A NOTE ON THE JOINT ESSENTIAL NUMERICAL RANGE OF ALUTHGE TRANSFORM

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Abstract: Denote by $T$ the bounded linear operator on a complex Hilbert space $X$ and let $T = U|T|$ be any polar decomposition of $T$ with $U$ a partial isometry and $|T| = (T^*T)^{1/2}$. Then, the Aluthge transform $\mathcal{T}$ of $T$ is the operator $|T|^{1/2}U|T|^{1/2}$. This study of the Aluthge transform $\mathcal{T}$ was introduced and studied by Aluthge in his study of $p$-hyponormal operators in 1990. This notion has received much attention in recent years for a single operator $T$. For instance, quite a lot has been researched on the essential numerical range of $T$ of an operator $T$. In contrast to this, nothing is known about the joint essential numerical range of Aluthge transform $\mathcal{T}$ of an $m$–tuple operator $T = (T_1, ..., T_m)$. The focus of this note is to study the properties of the joint essential numerical range of Aluthge transform for an $m$–tuple operator $T = (T_1, ..., T_m)$. This study is therefore helpful in the development of the research on hyponormal operators and semi-hyponormal operators.

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1. Introduction

The Aluthge transform is one of the basic concepts of operator theory. The study of the properties of the Aluthge transform and their generalizations have a long and distinguished history and has attracted much more attention of
many authors such as Guoxing Ji, Ni Liu and Ze Li [5] among others. This extensive research is because Aluthge transform is a very useful tool for studying some operator classes. Especially, it is used by many researchers in the study of \( p \)-hyponormal and semi-hyponormal operators. The much attention on the study of Aluthge transform is also due to its connection with the invariant subspace problem. Jung, Ko and Pearcy[6] showed that \( T \) has a nontrivial invariant subspace if and only if \( \tilde{T} \) does. In 2007, Guoxing Ji, Ni Liu and Ze Li [5] together showed that the essential numerical range of Aluthge transform is contained in the essential numerical range of \( T \). It is also known that \( \sigma(T) = \sigma(\tilde{T}) \) holds. This note studies the joint essential numerical range of \( \tilde{T} \) of an \( m \)-tuple operator \( T = (T_1, \ldots, T_m) \in B(X) \) and establishes some of its properties. Here, \( B(X) \) denotes the algebra of all bounded linear operators acting on a complex Hilbert space \( X \).

This study is divided into four sections. Section two is a brief exposition of the theory of the joint essential numerical range while section three is devoted to establishing some of the properties of the joint essential numerical range of \( \tilde{T} \) of an \( m \)-tuple operator \( T = (T_1, \ldots, T_m) \in B(X) \).

2. Joint Essential Numerical Range

The essential numerical range \( W_e(T) \) of a single operator \( T \) is defined as the numerical range of the coset \( T + \mathcal{K}(X) \) in the Calkin algebra \( B(X)/\mathcal{K}(X) \) where \( \mathcal{K}(X) \) is the ideal of all compact operators on \( X \). In [2], Bonsall and Dancan proved that \( W_e(T) \) is nonempty, compact and satisfies \( W_e(T + \beta) = W_e(T) + \beta \) for any scalar \( \beta \). Further, they showed that \( W_e(T) = 0 \) if and only if \( T \) is compact. It is also known that the essential numerical range of an operator \( T \) is contained in the closure of the numerical range. Recall here that \( \sigma_e(T) \subset W_e(T) \) where \( \sigma_e(T) \) is the essential spectrum of the operator \( T \in B(X) \). The essential spectrum \( \sigma_e(T) \) is defined as \( \sigma_e(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm} \} \). Here, we recall that an operator \( T \in B(X) \) is said to be Fredholm if it has closed range with finite dimensional null space and a range of finite co-dimension.

The joint essential numerical range \( W_{em}(T) \) of an \( m \)-tuple of operator \( T = (T_1, \ldots, T_m) \in B(X) \) was studied in [3] and [7] among others. It is denoted and defined as

\[
W_{em}(T) = \{ r_k \in \mathbb{C}^m : \langle T_k x_n, x_n \rangle \to r_k, \ x_n \to 0 \text{ weakly}; 1 \leq k \leq m \}.
\]

It is related to the joint numerical range by the formula

\[
W_{em}(T) = \bigcap \{ W_m(T + K) : K = (K_1, \ldots, K_m) \in \mathcal{K}(X) \}.
\]
It is clear that the joint essential numerical range of an $m$-tuple operator $T = (T_1, ..., T_m) \in B(X)$ is contained in the closure of the joint numerical range.

While discussing the properties of $W_{en}(T)$, Cyprian, Masibayi and Okelo [4] together showed that the properties of numerical ranges such as compactness, nonemptiness and convexity do hold for the joint essential numerical range.

### 3. Joint Essential Numerical Range Of Aluthge Transform

This section is devoted to the subject of the joint essential numerical range of the Aluthge transform and gives some of its properties. We defined Aluthge transform earlier in this paper. We recall this definition here and use it to come up with some results. Let $T = U|T|$ be any polar decomposition of $T$ with $U$ a partial isometry and $|T| = (T^*T)^{1/2}$, then the Aluthge transform $\tilde{T}$ of $T$ is the operator $|T|^{1/2}U|T|^{1/2}$. This was defined in [1] by Aluthge. Recall also that the $^*$-Aluthge transform $\tilde{T}^*$ is defined as $|T^*|^{1/2}U|T^*|^{1/2}$.

Let $T = (T_1, ..., T_m) \in B(X)$ and $K = (K_1, ..., K_m) \in K(X)$. Suppose $T = U|T|$ and $T + K = V|T + K|$ are the polar decompositions of $T$ and $T + K$ respectively with $V$ and $U$ partial isometries and $|T + K| = ((T + K)^*(T + K))^{1/2}$. We will in what follows define $\tilde{T} + \tilde{K}$ by $\tilde{T} + \tilde{K} = |T + K|^{1/2}V|T + K|^{1/2}$. It is clear from above that $K = V|T + K| - U|T| \in K(X)$.

We define the joint essential numerical range of the Aluthge transform as $W_{en}(\tilde{T}) = \{ r \in \mathbb{C}^m : \langle \tilde{T}x_n, x_n \rangle \to r_k, \ x_n \to 0 \text{ weakly} ; 1 \leq k \leq m \}.$

So as to proceed with our study of the joint essential numerical range of Aluthge transform, we first develop the following alternate definition of the joint essential numerical range of Aluthge transform.

**Theorem 1.** Let $T = U|T|$ be a polar decomposition of an $m$-tuple operator $T = (T_1, ..., T_m) \in B(X)$. Then, for a point $r = (r_1, ..., r_m) \in \mathbb{C}^m$, there exists an orthonormal sequence $\{ x_n \}_{n=1}^\infty \in X$ such that $\langle \tilde{T}x_n, x_n \rangle \to r_k ; 1 \leq k \leq m$ and $\|\tilde{T}x_n\| \to \|\tilde{T}\|_e$ if and only if $r \in W_{en}(\tilde{T})$

**Proof.** Assume that for a point $r = (r_1, ..., r_m) \in \mathbb{C}^m$ there exists an orthonormal sequence $\{ x_n \}_{n=1}^\infty \in X$ such that $\langle \tilde{T}x_n, x_n \rangle \to r_k ; 1 \leq k \leq m$
Thus, by Bessel's inequality, there is an integer \( n \) such that
\[
\|\tilde{T}x_n\| \to \|\tilde{T}\|_e. \quad \text{Since } \|x_n\| = 1 \text{ and every orthonormal sequence } \{x_n\}
\]
converges weakly to zero, it implies that \( r \in W_{e_m}(\tilde{T}) \).

To complete the proof, it remains to show that if \( r \in W_{e_m}(\tilde{T}) \) then there exists an orthonormal sequence \( \{x_n\}_{n=1}^\infty \) such that
\[
\langle \tilde{T}x_n, x_n \rangle \to r_k \quad 1 \leq k \leq m \text{ and } \|\tilde{T}x_n\| \to \|\tilde{T}\|_e.
\]

Now, \( r \in W_{e_m}(\tilde{T}) \) implies that there is a sequence \( \{x_n\} \) of vectors such that
\[
\langle \tilde{T}x_n, x_n \rangle \to r_k, \|x_n\| = 1, \; x_n \to 0 \text{ weakly and } \|\tilde{T}x_n\| \to \|\tilde{T}\|_e; \; 1 \leq k \leq m.
\]

Choosing the set \( \{x_1, ..., x_n\} \) which satisfy
\[
|\langle \tilde{T}x_n, x_n \rangle - r| < \frac{1}{i} \; \forall \; i \text{ and letting } M \text{ be the subspace spanned by } x_1, ..., x_n \text{ and } P \text{ be the projection onto } M \text{ then we have } \|Px_n\| \to 0 \text{ as } n \to \infty.
\]

Let \( z_n = \|(I - P)x_n\|^{-1}(I - P)x_n \). We obtain
\[
\tilde{T}z_n = \|(I - P)x_n\|^{-1}(I - P)x_n. \quad \text{This gives}
\]
\[
\langle \tilde{T}z_n, z_n \rangle = \|(I - P)x_n\|^{-1}(I - P)x_n, \|\tilde{T}(I - P)x_n\|^{-1}(I - P)x_n) = \|(I - P)x_n\|^{-2}\{\langle \tilde{T}x_n, x_n \rangle - \langle \tilde{T}x_n, P x_n \rangle - \langle \tilde{T}P x_n, x_n \rangle + \langle \tilde{T}P x_n, P x_n \rangle \} \to r_k
\]
as \( n \to \infty \).

We then choose \( n \) large enough such that
\[
|\langle \tilde{T}z_n, z_n \rangle - r| < \frac{1}{n+1}. \quad \text{If we let}
\]
\[
z_n = x_{n+1} \quad \text{we get } |\langle \tilde{T}x_{n+1}, x_{n+1} \rangle - r| < \frac{1}{n+1} \quad \text{which completes the proof. }
\]

The following theorem demonstrates that the set \( W_{e_m}(\tilde{T}) \) is invariant under the unitary equivalence of operators.

**Theorem 2.** If \( U \) is a unitary operator on \( X \), then \( W_{e_m}(U^*\tilde{T}U) = W_{e_m}(\tilde{T}) \).

**Proof.** Let \( U \) be unitary. The proof is complete if either \( W_{e_m}(U^*\tilde{T}U) = \emptyset \) or \( W_{e_m}(\tilde{T}) = \emptyset \). Now let \( W_{e_m}(\tilde{T}) \neq \emptyset \) and let \( r = (r_1, ..., r_m) \in W_{e_m}(\tilde{T}) \). There exists an orthonormal sequence \( \{x_n\} \in X \) such that
\[
\langle \tilde{T}k x_n, x_n \rangle \to r_k \quad \text{and } \|\tilde{T}k x_n\| \to \|\tilde{T}_k\|_e; \; 1 \leq k \leq m.
\]

Then, by passing to a subsequence we can assume that
\[
\sum_{n=1}^\infty |\langle \tilde{T}x_n, x_n \rangle|^2 < \infty \quad (1)
\]

Let \( n_1 = 1 \). Then
\[
\sum_{n=1}^\infty \langle \tilde{T}x_n, x_n \rangle |^2 \leq ||\tilde{T}x_n||^2 \quad \text{and } \sum_{n=1}^\infty |\langle \tilde{T}x_n, x_{n_1} \rangle|^2 \leq ||\tilde{T}^*x_{n_1}||^2.
\]

Thus, by Bessel’s inequality, there is an integer \( n_2 > n_1 \) such that
\[
\sum_{n=n_2}^\infty |\langle \tilde{T}x_n, x_n \rangle|^2 < \frac{1}{2} \quad \text{and } \sum_{n=n_2}^\infty |\langle \tilde{T}x_n, x_{n_1} \rangle|^2 < \frac{1}{2}.
\]
If this procedure is repeated, a strictly increasing sequence \( \{n_t\}_{t=1}^\infty \) of positive integers is obtained such that we have

\[
\sum_{n=n_t+1}^\infty |\langle \tilde{T}x_{n_t}, x_n \rangle|^2 < \frac{1}{2^t} \quad \text{and} \quad \sum_{n=n_t+1}^\infty |\langle \tilde{T}x_n, x_{n_t} \rangle|^2 < \frac{1}{2^t} \tag{2}
\]

(1) and (2) imply that

\[
\sum_{t,l=1}^\infty |\langle \tilde{T}x_{n_t}, x_{n_l} \rangle|^2 < \infty \tag{3}
\]

If \( P \) is an orthogonal projection onto the subspace \( M \) spanned by \( x_{n_1}, x_{n_2}, \ldots \), then

\[
\sum_{t,l=1}^\infty |\langle P\tilde{T}P x_{n_t}, x_{n_l} \rangle|^2 = \sum_{t,l=1}^\infty |\langle \tilde{T}x_{n_t}, x_{n_l} \rangle|^2 < \infty \quad \text{(by (3))},
\]

hence \( P\tilde{T}P \) is a Hilbert - Schmidt operator and therefore \( P\tilde{T}P \in \mathcal{K}(X) \). Since \( \tilde{T} \) is compact, \( U\tilde{T}U^* \) is compact. Now, let \( P \in B(X) \) be an infinite dimensional projection such that \( PU\tilde{T}U^*P \in \mathcal{K}(X) \). There is thus an orthonormal sequence \( \{x_n\} \in X \) such that \( Px_n = x_n \ \forall n \). Let \( K = (K_1, \ldots, K_m) \in \mathcal{K}(X) \). For any \( K_j : j \in [1, m] \), \( P\tilde{T}P = K_j \) and thus \( \langle PU\tilde{T}U^*P x_n, x_n \rangle = \langle K_j x_n, x_n \rangle \) implying \( \langle \tilde{T}j x_n, x_n \rangle = r_j + \langle K_j x_n, x_n \rangle \). From the orthonormality of sequence \( \{x_n\} \), we get \( K_j x_n \) converging weakly to 0 in norm as \( n \to \infty \), \( j \in [1, m] \). Therefore, \( \langle U\tilde{T}U^* x_n, x_n \rangle \longrightarrow r_j \) as \( n \to \infty \) implying \( r \in W_{e_m}(U^*\tilde{T}U) \). This implies that \( W_{e_m}(\tilde{T}) \subseteq W_{e_m}(U^*\tilde{T}U) \).

Replace \( \tilde{T} \) by \( U\tilde{T}U^* \) in the above to get \( W_{e_m}(U^*\tilde{T}U) \subseteq W_{e_m}(\tilde{T}) \) which completes the proof of the theorem.

We state the following theorem without proof.

**Theorem 3.** Let \( T = (T_1, \ldots, T_m) \in B(X) \). Then \( W_{e_m}(\tilde{T}^*) = (W_{e_m}(\tilde{T}))^* = \{r : \tilde{r} \in W_{e_m}(\tilde{T})\} \).

The following theorem will be used in the sequel hence we prove it before proceeding further. Here, we recall that the essential norm of \( \tilde{T} \) of an \( m \)-tuple of operators \( T = (T_1, \ldots, T_m) \in B(X) \) is denoted and defined by

\[
\|T\|_e = \inf\{\|\tilde{T} + K\| : K = (K_1, \ldots, K_m) \in \mathcal{K}(X)\}.
\]

**Theorem 4.** Let \( T = U|T| \) be a polar decomposition of an \( m \)-tuple of operators \( T = (T_1, \ldots, T_m) \in B(X) \). If \( \{x_n\} \in X, \|x_n\| = 1 \) and \( x_n \to 0 \) weakly, then \( \limsup \|\tilde{T}x_n\| \leq \|\tilde{T}\|_e \).
Proof. For every operator $K = (K_1, \ldots, K_m) \in \mathcal{K}(X)$, 
$\|\tilde{T} x_n\| = \|\tilde{T} + K\| x_n - K x_n\| \leq \|\tilde{T} + K\| x_n + \|K x_n\|$. Since $x_n \to 0$ weakly implies that $K x_n \to 0$ in the norm we have $\|\tilde{T} x_n\| \leq \|\tilde{T} + K\| x_n \leq \|\tilde{T} + K\|$. Then $\limsup \|\tilde{T} x_n\| \leq \|\tilde{T} + K\|$. The theorem follows when we take infimum over compact $K$. □

We also give proof to the following theorem.

Theorem 5. Let $T = U|T|$ be a polar decomposition of an $m$–tuple of operators $T = (T_1, \ldots, T_m) \in B(X)$. Then there exists an orthonormal sequence $\{x_n\}_{n=1}^\infty \in X$ such that $\|\tilde{T} x_n\| \to \|\tilde{T}\|_e$ and if $P$ is an infinite rank projection with $\tilde{T} = P x_n$ then $\{x_n\}_{n=1}^\infty$ can be chosen such that $P x_n = x_n$ for all $n$.

Proof. Suppose we have an orthonormal set of vectors $\{x_1, \ldots, x_{n-1}\}$ such that
$P x_m = x_m$ and $\|\tilde{T} x_m\| \geq \|\tilde{T}\|_e - m^{-1}$ where $m = 1, \ldots, n-1$.
Let $E$ be the orthogonal projection onto subspace $\mathcal{M}$ spanned by $\{x_1, \ldots, x_{n-1}\}$. Then $TE$ is compact since $E$ is compact. Note that $E P x_m = E x_m$ for all $m$ and $E P = E$. Thus $\|\tilde{T} (I - E) P\| = \|\tilde{T} (I - E)\| \geq \|\tilde{T} (I - E)\|_e = \|\tilde{T}\|_e$. Thus there exists a unit vector $x$ such that $(I - E) P x = x$ and $\|\tilde{T} x\| \geq \|\tilde{T}\|_e - n^{-1}$ implying that $\|\tilde{T} (I - E) P x\| \geq \|\tilde{T} (I - E) P\|_e - n^{-1}$. Therefore, $x_1, \ldots, x_n$ is constructed as orthonormal such that $P x_m = x_m$ for $m = 1, \ldots, n$ and $\|\tilde{T} x_m\| \geq \|\tilde{T}\|_e - m^{-1}$. The required sequence $\{x_n\}$ has thus been constructed. □

Theorem 6. Let $T = U|T|$ be a polar decomposition of $T = (T_1, \ldots, T_m) \in B(X)$ and $r = (r_1, \ldots, r_m) \in \mathbb{C}^m$. If $P$ is an infinite-dimensional projection such that $P(\tilde{T} - r_j I) P \in \mathcal{K}(X)$ for $j = 1, \ldots, k$ and $\|\tilde{T} P\|_e = \|\tilde{T}\|_e$ then $r \in W_{e_m}(\tilde{T})$.

Proof. Let $P \in B(X)$ be an infinite dimensional projection such that $P(\tilde{T} - r_j I) P \in \mathcal{K}(X)$ for $j = 1, \ldots, k$ and $\|\tilde{T} P\|_e = \|\tilde{T}\|_e$. There is thus an orthonormal sequence $\{x_n\} \in X$ such that $\|T P x_n\| \to \|\tilde{T} P\|_e$ and $P x_n = x_n \forall n$. Then, $\|\tilde{T} x_n\| \to \|\tilde{T}\|_e$ since $\|T P\|_e = \|\tilde{T}\|_e$.
Let $K = (K_1, \ldots, K_m) \in \mathcal{K}(X)$. For any $K_j : j \in [1, m]$ and $P T P = K_j$ we have $\langle P T P x_n, x_n\rangle = \langle K_j x_n, x_n\rangle$ implying $\langle \tilde{T} j x_n, x_n\rangle = r_j + \langle K_j x_n, x_n\rangle$. From the orthonormality of sequence $\{x_n\}$, we get $K_j x_n$ converging weakly to 0 in norm as $n \to \infty$, $j \in [1, m]$. Therefore, $\langle \tilde{T} j x_n, x_n\rangle \to r_j$ as $n \to \infty$ implying $r \in W_{e_m}(\tilde{T})$. □

Theorem 7. Let $T = (T_1, \ldots, T_m) \in B(X)$ and $\tilde{T} = |T|^{1/2} \cup |T|^{1/2}$. If $0 \in W_{e_m}(\tilde{T})$ then $\|\tilde{T}\|^2 + |r|^2 \leq \|\tilde{T} + r\|^2$ for any $r = (r_1, \ldots, r_m) \in \mathbb{C}^m$. 

Proof. Let \( 0 \in W_{e_m}(T) \). Then there exists a sequence \( \{x_n\} \in X \) such that \( \langle \tilde{T}_k x_n, x_n \rangle \to 0 \), \( \|x_n\| = 1 \) and \( \|\tilde{T}_k x_n\| \to \|\tilde{T}_k\| \) for \( 0 \leq k \leq m \). Note that \( \|\tilde{T}_k\|^2 + |r_k|^2 = \lim_{n \to \infty} \| (\tilde{T}_k + r_k) x_n \|^2 \leq \|\tilde{T}_k + r_k\|^2 \), \( 0 \leq k \leq m \). Now,
\[
\sum_{k=1}^{n} \|\tilde{T}_k\|^2 + \sum_{k=1}^{n} |r_k|^2 \leq \sum_{k=1}^{n} \|\tilde{T}_k + r_k\|^2.
\]
Thus \( \|\tilde{T}_k\|^2 + |r_k|^2 \leq \|\tilde{T}_k + r_k\|^2 \) for any \( r = (r_1, ..., r_m) \in \mathbb{C}^m \). \( \square \)

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References


