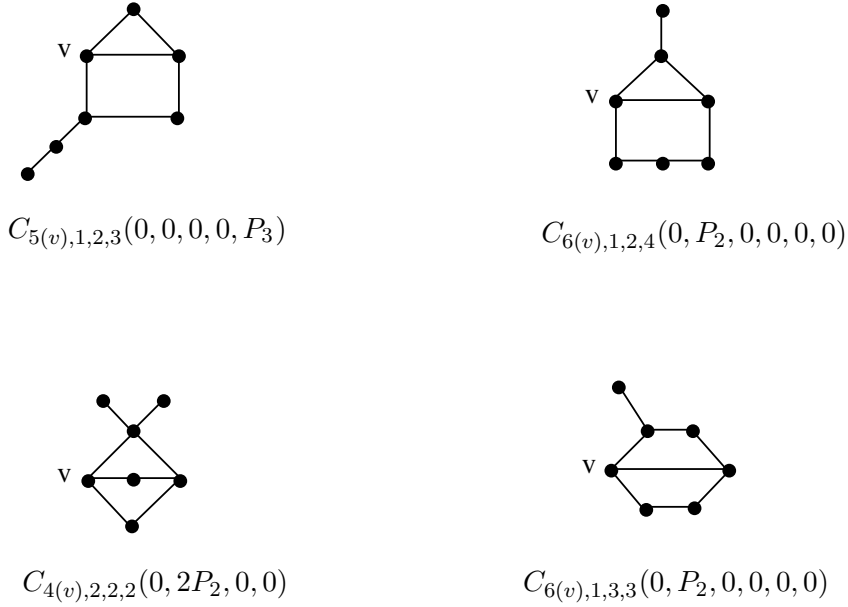


Figure 3.1

Note 3.2. A chord of length r divides the cycle of length n into two cycles of lengths $x + r$ and $n - x + r$. Clearly $C_{n(v_1),r,x,n-x}$ is a bicyclic but not two-cyclic graph. If each B_i is a tree, then $C_{n(v_1),r,x,n-x}(B_1, B_2, \dots, B_n)$ is also a bicyclic but not two-cyclic graph.

Theorem 3.3. Let G be a bicyclic but not two-cyclic graph of order $p = 2n + 1$ and let v be a non-cut vertex of G . Then v is a self vertex switching of G if and only if G is one of the following:

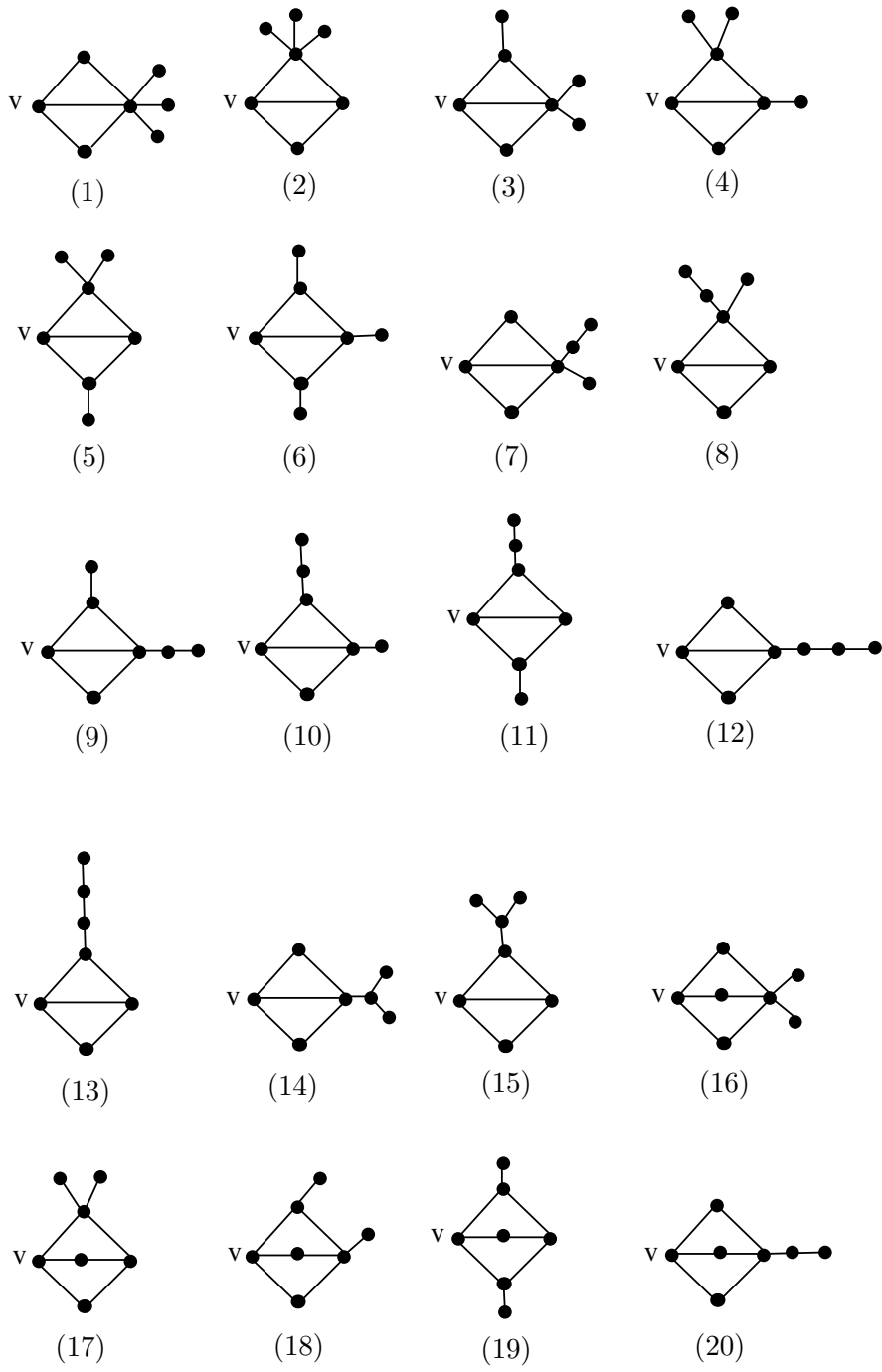
1. $C_{4(v),1,2,2}(0, P_4, 0, 0)$
2. $C_{4(v),1,2,2}(0, 0, K_{1,3}, 0)$
3. $C_{5(v),1,2,3}(0, 0, 0, 0, P_3)$
4. $C_{5(v),1,2,3}(0, 0, P_2, P_2, 0)$
5. $C_{6(v),1,2,4}(0, P_2, 0, 0, 0, 0)$
6. $C_{4(v),2,2,2}(0, 2P_2, 0, 0)$
7. $C_{6(v),1,3,3}(0, P_2, 0, 0, 0, 0)$

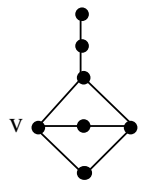
8. $C_{5(w),1,2,3}$, where w is a vertex adjacent to v and v lies on both the cycles C_3 and C_5 .

Proof. Let v be a self vertex switching of a bicyclic but not a two-cyclic graph G . Then $G \cong G^v$ and hence G has at least 5 vertices. Using Theorem 1.1, $d_G(v) = \frac{(p-1)}{2} = n$. Since v is not a cut vertex, using Theorem 2.1, either $G - \{v\}$ is acyclic and $d_G(v) = |V(G)| - 4$ or $G - \{v\}$ is unicyclic, $d_G(v) = |V(G)| - 3$ and either (i) the two non-adjacent vertices of v do not lie on the cycle of $G - \{v\}$ and at least two vertices of the cycle lie on the two paths connecting the two non-adjacent vertices in G or (ii) the two non-adjacent vertices lie on the cycle of $G - \{v\}$ or (iii) one of the two non-adjacent vertices lies on the cycle of $G - \{v\}$ and at least one vertex of the cycle lies on the two paths connecting the two non-adjacent vertices in G or $G - \{v\}$ is bicyclic but not two-cyclic and $d_G(v) = |V(G)| - 2$.

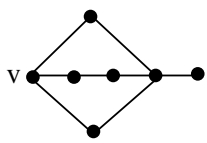
Case 1. $G - \{v\}$ is acyclic and $d_G(v) = |V(G)| - 4$.

Since G is connected, bicyclic but not two-cyclic and $G - \{v\}$ is acyclic, v must lie on all the three cycles of G . $d_G(v) = \frac{(p-1)}{2} = p - 4$ implies that $p = 7$ and hence $d_G(v) = 3$. The bicyclic but not two-cyclic graphs on 7 vertices are given in Figure 3.2. Among the 55 graphs given in Figure 3.2, there are seven graphs with v as a self vertex switching are given in Figure 3.3.

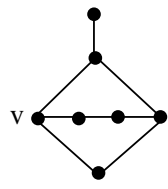




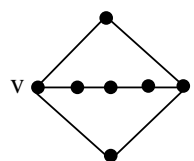
(21)



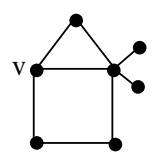
(22)



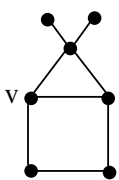
(23)



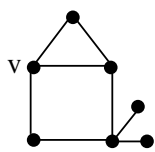
(24)



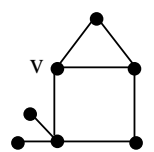
(25)



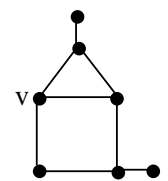
(26)



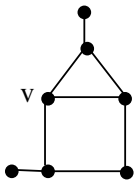
(27)



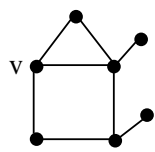
(28)



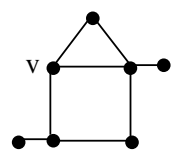
(29)



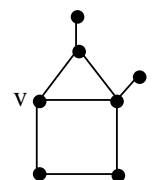
(30)



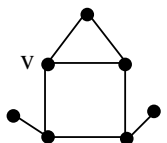
(31)



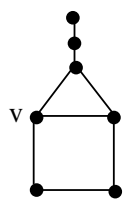
(32)



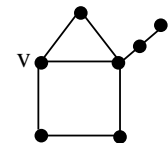
(33)



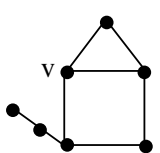
(34)



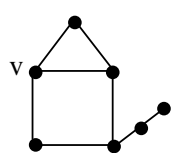
(35)



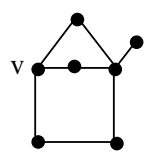
(36)



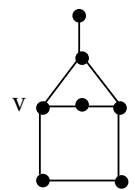
(37)



(38)



(39)



(40)

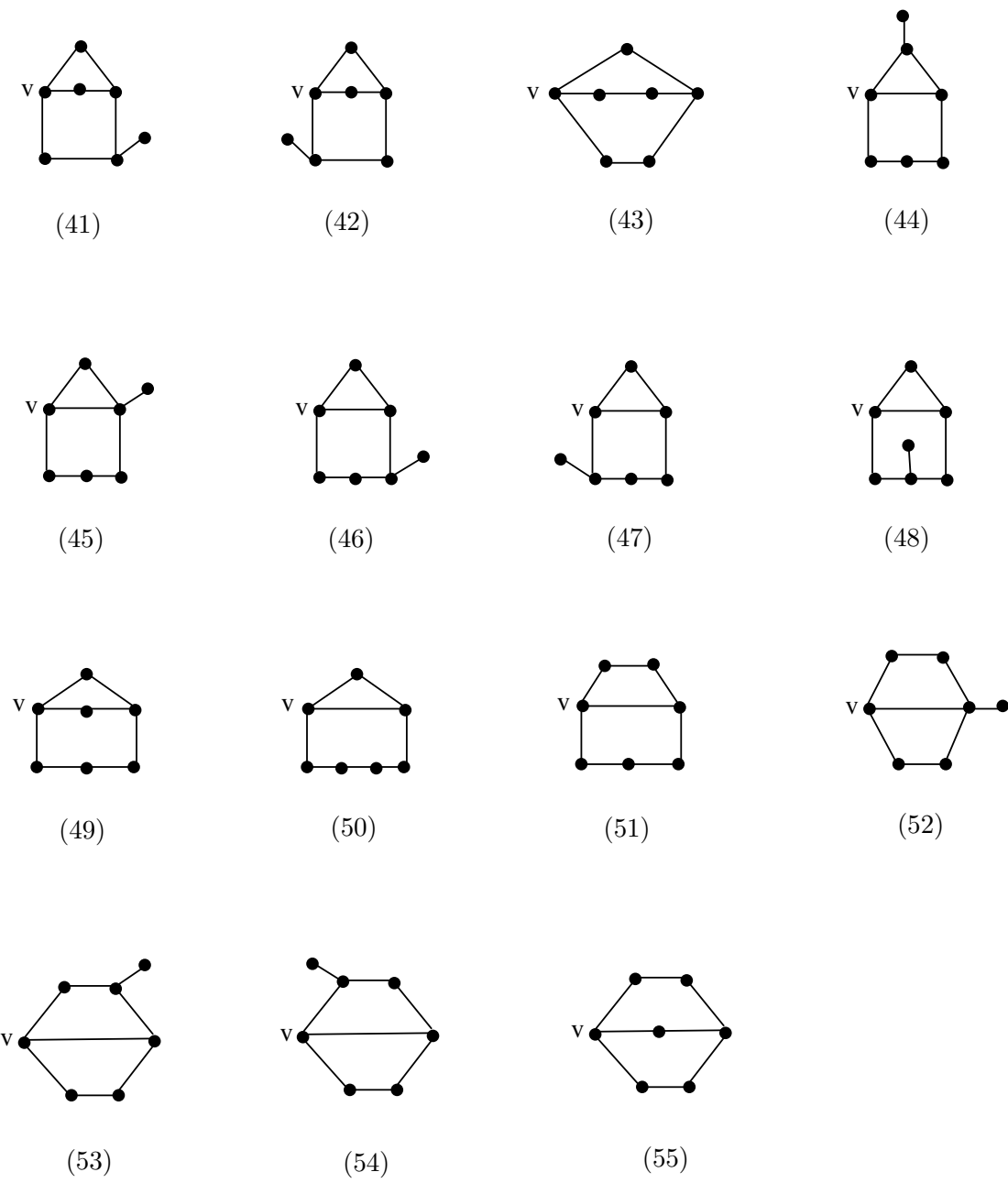


Figure 3.2.

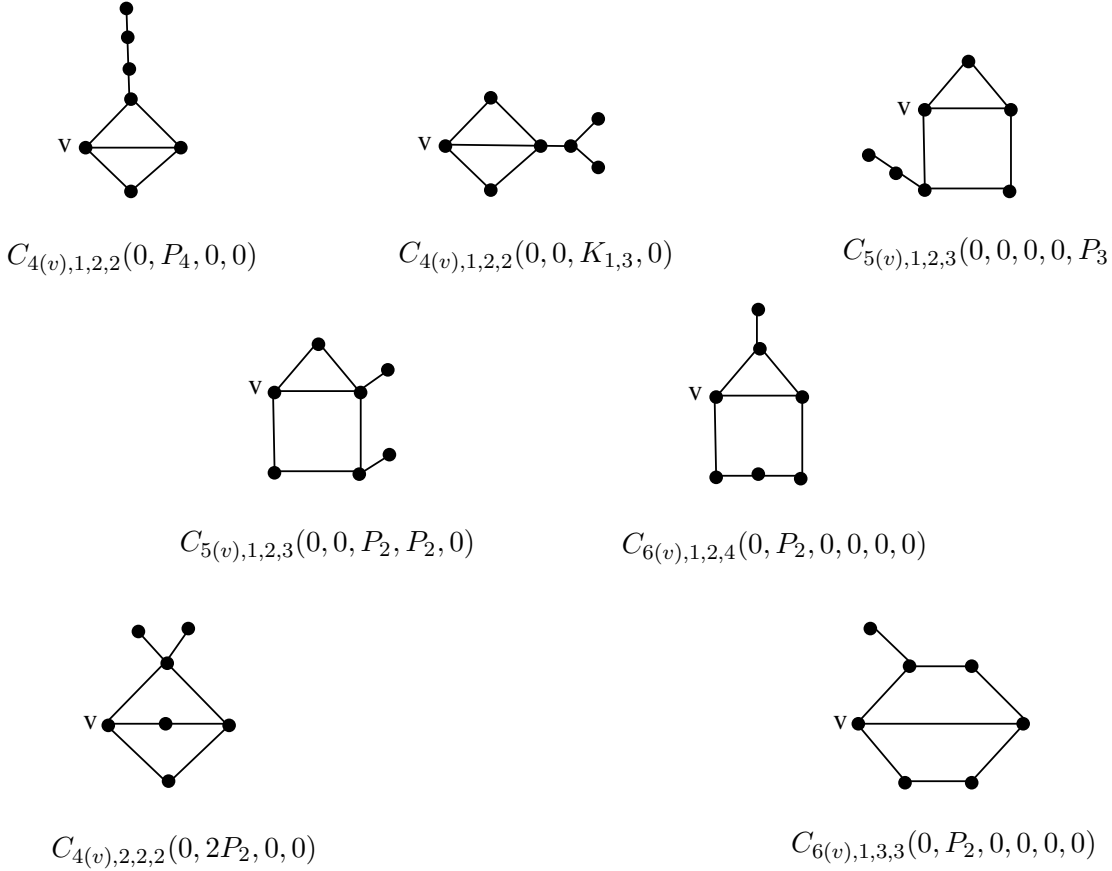


Figure 3.3.

Case 2. $G - \{v\}$ is unicyclic, $d_G(v) = |V(G)| - 3$ and either (i) the two non-adjacent vertices of v does not lie on the cycle of $G - \{v\}$ and at least two vertices of the cycle lie on the two paths connecting the two non-adjacent vertices in G or (ii) the two non-adjacent vertices lie on the cycle of $G - \{v\}$ or (iii) one of the two non-adjacent vertices lies on the cycle of $G - \{v\}$ and at least one vertex of the cycle lies on the two paths connecting the two non-adjacent vertices in G .

Since G is bicyclic but not two-cyclic and $G - \{v\}$ is unicyclic, v must lie on any two of the cycles, say $C1$ and $C2$, in G . $d_G(v) = \frac{(p-1)}{2} = p - 3$ gives $p = 5$.

The bicyclic but not two-cyclic graphs on 5 vertices are given in Figure 3.4.

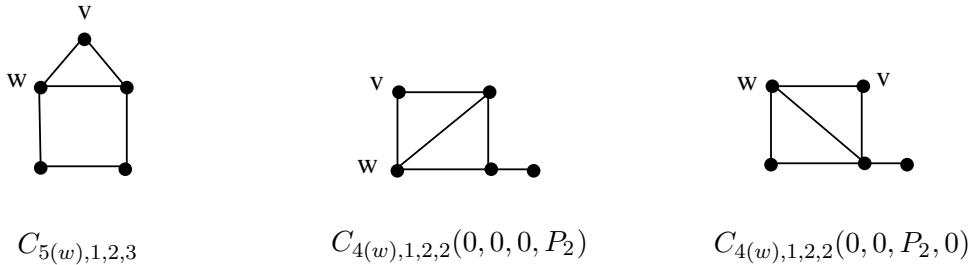


Figure 3.4.

Consider the graph $C_{5(w),1,2,3}$ given in Figure 3.4. If v lies on both the cycles C_3 and C_5 but not in C_4 , then v is a self vertex switching of G . Also, if v lies on both the cycles C_4 and C_5 but not in C_3 , then v is not a self vertex switching of G . The remaining two graphs $C_{4(w),1,2,2}(0,0,0,P_2)$, $C_{4(w),1,2,2}(0,0,P_2,0)$ given in Figure 3.4 have no self switching vertex. Thus G is the graph $C_{5(w),1,2,3}$, where w is a vertex adjacent to v and v lies on both the cycles C_3 and C_5 .

Case 3. $G - \{v\}$ is bicyclic but not two-cyclic and $d_G(v) = |V(G)| - 2$.

$d_G(v) = |V(G)| - 2$ implies that $p = 3$. But there is no bicyclic but not two-cyclic graph on 3 vertices. Hence there is no connected bicyclic but not two-cyclic graph G such that $G - \{v\}$ is bicyclic and $d_G(v) = |V(G)| - 2$. Therefore, G has no self vertex switching.

Thus from Case 1 and Case 2, if G has v as a self vertex switching, then G is either $C_{4(v),1,2,2}(0,P_4,0,0)$ or $C_{4(v),1,2,2}(0,0,K_{1,3},0)$ or $C_{5(v),1,2,3}(0,0,0,0,P_3)$ or $C_{5(v),1,2,3}(0,0,P_2,P_2,0)$ or $C_{6(v),1,2,4}(0,P_2,0,0,0,0)$ or $C_{4(v),2,2,2}(0,2P_2,0,0)$ or $C_{6(v),1,3,3}(0,P_2,0,0,0,0)$ or $C_{5(w),1,2,3}$ where w is a vertex adjacent to v and v lies on both the cycles C_3 and C_5 .

Conversely, let G be the graph given in the statement of the theorem. Then, clearly each case implies that v is a self vertex switching of G . This completes the proof. $\square$

Corollary 3.4. *Let G be a bicyclic but not a two-cyclic graph with non-cut vertices as self vertex switchings, if exist. Then $ss_1(G) = 0$ or 1 or 2 . $ss_1(G) = 1$ if and only if G is one of the following graphs:*

1. $C_{4(v),1,2,2}(0,0,K_{1,3},0)$
2. $C_{5(v),1,2,3}(0,0,0,0,P_3)$
3. $C_{5(v),1,2,3}(0,0,P_2,P_2,0)$
4. $C_{6(v),1,3,3}(0,P_2,0,0,0,0)$.

Also, $ss_1(G) = 2$ if and only if G is one of the following graphs:

1. $C_{4(v),1,2,2}(0,P_4,0,0)$
2. $C_{4(v),2,2,2}(0,2P_2,0,0)$
3. $C_{6(v),1,2,4}(0,P_2,0,0,0,0)$
4. $C_{5(w),1,2,3}$, where w is a vertex adjacent to v and v lies on both the cycles C_3 and C_5 .

4. Conclusion

In Section 2, we gave a necessary and sufficient condition for G^v to be a bicyclic but not two-cyclic graph. In Section 3, we characterized bicyclic but not two-cyclic graphs, each with a non-cut vertex as self vertex switching of G is given. Finally we have proved that $0 \leq ss_1(G) \leq 2$ for a bicyclic but not two-cyclic graph G with a non-cut vertex as self vertex switching, if exists.

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