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COUPLED FIXED POINTS AND COUPLED BEST PROXIMITY POINTS IN MODULAR FUNCTION SPACES

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Abstract: We generalize the notion of coupled fixed points and coupled best proximity points in the context of modular function spaces. We have found sufficient conditions for the existence and uniqueness of coupled fixed points and coupled best proximity points in modular function spaces.

We present an application of the main result, that generalizes well known examples about coupled best proximity points.

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1. Introduction

A fundamental result in fixed point theory is the Banach Contraction Principle in Banach spaces or in complete metric spaces. Fixed point theory is an important tool for solving equations $Tx = x$ for mappings T defined on subsets of metric or normed spaces. It is widely applied to nonlinear integral equations and differential equations.

The concept of coupled fixed point theorem is introduced in [3]. Later on Bhaskar and Lakshmikantham [1] introduced the notions of a mixed monotone mapping, studied the problems of uniqueness of a coupled fixed point in partially ordered metric spaces and applied their theorems to problems of the existence of solution for a periodic boundary value problem. Harjani, López and Sadarangani obtained in [5] some coupled fixed point theorems for a mixed

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monotone operator in a complete metric space endowed with a partial order by using altering distance functions. They applied their results to the study of the existence and uniqueness of a nonlinear integral equation. Recent results about the application of coupled fixed point theorems for solving integral equations are obtained in [8].

Other kind of a generalization of the Banach Contraction Principle is the notion of cyclic maps [12]. Because a non-self mapping $T : A \to B$ does not necessarily have a fixed point, one often attempts to find an element x which is in some sense closest to Tx. Best proximity point theorems are relevant in this perspective. The notion of best proximity point is introduced in [2]. This definition is more general than the notion of cyclic maps [12], in sense that if the sets intersect then every best proximity point is a fixed point. A sufficient condition for the existence and uniqueness of the best proximity points in uniformly convex Banach spaces is given in [2].

First results in the approximation of the sequence of successive iterations, which converges to the best proximity point for cyclic contractions is obtained in [25] and for coupled best proximity points in [6].

Besides the idea of defining a norm and considering a Banach space, another direction of generalization of the Banach Contraction Principle is based on considering an abstractly given functional defined on a linear space, which controls the growth of the members of the space. This functional is usually called modular and it defines a modular space. The theory of modular spaces was initiated by Nakano [20] in connection with the theory of ordered spaces, which was further generalized by Musielak and Orlicz [19]. Modular function spaces are subclass of the modular spaces. The study of the geometry of modular function spaces was initiated by Kozlowski [15, 13, 14]. Fixed point results in modular function spaces were obtained first by Khamsi, Kozlowski and Reich [10]. Further development of the theory of fixed points in modular function spaces can be found in the exhaustive references of the survey article [16] and in the book [9]. Kozlowski has contributed a lot towards the study of modular function spaces both on his own and with his collaborators. First results about best proximity points in modular function spaces are obtained in [7, 24].

We have tried to generalize the idea of coupled fixed points and coupled best proximity points in modular function spaces.

2. Preliminaries

Following [16, 9] we will recall some basic notions and facts about modular function spaces.

Let Ω be a nonempty set and Σ be a nontrivial σ–algebra of subsets of Ω. Let $\mathcal{P}$ be a δ–ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \cup K_n$. By $\mathcal{E}$ we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$ we will denote the space of all extended measurable functions, i.e. all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$. By $\mathbf{1}_A$ we denote the characteristic function of the set A.

Definition 1. Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial convex and even function. We say that ρ is a regular convex function pseudomodular if:

(i) $\rho(0) = 0$;

(ii) ρ is monotone, i.e., $|f(\omega)| \leq |g(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;

(iii) ρ is orthogonaly subadditive, i.e., $\rho(f\mathbf{1}_{A\cup B}) \leq \rho(f\mathbf{1}_A) + \rho(f\mathbf{1}_B)$, where $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{M}_\infty$;

(iv) ρ has the Fatou property, i.e., $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;

(v) ρ is order continuous in $\mathcal{E}$, i.e., $g_n \in \mathcal{E}$ and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

Similarly as in the case of measure spaces, we say that a set $A \in \Sigma$ is ρ–null if $\rho(g\mathbf{1}_A) = 0$ for every $g \in \mathcal{E}$. We say that a property holds ρ–almost everywhere if the exceptional set is ρ–null. As usual we identify any pair of measurable sets whose symmetric difference is ρ–null as well as any pair of measurable functions differing only on a ρ–null set. With this in mind we define

$$\mathcal{M}(\Omega, \sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty; |f(\omega)| < \infty\rho \ - \ a.e.\},$$

where each $f \in \mathcal{M}(\Omega, \sigma, \mathcal{P}, \rho)$ is actually an equivalence class of functions equal ρ a.e. rather than an individual function. Where no confusion exists we will write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \sigma, \mathcal{P}, \rho)$.

Definition 2. Let ρ be a regular convex function pseudomodular.

(1) We say that $\rho(0)$ is a regular convex function semimodular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies $f = 0$ ρ–a.e.;

(2) We say that ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0$ ρ–a.e.

The class of all nonzero regular convex function modular defined on Ω will be denoted by $\mathfrak{R}$.

Let us denote $\rho(f,E)=\rho(f1_E)$ for $f\in\mathcal{M}$, $E\in\Sigma$. It is easy to prove that $\rho(f,E)$ is a function pseudomodular in the sense of Definition 2.1.1 in [15] (more precisely, it is a function pseudomodular with the Fatou property). Therefore, we can use all results of the standard theory of modular function spaces as per the framework defined by Kozlowski [15, 13, 14], see also Musielk [19] for the basics of the general modular theory.

Definition 3. Let ρ be a convex function modular.

(a) A modular function space is the vector space $L_\rho(\Omega,\Sigma)$, or briefly L_ρ, defined by $L_\rho=\{f\in\mathcal{M}:\rho(\lambda f)\to 0 \text{ as } \lambda\to 0\}$;

(b) The following formula defines a norm in L_ρ (frequently called Luxemburg norm): $\|f\|_\rho=\inf\left\{\alpha>0:\rho\left(\frac{f}{\alpha}\right)\le 1\right\}$.

For the rest of the article if we state someting about a norm we will mean Luxemburg norm $\|\cdot\|_\rho$, which is generated by the modular ρ.

In this way, Lebesgue, Orlicz, Musielak–Orlicz, Lorentz, Orlicz–Lorentz are examples of modular function spaces.

In the following theorem, we recall some of the basic properties of modular function spaces.

Theorem 4. *Let* $\rho\in\mathfrak{R}$.

(1) $(L_\rho,\|f\|_\rho)$ *is a complete and the norm* $\|\cdot\|_\rho$ *is a monotone w.r.t the natural order in* $\mathcal{M}$.

(2) $\|f_n\|_\rho\to 0$ *iff* $\rho(\alpha f_n)\to 0$ *for every* $\alpha>0$.

(3) If $\rho(\alpha f_n)\to 0$ *for an* α*, then there exists a subsequence* $\{g_n\}$ *of* $\{f_n\}$ *such that* $g_n\to 0$ ρ*–a.e.*

(4) If $\{f_n\}$ *converges uniformly to* f *on a set* $E\in\mathcal{P}$*, then* $\rho(\alpha(f_n-f),E)\to 0$ *for every* $\alpha>0$.

(5) Let $f_n\to f$ ρ*–a.e. There exists a nondecreasing sequence of sets* $H_k\in\mathcal{P}$ *such that* $H_k\uparrow\Omega$ *and* f_n *converges uniformly to* f *on every* H_k *(Egoroff Theorem).*

(6) $\rho(f)\le\liminf\rho(f_n)$ *whenever* $f_n\to f$ ρ*–a.e. (Note that this property is equivalent to the Fatou property).*

(7) Define $L_\rho^0\{f \in L_\rho : \rho(f, \cdot)$ *is order continuous*$\}$ *and* $E_\rho = \{f \in L_\rho : \lambda f \in L_\rho^0$ *for every* $\lambda > 0\}$ *we have*

(a) $L_\rho \supset L_\rho^0 \supset E_\rho$*;*

(b) E_ρ *has the Lebesgue property, i.e.* $\rho(\alpha f, D_k) \to 0$ *for* $\alpha > 0$, $f \in E_\rho$ *and* $D_k \downarrow \emptyset$*;*

(c) E_ρ *is the closure of* $\mathcal{E}$ *(in the sense of* $\|\cdot\|_\rho$*).*

The next definition gives generalizations of the classical notions for normed spaces in the context of modular function spaces.

Definition 5. Let $L_\rho \in \mathfrak{R}$.

(a) We say that $\{f_n\}$ is a ρ–convergent to f and we write $f_n \to f(\rho)$ if and only if $\rho(f_n - f) \to 0$;

(b) A sequence $\{f_n\}_{n=1}^{\infty} \subset L_\rho$ is called ρ–Cauchy if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$, such that for any $m > n \geq N$ there holds the inequality $\rho(f_m - f_n) < \varepsilon$;

(c) The modular function space L_ρ is called ρ–complete if any ρ–Cauchy sequence is ρ–convergent.

(d) A set $B \subset L_\rho$ is called ρ–closed if for any sequence of $f_n \in B$, the convergence $f_n \to f(\rho)$ implies that f belongs to B;

(e) A set $B \subset L_\rho$ is called ρ–bounded if its ρ–diameter $\delta_\rho(B) = \sup\{\rho(f - g) : f, g \in B\} < \infty$;

(f) A set $B \subset L_\rho$ is called ρ–a.e. closed if for any $\{f_n\}$ in C which ρ–a.e. converges to some f, then we must have $f \in C$;

(g) Let $A, B \subset L_\rho$. We define the ρ–distance between the sets A and B by

$$d_\rho(A, B) = \inf\{\rho(f - g) : f \in A, g \in B\}$$

and $d_\rho(f, B) = \inf\{\rho(f - g) : g \in B\}$ if A consists of a single element f;

(h) We say that ρ has the Δ_2–property if $\sup_{n\in\mathbb{N}} \rho(2f_n, D_k) \to 0$, whenever $D_k \downarrow \emptyset$ and $\sup_{n\in\mathbb{N}} \rho(f_n, D_k) \to 0$.

(i) ([9], p.116) A function modular $\rho \in \mathfrak{R}$ is called uniformly continuous if for any $L > 0$ and $\varepsilon > 0$ there exists $\delta = \delta(L, \varepsilon) > 0$ such that if $\rho(x) \leq L$ and $\rho(y) < \delta$ there holds the inequality $|\rho(x + y) - \rho(x)| < \varepsilon$.

Theorem 6. *Let $\rho \in \mathfrak{R}$. Then L_ρ is ρ–complete.*

Theorem 7. *Let $\rho \in \mathfrak{R}$. The following conditions are equivalent*

(a) ρ has Δ_2;

(b) L_ρ^0 is a linear subspace of L_ρ;

(c) $L_\rho = L_\rho^0 = E_\rho$;

(d) If $\rho(f_n) \to 0$, then $\rho(2f_n) \to 0$;

(e) If $\rho(\alpha f_n) \to 0$ for an $\alpha > 0$, then $\|f_n\|_\rho \to 0$, i.e., the modular convergence is equivalent to the norm convergence.

Let us mention that the ρ–convergence do not imply ρ–Cauchy, since ρ– does not satisfy the triangle inequality. If ρ has Δ_2–property then ρ–convergence imply ρ–Cauchy.

Generalization of convexity properties for Banach spaces are investigated for modular function spaces in [11]. As demonstrated in [16] one concept of uniform convexity for Banach spaces generates several different types of uniform convexity in modular function spaces. This is due primarily to the fact that in general the modular function is not homogeneous.

Definition 8. Let $\rho \in \mathfrak{R}$ and $i \in \{1, 2\}$. Let $r > 0$, $\varepsilon > 0$. Define

$$D_i(r, \varepsilon) = \{(f, g) : f, g \in L_\rho, \rho(f) \le r, \rho(g) \le r, \rho\left(\frac{f-g}{i}\right) \ge \varepsilon r\}.$$

Let $\delta_i(r, \varepsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right) : (f, g) \in D_i(r, \varepsilon)\right\} > 0$ if $D_i(r, \varepsilon) \neq \emptyset$ and $\delta_i(r, \varepsilon) = 1$ if $D_i(r, \varepsilon) = \emptyset$.

(i) We say that ρ satisfies (UCi) if for any $r > 0$, $\varepsilon > 0$ there holds the inequality $\delta_i(r, s) > 0$.

(ii) We say that ρ satisfies (UUCi) if for every $s \ge 0$, $\varepsilon > 0$ there exists $\eta_i(s, \varepsilon) > 0$, depending on s and ε such that

$$\delta_i(r, s) > \eta_i(s, \varepsilon) > 0 \text{ for } r > s.$$

If ρ is (UC1) we obtain that the inequality

$$\rho\left(\frac{x+y}{2}\right) \le r\left(1 - \delta_1(r, \varepsilon)\right) \tag{1}$$

holds for every $\rho(x), \rho(y) \le r$ and $\rho\left(x - y\right) \ge r\varepsilon$.

Proposition 9. *The following conditions characterize relationship between the notions, that are defined in Definition 8*

(1) *(UUCi) implies (UCi) for* $i \in 1, 2$;

(2) $\delta_1(r, \varepsilon) \leq \delta_2(r, \varepsilon)$;

(3) *(UC1) implies (UC2);*

(4) *(UUC1) implies (UUC2);*

(5) *If* $\rho \in \mathfrak{R}$, *then (UUC1) and (UUC2) are equivalent;*

(6) *I* ρ *is homogeneous (e.g. is a norm) then all conditions (UC1), (UC2), (UUC1) and (UUC2) are equivalent.*

Lemma 10. *([24]) Let* $\rho \in \mathfrak{R}$. *Let* ρ *be (UC1), has the* Δ_2*–property,* $A \subset L_\rho$ *be a* ρ*–closed and convex subset,* $B \subset L_\rho$ *be* ρ*–closed subset and* $A \cup B$ *be* ρ*–bounded. If the sequences* $\{x_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty} \subset A$ *and* $\{y_n\}_{n=1}^{\infty} \subset B$ *be such that:*

(i) $\lim_{n\to\infty} \rho(z_n - y_n) = d_\rho$;

(ii) *for every* $\varepsilon > 0$ *there exists* $N_0 \in \mathbb{N}$ *such that for every* $m > n \geq N_0$ *there holds the inequality* $\rho(x_m - y_n) \leq d_\rho + \varepsilon$.

Then for every $\varepsilon > 0$ *there exists* $N_1 \in \mathbb{N}$ *such that for every* $m > n \geq N_1$ *there holds the inequality* $\rho(x_m - z_n) < \varepsilon$.

Lemma 11. *([24]) Let* $\rho \in \mathfrak{R}$. *Let* ρ *be (UC1), has the* Δ_2*–property,* A *be a* ρ*–closed and convex subset of* L_ρ, B *be* ρ*–closed subset of* L_ρ *and* $A \cup B$ *be* ρ*–bounded. If the sequences* $\{x_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty} \subset A$ *and* $\{y_n\}_{n=1}^{\infty} \subset B$ *be such that:*

(i) $\lim_{n\to\infty} \rho(z_n - y_n) = d_\rho$;

(ii) $\lim_{n\to\infty} \rho(x_n - y_n) = d_\rho$.

Then $\lim_{n\to\infty} \rho(x_n - z_n) = 0$.

Lemma 12. *([24]) Let* $\rho \in \mathfrak{R}$. *Let* ρ *has the* Δ_2*–property, be uniformly continuous,* $A, B \subset L_\rho$ *be subsets and* $A \cup B$ *be* ρ*–bounded. If the sequences* $\{x_n\}_{n=1}^{\infty}, \{z_n\}_{n=1}^{\infty} \subset A$ *and* $\{y_n\}_{n=1}^{\infty} \subset B$ *be such that:*

(i) $\lim_{n\to\infty} \rho(z_n - x_n) = 0$;

(ii) $\lim_{n\to\infty} \rho(z_n - y_n) = d_\rho$.

Then $\lim_{n\to\infty} \rho(x_n - y_n) = d_\rho$.

We recall that M is called an Orlicz function, provided M is even, convex, continuous nondecreasing in $[0,\infty)$ function with $M(0) = 0$, $M(t) > 0$ for any $t \neq 0$. Let M be an Orlicz function and let (Ω, Σ, μ) be a measure space. Let us consider the space $L^0(\Omega)$ consisiting of all measurable real–velued functions on Ω and define for every $f \in L^0(\Omega)$ the Orlicz function modular $\widetilde{M}(f) = \int_\Omega M(f(t))d\mu(t)$.

Definition 13. The Orlicz space $L_M(\Omega, \Sigma, \mu)$ is the space of all classes of equivalent μ–measurable functions $f : \Omega \to \mathbb{R}$ over the measure space (Ω, Σ, μ) such that $\widetilde{M}(\lambda f) \to 0$ as $\lambda \to 0$ or equivalently $\widetilde{M}\left(\frac{f}{\lambda}\right) < \infty$ for some $\lambda > 0$.

The function $\widetilde{M}$ is a regular convex function modular and it is called Orlicz function modular. An extensive study of Orlicz spaces can be found in [17, 18, 21, 22].

If $M(t) = |t|^p$, $p \geq 1$ we obtain the space $L_p(\Omega, \Sigma, \mu)$. The most common examples of Orlicz spaces are the sequence spaces ℓ_M, the function spaces $L_M(0,1)$ and $L_M(0,\infty)$ that correspond to the cases: Ω countable union of atoms of equal mass, $\Omega = [0,1]$ and $\Omega = (0,\infty)$, μ the usual Lebesgue measure.

We say that M satisfies the Δ_2–condition if there exist constants $C, t_0 > 0$, such that $M(2t) \leq CM(t)$ for any $t \geq t_0$. It is easy to observe that if M satisfies the Δ_2–condition, then the Orlicz function modular $\widetilde{M}$ has the Δ_2 property.

If we restrict to the Orlicz space $L_M(0,1)$, then the Orlicz function modular is defined by $\widetilde{M}(f) = \displaystyle\int_0^1 M(f(s))d\mu(s)$. We will denote the corresponding modular function space by $L_{\widetilde{M}}(0,1)$. When $M = |t|^p$ we will denote $L_{\widetilde{M}}(0,1)$ by $L_{\widetilde{p}}(0,1)$.

3. Main Results

Coupled best proximity point and coupled fixed points in metric spaces are defined in [23] and [3]. We will give definitions for coupled best proximity points following and coupled fixed points in modular function spaces.

Definition 14. Let A and B be nonempty subsets of a modular function space X, $F : A \times A \to B$. An ordered pair $(x,y) \in A \times A$ is called a coupled best proximity point of F if

$$\rho(x - F(x,y)) = \rho(y - F(y,x)) = d.$$

Definition 15. Let A be nonempty subset of a modular function space X, $F : A \times A \to A$. An ordered pair $(x, y) \in A \times A$ is said to be a coupled fixed point of F in A if $x = F(x, y)$ and $y = F(y, x)$.

It is easy to see that if $A = B$ in Definition 14, then a coupled best proximity point reduces to a coupled fixed point.

An iterated sequences for investigation of existence of coupled best proximity point and coupled fixed points in metric spaces is defined in [3, 23]. We will give definitions for coupled best proximity points and coupled fixed points in modular functional spaces.

Definition 16. ([23])Let A and B be nonempty subsets of a modular function space X. Let $F : A \times A \to B$ and $G : B \times B \to A$. For any pair $(x, y) \in A \times A$ we define the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ by $x_0 = x$, $y_0 = y$ and

$$\begin{aligned} x_{2n+1} &= F(x_{2n}, y_{2n}), & y_{2n+1} &= F(y_{2n}, x_{2n}) \\ x_{2n+2} &= G(x_{2n+1}, y_{2n+1}), & y_{2n+2} &= G(y_{2n+1}, x_{2n+1}) \end{aligned}$$

for all $n \geq 0$.

We will generalize the notion of cyclic contraction pair (F, G) of maps in metric spaces [4, 23] for modular function spaces.

Definition 17. Let A and B be nonempty subsets of a modular function space X, $F : A \times A \to B$ and $G : B \times B \to A$. The ordered pair (F, G) is said to be a cyclic contraction if there exist non-negative numbers α, β, such that $\alpha + \beta < 1$ and there holds the inequality

$$\rho(F(x, y) - G(u, v)) \leq \alpha\rho(x - u) + \beta\rho(y - v) + (1 - (\alpha + \beta))d(A, B)$$

for all $(x, y) \in A \times A$ and $(u, v) \in B \times B$.

Definition 18. Let A be nonempty subsets of a modular function space X, $F : A \times A \to A$ is said to be a ρ–contraction if there exist non-negative numbers α, β, such that $\alpha + \beta < 1$ and there holds the inequality

$$\rho(F(x, y) - F(u, v)) \leq \alpha\rho(x - u) + \beta\rho(y - v)$$

for all $x, y, u, v \in A$.

Theorem 19. *Let $\rho \in \mathfrak{R}$. Let $A \subset L_\rho$ be nonempty, ρ–closed and ρ–bounded. Let $F : A \times A \to A$ be a ρ–contraction. Then F has unique coupled fixed points $(x, y) \in A$. Moreover for any $(x_0, y_0) \in A$ the sequences $\{x_n\}$, $\{y_n\}$ defined by the equations:*

$$\begin{aligned} x_1 = F(x_0, y_0),\ y_1 = F(y_0, x_0), \\ x_{n+1} = F(x_n, y_n),\ y_{n+1} = F(y_n, x_n) \end{aligned} \tag{2}$$

converge to the unique coupled fixed points $(x,y)\in A$.

Proof. : From the definition of (2) we get

$$\begin{array}{rcl}\rho(x_{n+1}-x_n) & = & \rho(F(x_n,y_n)-F(x_{n-1},y_{n-1}))\\ & \leq & \alpha\rho(x_n-x_{n-1})+\beta\rho(y_n-y_{n-1})\end{array}$$

and

$$\begin{array}{rcl}\rho(y_{n+1}-y_n) & = & \rho(F(y_n,x_n)-F(y_{n-1},x_{n-1}))\\ & \leq & \alpha\rho(y_n-y_{n-1})+\beta\rho(x_n-x_{n-1}).\end{array}$$

Therefore after summing the above inequalities we get

$$\begin{array}{rcl}S_1 & = & \rho(x_{n+1}-x_n)+\rho(y_{n+1}-y_n)\\ & \leq & (\alpha+\beta)(\rho(x_n-x_{n-1})+\rho(y_n-y_{n-1}))\\ & \leq & (\alpha+\beta)^2(\rho(x_{n-1}-x_{n-1})+\rho(y_{n-1}-y_{n-2})\\ & & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ & \leq & (\alpha+\beta)^n(\rho(x_1-x_0)+\rho(y_1-y_0)).\end{array}$$

From the last chain of inequalities we get that for any $n,p\in\mathbb{N}$ there holds

$$\rho(x_{n+p}-x_n)+\rho(y_{n+p}-y_n)\leq(\alpha+\beta)^n(\rho(x_p-x_0)+\rho(y_p-y_0)).$$

From the assumption that A ia a ρ bounded set it follows that existence of $M>0$, such that $\rho(u-v)\leq M$ for any $u,v\in A$. Consequently

$$\begin{array}{rcl}S_2 & = & \rho(x_{n+p}-x_n)+\rho(y_{n+p}-y_n)\\ & \leq & (\alpha+\beta)^n(\rho(x_p-x_0)+\rho(y_p-y_0))\\ & \leq & 2M(\alpha+\beta)^n<\varepsilon.\end{array}$$

From $\alpha+\beta\in[0,1)$ it follows that for any $\varepsilon>0$ there is N_0 such that for any $n\geq N_0$, there holds $(\alpha+\beta)^n<\frac{\varepsilon}{2M}$. Then for any $n_0\geq\mathbb{N}$ and any $p\geq1$ the inequality $\max\{\rho(x_{n+p}-x_n),\rho(y_{n+p}-y_n)\}\leq 2M(\alpha+\beta)^n<\varepsilon$ holds. Thus for every $\varepsilon>0$ there exists N_0 such that for any $n\geq N_0$ the inequality $\rho(z_{n+p}-z_n)<\varepsilon$ holds, for $p\in\mathbb{N}$ and $z=x$ or $z=y$. Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. From the completeness of L_ρ and the closeness of A it follows that there are $x,y\in A$, such that $\lim_{n\to\infty}\rho(x_n-x)=0$ and $\lim_{n\to\infty}\rho(y_n-y)=0$.

We will proof that $F(x,y)=x$ and $F(y,x)=y$.

$$\begin{array}{rl}\rho\left(\dfrac{x-F(x,y)}{2}\right) & =\rho\left(\dfrac{x-F(x_n,y_n)+F(x_n,y_n)-F(x,y)}{2}\right)\\ & \leq\rho(x-F(x_n,y_n))+\rho(F(x_n,y_n)-F(x,y))\\ & \leq\rho(x-F(x_n,y_n))+\alpha\rho(x_n-x)+\beta\rho(y_n-y)\\ & \leq\rho(x-x_{n+1})+\alpha\rho(x_n-x)+\beta\rho(y_n-y).\end{array}$$

But $\lim_{n\to\infty} \rho(x - x_n) = 0$ and $\lim_{n\to\infty} \rho(y - y_n) = 0$ and thus $\rho(x - F(x, y)) = 0$ i.e. $x = F(x, y)$. In a similar fashion we get

$$\begin{aligned}\rho\left(\frac{y - F(y, x)}{2}\right) &= \rho\left(\frac{y - F(y_n, x_n) + F(y_n, x_n) - F(y, x)}{2}\right)\\ &\le \rho(y - F(y_n, x_n)) + \rho(F(y_n, x_n) - F(y, x))\\ &\le \rho(x - F(x_n, y_n)) + \alpha\rho(y_n - y) + \beta\rho(x_n - x)\\ &\le \rho(y - y_{n+1}) + \alpha\rho(y_n - y) + \beta\rho(x_n - x).\end{aligned}$$

From $\lim_{n\to\infty} \rho(x-x_n) = 0$ and $\lim_{n\to\infty} \rho(y-y_n) = 0$ it follows that $\rho(y-F(y, x)) = 0$ i.e. $y = F(y, x)$. Therefore (x, y) is coupled fixed points of F in A.

Suppose (u, v) is also another coupled fixed points of F in A i.e. $u = F(u, v)$ and $v = F(v, u)$. From the inequalities $\rho(x - u) = \rho(F(x, y) - F(u, v) \le \alpha\rho(x-u)+\beta\rho(y-v)$ and $\rho(y-v) = \rho(F(y, x)-F(v, u) \le \alpha\rho(y-v)+\beta\rho(x-u)$ we get that $\rho(x-u)+\rho(y-v) \le (\alpha+\beta)(\rho(x-u)+\rho(y-v)) < \rho(x-u)+\rho(y-v)$, which is a contradiction. So the assumption that there exists second coupled fixed points is not true.

By what we have just proved for any initial guess (u_0, v_0) the sequences $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ defined in (2) converge to a coupled fixed point (u, v) of F in A, which is unique. Thus $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ converge to the coupled fixed point (x, y) □

Theorem 20. *Let $\rho \in \mathfrak{R}$. Assume that ρ satisfies (UC1), has the Δ_2–property and be uniformly continuous. Let $A, B \subseteq L_\rho$ be ρ–closed, ρ–bounded, convex subsets and $F : A \times A \to B$ and $G : B \times B \to A$ be an order cyclic ρ–contraction pair (F, G). Then there exists a unique order pair $(x, y) \in A \times A$ such that (x, y) is a coupled ρ–best proximity point of F in A (i.e. $\rho(x - F(x, y)) + \rho(y - F(y, x)) = 2d(A, B)$). There holds $x = G(F(x, y), F(y, x))$, $y = G(F(y, x), F(x, y))$ the order pair $(F(y, x), F(x, y))$ is a coupled ρ–best proximity point of G in B. More over for any initial guess $(x_0, y_0) \in A \times A$ the iterated sequences $\{x_n\}$, $\{y_n\}$ defined by*

$$\begin{aligned}&x_{2n+1} = F(x_{2n}, y_{2n}),\ y_{2n+1} = F(y_{2n}, x_{2n}),\\ &x_{2n+2} = G(x_{2n+1}, y_{2n+1}), y_{2n+2} = G(y_{2n+1}, x_{2n+1}),\\ &n = 0, 1, 2 \ldots\end{aligned} \tag{3}$$

satisfied $\lim_{n\to\infty} \rho(x_{2n} - x) = 0$, $\lim_{n\to\infty} \rho(y_{2n} - y) = 0$, $\lim_{n\to\infty} \rho(x_{2n+1} - F(x, y)) = 0$, $\lim_{n\to\infty} \rho(y_{2n+1} - F(y, x)) = 0$.

Lemma 21. *Let $\rho \in \mathfrak{R}$. Assume that ρ satisfies (UC1), has the Δ_2–property and be uniformly continuous. Let $A, B \subset L_\rho$ be nonempty, ρ–closed* *and ρ–bounded, convex subsets and $F : A \times A \to B$ and $G : B \times B \to A$ be an order cyclic ρ–contraction pair (F, G). If $(x_0, y_0) \in A$ and we define:*

$$\begin{array}{ll} x_{2n+1} = F(x_{2n}, y_{2n}), & y_{2n+1} = F(y_{2n}, x_{2n}) \\ x_{2n+2} = G(x_{2n+1}, y_{2n+1}), & y_{2n+2} = G(y_{2n+1}, x_{2n+1}) \end{array}$$

for all $n \in \mathbb{N} \cup \{0\}$, then $\lim_{n\to\infty} \rho(x_{2n} - x_{2n+1}) = d(A, B)$, $\lim_{n\to\infty} \rho(y_{2n} - y_{2n+1}) = d(A, B)$.

Proof. From the assumption that (F, G) is an order cyclic ρ–contraction pair we get

$$\begin{array}{rcl} \rho(x_{2n+1} - x_{2n}) & = & \rho(F(x_{2n}, y_{2n}) - G(x_{2n-1}, y_{2n-1})) \\ & \leq & \alpha\rho(x_{2n} - x_{2n-1}) + \beta\rho(y_{2n} - y_{2n-1}) \\ & & +(1 - (\alpha + \beta))d(A, B) \end{array}$$

and

$$\begin{array}{rcl} \rho(y_{2n+1} - y_{2n}) & = & \rho(F(y_{2n}, x_{2n}) - G(y_{2n-1}, x_{2n-1})) \\ & \leq & \alpha\rho(y_{2n} - y_{2n-1}) + \beta\rho(x_{2n} - x_{2n-1})) \\ & & +(1 - (\alpha + \beta))d(A, B). \end{array}$$

Just to fit some formulas in the text field let us denote $d_{n+1} = \rho(x_{n+1} - x_n) + \rho(y_{n+1} - y_n)$. From the above two inequalities we get

$$d_{2n+1} \leq (\alpha + \beta)d_{2n} + 2(1 - (\alpha + \beta))d(A, B). \tag{4}$$

In a similar fashion we get

$$\begin{array}{rcl} S_3 & = & \rho(x_{2n} - x_{2n-1}) = \rho(F(x_{2n-1}, y_{2n-1}) - G(x_{2n-2}, y_{2n-2})) \\ & \leq & \alpha\rho(x_{2n-1} - x_{2n-2}) + \beta\rho(y_{2n-1} - y_{2n-2}) \\ & & +(1 - (\alpha + \beta))d(A, B) \end{array}$$

and

$$\begin{array}{rcl} S_4 & = & \rho(y_{2n} - y_{2n-1}) = \rho(F(y_{2n-1}, x_{2n-1}) - G(y_{2n-2}, x_{2n-2})) \\ & \leq & \alpha\rho(y_{2n-1} - y_{2n-2}) + \beta\rho(x_{2n-1} - x_{2n-2})) \\ & & +(1 - (\alpha + \beta))d(A, B). \end{array}$$

From the above two inequalities we get

$$d_{2n} \leq (\alpha + \beta)d_{2n-1} + 2(1 - (\alpha + \beta))d(A, B). \tag{5}$$

Let us denote $k = \alpha+\beta$. Using (4) and (5) we obtain the chain of inequalities

$$\begin{aligned}
d_{n+1} &\leq kd_n + 2(1-k)d(A,B)\\
&\leq k(kd_{n-1} + 2(1-k)d(A,B)) + 2(1-k)d(A,B)\\
&= k^2 d_{n-1} + 2(1-k)d(A,B)(1+k)\\
&\leq k^2(kd_{n-2} + 2(1-k)d(A,B)) + 2(1-k)d(A,B)(1+k)\\
&= k^3 d_{n-2} + 2(1-k)d(A,B)(1+2k+k^2)\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\leq k^n d_0 + 2(1-k)\tfrac{1-k^n}{1-k}d(A,B) = k^n d_0 + 2(1-k^n)d(A,B).
\end{aligned}$$

Therefore $\lim_{n\to\infty} d_{n+1} = 2d(A,B)$. From the inequalities $\rho(x_{2n} - x_{2n+1}) \leq d(A,B)$ and $\rho(y_{2n} - y_{2n+1}) \leq d(A,B)$ it follows that $\lim_{n\to\infty} \rho(x_{2n} - x_{2n+1}) = d(A,B)$ and $\lim_{n\to\infty} = \rho(y_{2n} - y_{2n+1}) = d(A,B)$. □

Lemma 22. *Let $\rho \in \mathfrak{R}$. Assume that ρ satisfies (UC1), has the Δ_2–property and be uniformly continuous. Let $A, B \subseteq L_\rho$ be ρ–closed, ρ–bounded, convex subsets and $F : A \times A \to B$ and $G : B \times B \to A$ be an order cyclic ρ–contraction pair (F, G). If $(x_0, y_0) \in A \times A$ and we define:*

$$\begin{aligned}
x_{2n+1} &= F(x_{2n}, y_{2n}), & y_{2n+1} &= F(y_{2n}, x_{2n})\\
x_{2n+2} &= G(x_{2n+1}, y_{2n+1}), & y_{2n+2} &= G(y_{2n+1}, x_{2n+1})
\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$, then for $\varepsilon > 0$, there exists a positive integer N_0 such that for all $m > n \geq N_0$ such that $n + m$ is an odd number we have $\rho(x_m - x_n) + \rho(y_m - x_n) < 2d(A,B) + \varepsilon$.

Proof. : From the assumption that (F, G) is an order cyclic ρ–contraction pair we get

$$\begin{aligned}
\rho(x_{2m} - x_{2n+1}) &= \rho(G(x_{2m-1}, y_{2m-1}) - F(x_{2n}, y_{2n}))\\
&\leq \alpha\rho(x_{2m-1} - x_{2n}) + \beta\rho(y_{2m-1} - y_{2n})\\
&\quad +(1-(\alpha+\beta))d(A,B)
\end{aligned}$$

and

$$\begin{aligned}
\rho(y_{2m} - y_{2n+1}) &= \rho(G(y_{2m-1}, x_{2m-1}) - F(y_{2m}, x_{2n}))\\
&\leq \alpha\rho(y_{2m-1} - y_{2n}) + \beta\rho(x_{2m-1} - x_{2n})\\
&\quad +(1-(\alpha+\beta))d(A,B)
\end{aligned}$$

Let us denote $d_{m,n} = \rho(x_m - x_n) + \rho(y_m - y_n)$ and $k = \alpha + \beta$. After summing the above two inequalities we get

$$\begin{aligned}
S_4 = d_{2m,2n+1} &\leq 2kd_{2m-1,2n} + 2(1-k)d(A,B)\\
&\leq k(kd_{2m-2,2n-1} + 2(1-k)d(A,B)) + 2(1-k)d(A,B)\\
&= k^2 d_{2m-2,2n-1} + 2(1-k)d(A,B))(1+k)\\
&\leq k^2(kd_{2m-3,2n-2} + 2(1-k)d(A,B)) + 2(1-k)d(A,B)(1+k)\\
&= k^3 d_{2m-3,2n-2} + 2(1-k)d(A,B))(1+k+k^2)\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\leq k^{2n+1}d_{2m-2n-1,0} + \tfrac{1-k^{2n+1}}{1-k}2(1-k)d(A,B)\\
&= k^{2n+1}d_{2m-2n-1,0} + 2(1-k^{2n+1})d(A,B).
\end{aligned}$$

In a similar fashion we get

$$\begin{aligned}
\rho(x_{2m+1} - x_{2n}) &= \rho(F(x_{2m}, y_{2m}) - G(x_{2n-1}, y_{2n-1}))\\
&\leq \alpha\rho(x_{2m} - x_{2n-1}) + \beta\rho(y_{2m} - y_{2n-1})\\
&\quad +(1-(\alpha+\beta))d(A,B)
\end{aligned}$$

and

$$\begin{aligned}
\rho(y_{2m+1} - y_{2n}) &= \rho(F(y_{2m}, x_{2m}) - F(y_{2n-1}, x_{2n-1}))\\
&\leq \alpha\rho(y_{2m} - y_{2n-1}) + \beta\rho(x_{2m} - x_{2n-1})]\\
&\quad +(1-(\alpha+\beta))d(A,B)
\end{aligned}$$

After summing the above two inequalities we get

$$\begin{aligned}
S_5 = d_{2m+1,2n} &\leq kd_{2m,2n-1} + 2(1-k)d(A,B)\\
&\leq k(kd_{2m-1,2n-2} + 2(1-k)d(A,B)) + 2(1-k)d(A,B)\\
&= k^2 d_{2m-1,2n-2} + 2(1-k)d(A,B)(1+k)\\
&\leq k^2(kd_{2m-2,2n-3} + 2(1-k)d(A,B)) + 2(1-k)d(A,B)(1+k)\\
&= k^3 d_{2m-2,2n-3} + 2(1-k)d(A,B)(1+k+k^2))\\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\leq k^{2n}d_{2m-2n+1,0} + \tfrac{1-k^{2n}}{1-k}2(1-k)d(A,B)\\
&= k^{2n}d_{2m-2n+1,0} + 2(1-k^{2n})d(A,B).
\end{aligned}$$

Consequently for any $n < m$ such that $n+m$ is an odd number there holds

$$d_{m,n} \leq k^n d_{m-n,0} + 2(1-k^n)d(A,B) \leq k^n d_{m-n,0} + 2d(A,B),$$

from the ρ–boundedness of $A \cup B$ it follows that there is M, such that $d_{m,n} \leq M$. For any $\varepsilon > 0$ there is N_0 such that for any $n \geq N_0$ holds $k^n < \frac{\varepsilon}{M}$. Thus

$$\begin{aligned}
2d(A,B) &\leq d_{m,n} \leq k^n M + 2d(A,B)\\
&\leq \tfrac{\varepsilon}{M}M + 2d(A,B) < 2d(A,B) + \varepsilon.
\end{aligned}$$

□

Lemma 23. *Let $\rho \in \mathfrak{R}$. Assume that ρ satisfies (UC1), has the Δ_2–property and be uniformly continuous. Let $A, B \subseteq L_\rho$ be ρ–closed, ρ–bounded, convex subsets and $F : A \times A \to B$ and $G : B \times B \to A$ be an order cyclic ρ–contraction pair (F, G). If $(x_0, y_0) \in A \times A$ and we define:*

$$\begin{array}{ll} x_{2n+1} = F(x_{2n}, y_{2n}), & y_{2n+1} = F(y_{2n}, x_{2n}) \\ x_{2n+2} = G(x_{2n+1}, y_{2n+1}), & y_{2n+2} = G(y_{2n+1}, x_{2n+1}). \end{array}$$

for all $n \in \mathbb{N} \cup \{0\}$. If the sequences $\{x_{2n}\}_{n=0}^\infty$ and $\{y_{2n}\}_{n=0}^\infty$ are ρ–convergent to x and y then $\rho(x - F(x, y)) = d(A, B)$ and $\rho(y - F(y, x)) = d(A, B)$.

Proof. From Lemma 21 we have that there hold $\lim_{n\to\infty} \rho(x_{2n} - x_{2n+1}) = d(A, B)$ and $\lim_{n\to\infty} \rho(x_{2n+2} - x_{2n+1}) = d(A, B)$. Thus from Lemma 11 we obtain $\lim_{n\to\infty} \rho(x_{2n} - x_{2n+2}) = 0$. From $\lim_{n\to\infty} \rho(x_{2n} - x_{2n-1}) = d(A, B)$ and $\lim_{n\to\infty} \rho(x_{2n} - x) = 0$ applying Lemma 12 we get $\lim_{n\to\infty} \rho(x_{2n-1} - x) = d(A, B)$.

From Lemma 21 we have that there hold $\lim_{n\to\infty} \rho(y_{2n} - y_{2n+1}) = d(A, B)$ and $\lim_{n\to\infty} \rho(y_{2n+2} - y_{2n+1}) = d(A, B)$. Thus from Lemma 11 we obtain $\lim_{n\to\infty} \rho(y_{2n} - y_{2n+2}) = 0$. From $\lim_{n\to\infty} \rho(y_{2n} - y_{2n-1}) = d(A, B)$ and $\lim_{n\to\infty} \rho(y_{2n} - y) = 0$ applying Lemma 12 we get $\lim_{n\to\infty} \rho(y_{2n-1} - y) = d(A, B)$.

Since

$$\begin{aligned} S_6 = d(A, B) &\le \lim_{n\to\infty} \rho(F(x, y) - x_{2n}) \\ &= \lim_{n\to\infty} \rho(F(x, y) - G(x_{2n-1}, y_{2n-1})) \\ &\le \lim_{n\to\infty} [\alpha\rho(x - x_{2n-1}) + \beta\rho(y - y_{2n-1}) + (1 - (\alpha + \beta))d(A, B)] \\ &= d(A, B). \end{aligned}$$

Using the uniform continuity of ρ we get

$$\rho(F(x, y) - x) = \lim_{n\to\infty} \rho(F(x, y) - x_{2n}) = d(A, B).$$

Since

$$\begin{array}{rcl} d(A, B) & \le & \lim_{n\to\infty} \rho(F(y, x) - y_{2n}) \\ & = & \lim_{n\to\infty} \rho(F(y, x) - G(y_{2n-1}, x_{2n-1})) \\ & \le & \lim_{n\to\infty} [\alpha\rho(y - y_{2n-1}) + \beta\rho(x - x_{2n-1}) \\ & & +(1 - 2(\alpha + \beta))d(A, B)] \\ & = & d(A, B). \end{array}$$

Using the uniform continuity of ρ we get

$$\rho(F(y, x) - y) = \lim_{n\to\infty} \rho(F(y, x) - y_{2n}) = d(A, B).$$

□

Proof. of Theorem 20 For and pair $(x_0, y_0) \in (A \times A)$ we consider the sequences $\{x_n\}$, $\{y_n\}$. From Lemma 21 we have that $\lim_{n\to\infty} \rho(x_{2n} - x_{2n+1}) = d(A, B)$. By Lemma 22 we have that for every $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$, such that there holds the inequality $\rho(x_{2m} - x_{2n+1}) < d(A, B) + \varepsilon$ for every $m, n \geq N_0$. From Lemma 10 by setting $z_n = x_{2n}$, $x_m = x_{2m}$ and $y_n = x_{2n+1}$ it follows there exists $N_1 \in \mathbb{N}$, such that the inequality $\rho(T^{2m}x - T^{2n}x) < \varepsilon$ holds for every $m > n \geq N_1$. Therefore the sequence $\{x_{2n}\}_{n=1}^{\infty}$ is a ρ–Cauchy sequence. The proof that $\{y_{2n}\}_{n=1}^{\infty}$ is a ρ–Cauchy sequence can be made in a similar fashion.

From the completeness of L_ρ and closeness of A is follows that there are $x, y \in A$, such that $\lim_{n\to\infty} \rho(x_{2n} - x) = 0$ and $\lim_{n\to\infty} \rho(y_{2n} - y) = 0$. By Lemma 23 it follows that (x, y) is a coupled ρ–best proximity point.

First we will prove that if (x, y) is a coupled ρ–best proximity point then $x = G(F(x, y), F(y, x))$ and $y = G(F(y, x), F(x, y))$. From the inequality

$$\begin{aligned} S_7 = & \, d(A, B) \leq \rho(F(x, y) - G(F(x, y), F(y, x))) \\ \leq & \, \alpha\rho(x - F(x, y)) + \beta\rho(y - F(y, x)) + (1 - (\alpha + \beta))d(A, B) \\ = & \, \alpha d(A, B) + \beta d(A, B) + (1 - (\alpha + \beta))d(A, B) = d(A, B) \end{aligned}$$

it follows that $\rho(F(x, y) - G(F(x, y), F(y, x))) = d(A, B)$. From the equality $\rho(F(x, y) - x) = d(A, B)$ and Lemma 11 it follows that $x = G(F(x, y), F(y, x))$. By similar arguments from the inequality

$$\begin{aligned} S_8 = & \, d(A, B) \leq \rho(F(y, x) - G(F(y, x), F(x, y))) \\ \leq & \, \alpha\rho(y - F(y, x)) + \beta\rho(x - F(x, y)) + (1 - (\alpha + \beta))d(A, B) \\ = & \, \alpha d(A, B) + \beta d(A, B) + (1 - (\alpha + \beta))d(A, B) = d(A, B) \end{aligned}$$

it follows that $\rho(F(y, x) - G(F(y, x), F(x, y))) = d(A, B)$. From the equality $\rho(F(y, x) - y) = d(A, B)$ and Lemma 11 it follows that $y = G(F(y, x), F(x, y))$.

Consequently $(F(x, y), F(y, x))$ is a coupled best proximity point of G in B.

It remains to prove that the coupled best proximity point is unique. Let us suppose the contrary, i.e. there exists $(u, v) \in A \times A$ such that $\rho(x - u) + \rho(y - v) > 0$. Just to fit the next four inequalities in the text field let us denote $d = d(A, B)$. We can write the inequalities

$$\begin{aligned} S_9 = & \, \rho(F(x, y) - u) = \rho(F(x, y) - G(F(u, v), F(v, u))) \\ \leq & \, \alpha\rho(x - F(u, v)) + \beta\rho(y - F(v, u)) + (1 - (\alpha + \beta))d, \end{aligned} \tag{6}$$

$$\begin{aligned} S_{10} = & \, \rho(F(y, x) - v) = \rho(F(y, x) - G(F(v, u), F(u, v))) \\ \leq & \, \alpha\rho(y - F(v, u)) + \beta\rho(x - F(u, v)) + (1 - (\alpha + \beta))d, \end{aligned} \tag{7}$$

$$\begin{aligned} S_{11} &= \rho(F(u,v)-x) = \rho(F(u,v)-G(F(x,y),F(y,x))) \\ &\leq \alpha\rho(u-F(x,y)) + \beta\rho(v-F(y,x)) + (1-(\alpha+\beta))d, \end{aligned} \tag{8}$$

$$\begin{aligned} S_{12} &= \rho(F(v,u)-y) = \rho(F(v,u)-G(F(y,x),F(x,y))) \\ &\leq \alpha\rho(v-F(y,x)) + \beta\rho(u-F(x,y)) + (1-(\alpha+\beta))d. \end{aligned} \tag{9}$$

Let us denote

$$S_{13} = \rho(F(x,y)-u) + \rho(F(y,x)-v) + \rho(F(u,v)-x) + \rho(F(v,u)-y).$$

Summing (6), (7), (8) and (9) we get the inequality

$$(1-(\alpha+\beta))S_{13} \leq 4(1-(\alpha+\beta))d(A,B).$$

Therefore

$$\begin{aligned} \rho(F(x,y)-u) &= \rho(F(y,x)-v) = \rho(F(u,v)-x) \\ &= \rho(F(v,u)-y) = d(A,B). \end{aligned}$$

From

$$\begin{aligned} \rho(F(x,y)-x) &= \rho(F(y,x)-y) = \rho(F(u,v)-u) \\ &= \rho(F(v,u)-v) = d(A,B). \end{aligned}$$

and Lemma 11 it follows that $x = u$ and $y = v$.

From Lemma 23 it follows that for every $\varepsilon > 0$ there exists N, such that for any $n, m \geq N$ there holds $\rho(x_{2n} - x_{2m+1}) \leq d(A,B) + \varepsilon$. Therefore applying Lemma 10 and $\lim_{n\to\infty} \rho(x_{2n} - x_{2n+1}) = d(A,B)$ it follows that $\{x_{2n+1}\}_{n=1}^{\infty}$ is a Cauchy sequence. From the completeness of L_ρ and closeness of B it follows that there is $\overline{x} \in B$ such that $\lim_{n\to\infty} \rho(x_{2n+1} - \overline{x}) = 0$.

By similar arguments we get from Lemma 23 it follows that for every $\varepsilon > 0$ there exists N, such that for any $n, m \geq N$ there holds $\rho(y_{2n} - y_{2m+1}) \leq d(A,B)+\varepsilon$. Therefore applying Lemma 10 and $\lim_{n\to\infty} \rho(y_{2n} - y_{2n+1}) = d(A,B)$ it follows that $\{y_{2n+1}\}_{n=1}^{\infty}$ is a Cauchy sequence. From the completeness of L_ρ and closeness of B it follows that there is $\overline{y} \in B$ such that $\lim_{n\to\infty} \rho(y_{2n+1} - \overline{y}) = 0$. We will show that $\overline{x} = F(x,y)$ and $\overline{y} = F(y,x)$.

From $\lim_{n\to\infty} \rho(x_{2n+1} - \overline{x}) = 0$, $\lim_{n\to\infty} \rho(x_{2n+1} - x_{2n}) = d(A,B)$ and Lemma 12 it follows that $\lim_{n\to\infty} \rho(x_{2n} - \overline{x}) = d(A,B)$. Using the uniform continuity of ρ we get $\rho(x - \overline{x}) = \lim_{n\to\infty} \rho(x_{2n} - \overline{x}) = d(A,B)$. By $\rho(x - F(x,y)) = d(A,B)$ and Lemma 10 it follows that $x = F(x,y)$.

From $\lim_{n\to\infty} \rho(y_{2n+1} - \overline{y}) = 0$, $\lim_{n\to\infty} \rho(y_{2n+1} - y_{2n}) = d(A,B)$ and Lemma 12 it follows that $\lim_{n\to\infty} \rho(y_{2n} - \overline{y}) = d(A,B)$. Using the uniform continuity of ρ we get $\rho(y - \overline{y}) = \lim_{n\to\infty} \rho(y_{2n} - \overline{y}) = d(A,B)$. By $\rho(y - F(y,x)) = d(A,B)$ and Lemma 10 it follows that $y = F(y,x)$. □

4. Application

Let $p \in [1, +\infty)$, $a > 0$, $\alpha, \beta \in (0, 1)$ be such that $\alpha+\beta < 1$ and $(1-\alpha-\beta)a = \gamma$. Let us consider the system of equations

$$\left| \begin{array}{l} |(1+\alpha)x + \beta y + \gamma|^p = (2a)^p \\ |\alpha x + (1+\beta)y + \gamma|^p = (2a)^p \\ x \geq 0, \ y \geq 0. \end{array} \right. \tag{10}$$

It is easy to check that by using a Computer Algebra Software that the ordered pair (a, a) is a solution of (10) if $p \in \mathbb{N}$. If we try to solve this system for $p \notin \mathbb{N}$ then the computer will give no answer.

Let us consider the space $\mathbb{R}_{|\cdot|^p}$ of all reals endowed with the function modular $\rho_p(\cdot) = |\cdot|^p$. For $p = 1$ we get that the space $\mathbb{R}_{|\cdot|^1}$, which is a normed space and from [6] it follows that it is uniformly convex. Consequently if considered as modular function space we get that ρ_1 satisfies (UC1), has the Δ_2–property and is uniformly continuous.

Let us denote with $\delta_1(r, \varepsilon)$ the infimum

$$\inf\left\{1 - \frac{1}{r}\left|\frac{x+y}{2}\right|^p : x, y \in \mathbb{R}_{|\cdot|^p}, |x|^p \leq r, |y|^p \leq r, |x-y|^p \geq \varepsilon r\right\}.$$

It is easy to observe that the infimum is attained for $x = r^{1/p}$ and $y = r^{1/p} - (\varepsilon r)^{1/p}$ and we get

$$1 - \frac{1}{r}\left|r^{1/p} - \frac{(\varepsilon r)^{1/p}}{2}\right|^p = 1 - \left(1 - \frac{\varepsilon^{1/p}}{2}\right)^p > 0.$$

Thus $\delta_1(r, \varepsilon) > 0$ and consequently ρ_p satisfies (UC1). It is easy to check that ρ_p has the Δ_2–property and is uniformly continuous. Thus we can apply Theorem 20 in $\mathbb{R}_p$.

Let us consider the subsets of $\mathbb{R}_p$: $A = [a, b]$, $B = [-b, -a]$ for $0 < a < b$. It is easy to calculate that $d(A, B) = |a - (-a)|^p = (2a)^p$. Let us consider the functions $F(x, y) = -\alpha x - \beta y - \gamma$ and $G(x, y) = -\alpha x - \beta y + \gamma$. From the assumptions it follows that $F : A \times A \to B$ and $G : B \times B \to A$, because $G(x, y) \in A$ for $x, y \in B$ and $F(x, y) \in B$ for $x, y \in A$.

For arbitrary $x, y \in A$ and $u, v \in B$ using the convexity of the function $|\cdot|^p$

we get the inequality

$$\begin{aligned} S_{12} &= \rho(F(x,y) - G(u,v)) = |F(x,y) - G(u,v)|^p \\ &= |-\alpha x - \beta y - \gamma - (-\alpha u - \beta v + \gamma)|^p \\ &= |\alpha(x-u) + \beta(y-v) + 2\gamma|^p \\ &= |\alpha(x-u) + \beta(y-v) + (1-\alpha-\beta)2a|^p \\ &\leq \alpha|x-u|^p + \beta|y-v|^p + (1-\alpha-\beta)(2a)^p \\ &= \alpha\rho_p(x-u) + \beta\rho_p(y-v) + (1-\alpha-\beta)d(A,B). \end{aligned}$$

Thus (F,G) is an order cyclic ρ–contraction pair and from Theorem 20 it follows that there exists a unique order pair $(x,y) \in A \times A$ such that (x,y) is a coupled ρ–best proximity point of F in A (i.e. $\rho_p(x - F(x,y)) = (2a)^p$ and $\rho_p(y - F(y,x)) = (2a)^p$, which just (10)). The solution can be approximated by using the sequence of consecutive iterations $\{x_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ defined in Theorem 20 starting with an arbitrary guess points x_0 and y_0.

If we put $\alpha = \beta = \frac{1}{4}$, $\gamma = \frac{1}{2}$, $a = 1$, $b = 2$ and $p = 1$ we get Example 2 from [6] and Example 3.12 from [23]. If we put $\alpha = \frac{1}{4}$, $\beta = \frac{1}{4}$, $\gamma = 1$, $a = 2$, $b = 4$ and $p = 1$ we get Example 3.5 [23]. If we put $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$, $\gamma = \frac{1}{6}$, $a = 1$, $b = 2$ and $p = 1$ we get Example 2 from [6] Example 17 from [4].

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