

COMMON SUBSPACE-HYPERCYCLIC VECTORS

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Abstract: In this article, we show that hypercyclic operators have a dense invariant subspace, consisting except for zero, of subspace-hypercyclic vectors. Also, we prove that any finite set of hypercyclic operators has common subspace-hypercyclic vectors. Moreover, we show that for a countable family of hypercyclic operators, there exists a dense G_δ -set of X such that any of its members are subspace-hypercyclic vectors for any operator of this family.

AMS Subject Classification: 47A16, 47B37, 37B99

Key Words: hypercyclic vectors, subspace-hypercyclic operators, subspace-hypercyclic vectors

1. Introduction and Preliminaries

Let X be a separable, infinite dimensional Banach space over \mathbb{C} , the complex field, and let $B(X)$ be the algebra of operators $T : X \rightarrow X$, that are bounded and linear. An operator $T \in B(X)$ is called a hypercyclic operator, if there is a vector x in X whose orbit, $Orb(T, x) = \{x, Tx, T^2x, \dots\}$, is dense in X . In this case, we say that x is a hypercyclic vector for T .

We denote the set of all hypercyclic vectors for T , by $HC(T)$.

The first example of hypercyclic operators was constructed by Rolewicz in 1969. Rolewicz in [11] proved that if B be the backward shift on $l^2(\mathbb{N})$, then for any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, λB is hypercyclic. An excellent book about hyper-

Received: February 21, 2017

Revised: April 4, 2018

Published: May 23, 2018

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url: www.acadpubl.eu

cyclicity is the book of Grosse-Erdmann and Peris ([5]). For more interesting facts about hypercyclicity, one can read [2].

Madore and Martinez-Avendano in 2011, defined subspace-hypercyclic operators as follows.

Definition 1.1. Let $T \in B(X)$. We say that T is subspace-hypercyclic with respect to a closed subspace M of X , if there exists a vector $x \in X$ such that $orb(T, x) \cap M$ is dense in M . In this case, x is called an M -hypercyclic vector for T .

One can read more about this concept in [9] and [10].

Madore and Martinez-Avendano asked this question in [7]:

”If T is hypercyclic, must there be a proper subspace M such that T is subspace-hypercyclic for M ?”

Le answered positively to this question under some conditions in [6]. Martinez-Avendano and Zatarian-Vera in [8], showed that for every finite-codimensional subspace, hypercyclic coanalytic Toeplitz operators are subspace-hypercyclic. Finally in 2016, Bamerni, Kadets, and Kilicman answered yes to this question. Theorem1.2 is the main theorem of their paper.

Theorem 1.2. ([1]) *If A is a dense subset of a Banach space X , then there is a nontrivial closed subspace M such that $A \cap M$ is dense in M .*

Let x be a hypercyclic vector for T . Then $orb(T, x)$ is dense in M . By Theorem1.1, we can find a closed and proper subspace M , such that $orb(T, x) \cap M$ is dense in M . That means x is a subspace-hypercyclic vector for T with respect to M . So we have the following corollary.

Corollary 1.3. ([1]) *If T is a hypercyclic operator on a Banach space X , then there is a nontrivial closed subspace M of X such that T is M -hypercyclic.*

In this paper, we show that hypercyclic operators have a dense invariant subspace, consisting except for zero, of subspace-hypercyclic vectors. Also, we use Theorem1.2 and Corollary1.3 and prove that for every finite set of hypercyclic operators, there exists common subspace-hypercyclic vectors. Moreover, we show that if T is a hypercyclic operator, then common subspace-hypercyclic vectors for T^m 's make a dense G_δ -set.

2. Main results and Examples

Let us recall Herrero-Bourdon theorem from [5].

Theorem 2.1. *If x is a hypercyclic vector for T , then*

$$\{P(T)x : P \text{ is a polynomial}\} \setminus \{0\}$$

is a dense set of hypercyclic vectors. Specially, any hypercyclic operator admits a dense invariant subspace, consisting except for zero, of hypercyclic vectors.

While T , must always contain a dense subspace, it not always contains a closed infinite dimensional one. For more information, one can see [12].

Now we extend Theorem2.1 for subspace-hypercyclic vectors as follows.

Theorem 2.2. *Let $T \in B(X)$ be a hypercyclic operator. Then T has a dense invariant subspace, that any of its nonzero members, is a subspace-hypercyclic vector for T .*

Proof. T is hypercyclic. So there exists $x \in X$ such that $orb(T, x)$ is dense in X . By Corollary1.3, there is a proper and closed subspace M of X such that T is M -hypercyclic. So x is a subspace-hypercyclic vector for T too. By Theorem2.1, with respect to T , we can find a dense invariant subspace, such that any of its nonzero members is a hypercyclic vectors for T . As we see, any of these vectors are subspace-hypercyclic vectors and this completes the proof. \square

The following examples are two classic operators that satisfy in condition of Theorem2.2.

Example 2.3. (Birkhoff's operators) Let

$$T_a f(z) = f(z + a), \quad a \neq 0$$

be Birkhoff's operators on $H(\mathbb{C})$ of entire functions.

The Birkhoff's operators are hypercyclic ([5], p. 275). So by Theorem2.2, for any $a \in \mathbb{C}$, the operator T_a has a dense invariant subspace, such that any of its nonzero members is a subspace-hypercyclic vectors for T_a .

Example 2.4. (MacLane's operator) The operator of differentiation, $D : f \rightarrow f'$ on $H(\mathbb{C})$, is hypercyclic ([5], p. 275). So by Theorem2.1, it has a dense invariant subspace, such that any of its nonzero members is a subspace-hypercyclic vector for D .

One can find another example by setting $T = \lambda B$, where B is the backward shift on l^2 , and λ is a complex number with $|\lambda| > 1$.

If T be a hypercyclic operator, then $HC(T)$ is a dense G_δ -set ([5], p. 39). If Λ is a countable family Λ of hypercyclic operators, Baire Category Theorem implies that $\bigcap_{T \in \Lambda} HC(T)$ is also a dense G_δ -set ([3]).

Theorem 2.5. ([5]) *Let Λ be a countable set. Then the set of common hypercyclic vectors for $(T_\lambda)_{\lambda \in \Lambda}$ is a dense G_δ -set.*

Costaki and Sambarino in [4] and Shakrin in [12], have proved sufficient conditions for uncountable families of operators to have a dense G_δ -set of common hypercyclic vectors.

Theorem 2.6. *Let Λ be a countable set. Then the set of common subspace-hypercyclic vectors for $(T_\lambda)_{\lambda \in \Lambda}$, where $T_\lambda \in B(X)$ for any $\lambda \in \Lambda$, form a dense G_δ -set of X .*

Proof. By hypothesis T_λ is hypercyclic, for any $\lambda \in \Lambda$. Theorem 2.5 asserts that there exists a dense G_δ -set of common hypercyclic vectors for T_λ 's, where $\lambda \in \Lambda$. Let's put its name A .

T_λ 's, where $\lambda \in \Lambda$, have a dense G_δ -set of common hypercyclic vectors like A .

Let x be a member of the A . Then by Corollary 1.3, correspond to any natural number n , we can find a closed nontrivial subspace $M_{x,\lambda}$ such that T_λ be $M_{x,\lambda}$ -hypercyclic. So each $x \in A$ is a subspace-hypercyclic vector for any T_λ . Hence the members of the A , are subspace-hypercyclic vectors.

Note that the members of A , may be subspace-hypercyclic vectors with respect to different subspaces. \square

By Theorem 2.6, we can assert a corollary about common vectors of T^n 's, where T is hypercyclic as follows.

Corollary 2.7. *Let $T \in B(X)$ be hypercyclic. Then common subspace-hypercyclic vectors for $\{T^n : n \in \mathbb{N}\}$, make a dense G_δ -set of X .*

In 2016, Bamerni, Kadets and Kilicman in [1] asked the following question:

"if T^n is M_n -hypercyclic for all $n \geq 1$, is there any relation among all M_n ?"

In what follows we try to answer this question. First, we show that for any $n \in \mathbb{N}$, one can find a common subspace M such that T and T^n are both M -hypercyclic.

Theorem 2.8. *Let $T \in B(X)$ be a hypercyclic operator and let n be a positive integer. Then there exists a closed nontrivial subspace M such that T and T^n are both M -hypercyclic.*

Especially T and T^n have a dense subset of common M -hypercyclic vectors.

Proof. Let n be a positive integer. T is hypercyclic hence, T^n is also a hypercyclic operator. By Corollary 1.3, corresponding to this n , we can find a nontrivial closed subspace M_n such that T^n is M_n -hypercyclic.

Let x be an M_n -hypercyclic vector for T^n . So $orb(T^n, x)$ is dense in M_n . But:

$$orb(T^n, x) \cap M_n \subseteq orb(T, x) \cap M_n. \quad (2.1)$$

Hence $orb(T, x) \cap M_n$ is also dense in M_n . That means x is a subspace-hypercyclic vector for T with respect to M_n . Therefore T is an M_n -hypercyclic operator. If we consider $M := M_n$, then both T and T^n are M -hypercyclic operators.

Also relation (2.1) show us that any M -hypercyclic vector for T^n is an M -hypercyclic vector for T . But the M -hypercyclic vectors for T^n or $HC(T^n, M)$ is dense in M ([7]). So we have a dense set of common M -hypercyclic vectors for T and T^n in M .

□

In the next theorem, we extend Theorem 2.8 to finite number of operators as follows.

Theorem 2.9. *Let T be a hypercyclic operator. Let $\{n_1, n_2, \dots, n_k\}$ be a finite sequence of natural numbers. Then there is a closed nontrivial subspace M such that T^{n_i} is M -hypercyclic for any $1 \leq i \leq k$. Especially the set of common M -hypercyclic vectors for T^{n_i} 's, is dense in M .*

Proof. T is a hypercyclic operator, hence T^{n_i} 's is also a hypercyclic operator. Now Corollary 1.3, asserts that for any i , there exists a closed nontrivial subspace M_i , such that T^{n_i} is M_i -hypercyclic. Let $n_\alpha = \max\{n_1, n_2, \dots, n_k\}$. Let x be an M_{n_α} -hypercyclic vector for T^{n_α} . Then $orb(T^{n_\alpha}, x) \cap M_{n_\alpha}$ is dense in M_{n_α} . But $n_i \leq n_\alpha$ for any $1 \leq i \leq k$. Hence:

$$orb(T^{n_\alpha}, x) \cap M_{n_\alpha} \subseteq orb(T^{n_i}, x) \cap M_{n_\alpha}, \quad 1 \leq i \leq k. \quad (2.2)$$

So $orb(T^{n_i}, x) \cap M_{n_\alpha}$ is dense in M_{n_α} , for any $1 \leq i \leq k$. If we consider $M := M_{n_\alpha}$, then any T^{n_i} is M -hypercyclic.

Also by relation (2.2), we conclude that any M -hypercyclic vector for T^{n_α} is an M -hypercyclic vector for T^{n_i} ($1 \leq i \leq k$). By (Madore and Martinez-Avendano (2011)), the set of M -hypercyclic vectors for T^{n_α} are dense in M . Therefore common M -hypercyclic vectors of T^{n_i} 's are dense in M .

□

If in Theorem 2.9 we consider $n_1 = 1, n_2 = 2, \dots, n_{k-1} = k - 1$ and $n_k = k$, then we have the following corollary.

Corollary 2.10. *Let $T \in B(X)$ be hypercyclic. Then there exists a closed nontrivial subspace M such that T, T^2, \dots, T^{k-1} and T^k are M -hypercyclic. Especially there is a dense subset of common M -hypercyclic vectors for T, T^2, \dots, T^{k-1} and T^k , in M .*

As a consequence of Corollary 2.10, Question 3 of Bamerni, Kadets and Kilicman in [1], is answered partially. In fact, we show that T, T^2, \dots, T^{n-1} and T^n can have a common nontrivial closed subspace for subspace-hypercyclicity with many common subspace-hypercyclic vectors.

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