



THE STRUCTURE OF LATTICES OF EXISTENTIAL FORMULAS FOR FRAGMENT OF JONSSON SETS

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Abstract: This article is related to one of the branches of Model Theory, more precisely to the study of Jonsson sets. This part of Model Theory is concerned with the study of incomplete inductive theories, namely with Jonsson theories and some of their generalizations. It examines the model-theoretical properties of Jonsson theories. In particular the lattice of special formulas is considered. In the study of complete theories one of the main methods is to use the properties of a topological space $S_n(T)$. In the case of Jonsson theory, we can consider the lattice $E_n(Fr(X))$ of existential formulas, which is a sublattice of the Boolean algebra $F_n(Fr(X))$. The main aim of this article is to develop the basic concepts and methods of that part of the Model Theory which will provide an opportunity to fruitful studies of Jonsson theories and some of its generalizations.

Our technique is a standard method of the study of incomplete theories. It involves the translation of the elementary properties of the center of a Jonsson theory into the theory itself.

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1. Introduction

We continue in this article our work with lattice of existential formulas which was started in [1], where one of the author studied similar questions but only

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for Jonsson theories. Our research interests are related to the description of model-theoretic properties of certain general classes of partial theories which are a subclass of inductive theories. These theories are well studied in algebra and model theory.

As a rule, we always deal with two subjects:

- 1) Jonsson theory,
- 2) a class of its existentially closed models.

It is well known that the perfect Jonsson theory is comfortable enough for the model-theoretic research. Practically, in the case of perfectness we can say that by using semantic method we can give a definite description of the above objects (Jonsson theory class and its existentially closed models). Therefore, it would be interesting to learn how to allocate in an arbitrary theory fragment which will be Jonsson theory. This approach is non-trivial, at least because of the fact that any theory of the set of its universal existential consequences, not necessarily will be Jonsson theory. On the other hand, for any theory in some special enrichment always be achieved firstly Jonssonness, and also one is perfect theory. We have proposed the idea of considering a new approach to the so-called Jonsson subsets of semantic models that allows firstly to expand semantic aspect, and secondly to transfer many of the ideas of art complete theories for Jonsson fragments, which is a generalization of the considered problems and challenges. This work is devoted to the study of some properties of a fragment of Jonsson sets.

The well-known specialist in mathematical logic H. J. Keisler in the review “Fundamentals of Model Theory” in [2] has defined the basic concepts and directions of the development of model theory. He has defined two historical directions in the development of model theory. They are western and eastern model theories. Such division is connected with the fact that A. Tarski lived on the western coast of the USA, while A. Robinson lived on the eastern coast of the USA. This distinction has lost its geographical value long ago; however, it is useful from the mathematical point of view. The western model theory to a greater degree was based on problems in number theory, analysis, set theory, and in all formulas of the first-order logic. The Eastern model theory was usually based on problems in abstract universal algebra, where formulas with prefix length at most 2 were considered. In the western variant, morphisms are elementary ones while in the eastern variant the morphisms are isomorphic embeddings and homomorphisms. On the other hand, the theory of models is uniform, i.e., theorems of “eastern” character naturally are supplement to the western analogues, for example, A. Macintyres “eastern” theorem on omitting

types is supplement to the well-known theorem on omitting types in the “western” sense. More information on the connection of these two directions can be found in [2]. More precisely, it examines the model-theoretical properties of a fragment of Jonsson sets. In particular, the lattice of existential formulas is considered. In the study of complete theories one of the main methods is to use the properties of a topological space $S_n(T)$. In the language of ultrafilters of the Boolean algebra $F_n(T)$, where T is a fixed theory, the classical concepts of Model Theory such as the stability of the model and the theory, the saturation of the model, the homogeneity of the model, diagrams of models and etc are studied.

In [1] it has been investigated and described the properties of lattices of existential formulas for Jonsson theory. Later in the following sources [3, 4, 5, 6] one of the authors of this article defined Jonsson sets and started working on the implementation of the program of the description of the model-theoretic properties of these sets. In the case of a fragment of Jonsson sets, we can consider the lattice $E_n(Fr(x))$ of existential formulas which is a sublattice of the Boolean algebra $F_n(Fr(x))$. Since in general the existential formulas are not closed with respect to the Boolean operations, the topological space of existential types differs significantly from the complete cases. It is clear that such an approach (restriction of $F_n(Fr(X))$ to $E_n(Fr(X))$) is a generalization of the case when we deal with the complete theory. Since in general the Jonsson sets are incomplete it would be interesting to consider the properties of the lattice of existential formulas in connection with the above mentioned context. The main tool of research for a Jonsson sets is a semantic method. The essence of this method is the translation of the properties of the central completion to Jonsson prototype. In this article in addition to the semantic method and other outcomes of Jonsson sets we use notions and results from the work of Volker Weispfenning[7].

The mainstream of investigations in Model Theory belongs to western Model Theory, nevertheless we can observe a number of papers where arise similar issues in the study of a fragment of Jonsson sets. For example, in [8] it was considered the properties of the class of simple theories of existentially closed models of universal sentences. In this case it was considered the Robinsonien class of fragment of Jonsson sets and we can see that this class of theories is a natural subclass of the Jonsson sets. Usually for investigating such sets we use semantic method, but even in the case where a fragment of Jonsson sets is not Jonssonien, the idea of a semantic generalization of the method for the Jonsson sets is useful. The essence of this generalization is the translation of properties of a fragment of Jonssons sets-central completion to a fragment of Jonsson sets

- prototype. A series of results that establish the relationship between properties of a fragment of Jonsson sets, a fragment of Jonsson sets - center of a fragment of Jonsson sets and properties of the lattice of equivalence classes of existential formulas with respect to these sets is proved. It is essential that the existence of a semantic model of a fragment of Jonsson sets does not depend on additional axioms of set theory. The obtained results are analogues of the results obtained in [7, 9, 10]. In [9] it is also examined other properties of such lattices.

Furthermore, we consider the notion of a fragment of Jonsson sets and establish a connection between properties of a fragment of Jonsson sets, the central completion of a fragment of Jonsson sets and properties of the lattice of equivalence classes of existential formulas with respect to this theory. We assume familiarity with definitions and notations on Jonsson sets. All the necessary concepts for Jonsson sets can be found in the following source [9]. Jonsson conditions are natural algebraic requirements that arise in the study of a wide class of algebras. Jonsson properties are satisfied by a fragment of Jonsson sets such as group theory, the theory of Abelian groups, the theory of fields of fixed characteristic, the theory of Boolean algebra, the theory of ordered sets, the theory of polygons, and many others. As it is evident from the list, the obtaining of technical apparatus of results for a fragment of Jonsson sets, with respect to applications, would be wide enough. In this paper, the object of our research will focus on a new class of theories related to the notion of a fragment of Jonsson sets. This work is a review of results concerning researches of the notion of a fragment of Jonsson sets.

2. Concepts and results concerning Jonsson theories

Let us give crucial definitions concerned Jonsson Theories and Jonsson sets, videlicet convex and existentially prime theories.

Definition 1. [9] The theory T is called convex if for any of its model $\mathfrak{A}$ and for any family $\{B_i \mid i \in \mathbb{N}\}$ of substructures of $\mathfrak{A}$, which are models of the theory T , the intersection $\bigcap_{i \in I} B_i$ is a model of T . It is assumed that this intersection is not empty. If this intersection is never empty, then the theory is called strongly convex.

Definition 2. [9] An inductive theory T is called the existentially prime if

- 1) It has a prime algebraic (AP) model (the class of its AP is denoted by

$T_{AP})$,

- 2) The intersection of the class (E_T) with the class AP is not empty, i.e.

$$T_{AP} \cap E_T \neq \emptyset$$

.

Let L be a first-order language.

Definition 3. [9] We say that a theory T admits *JEP*, if for any two $A, B \in \text{Mod} T$ there exist $C \in \text{Mod} T$ and homomorphisms $h_1 : A \rightarrow C$, $h_2 : B \rightarrow C$.

Definition 4. [9] We say that a theory T admits *AP*, if for any $A, B, C \in \text{Mod} T$ such that $h_1 : A \rightarrow C$, $g_1 : A \rightarrow B$, where h_1, g_1 are homomorphisms, there exist a $D \in \text{Mod} T$ and $h_2 : C \rightarrow D$, $g_2 : B \rightarrow D$, where h_2, g_2 are Δ -homomorphisms such that $h_2 \circ h_1 = g_2 \circ g_1$.

Definition 5. [9] The theory T is called Jonsson theory if it satisfies the following conditions:

- 1) T has an infinite model;
- 2) T is $\forall\exists$ -axiomatizable;
- 3) T has a joint embedding property *JEP*;
- 4) T has a property of amalgam (*AP*).

Definition 6. [9] A model C of a Jonsson theory T is called the semantic model of the theory T , if its ω^+ is homogenous and universal.

Definition 7. [9] A Jonsson theory T is called perfect if every semantic model of T is a saturated model of the theory T^* , where T^* is a center of T i.e. $T^* = Th(C)$ and C is a semantic model of the theory T .

One can see that all semantic models are elementarily equivalent with each other.

Definition 8. [9] We say that a set X is Σ -definable if it is definable by some existential formula. A set X is called Jonsson in a theory T if it satisfies the following properties:

- 1) X is a Σ -definable subset of C ;
- 2) $Dcl(X)$ is a carrier of some existentially closed submodel C .

Now we want to define the notion of a fragment of Jonsson sets. Let X be a Jonsson set and M an existentially closed model, where $Dcl(X) = M$. Let us consider $Th_{\forall\exists}(M) = Fr(X)$. This set is called a fragment of the Jonsson set X .

In the following we shall have to deal with a fixed convex perfect existentially prime Jonsson theory T which is complete for existential sentences and its semantic model is C .

Let $E_n(Fr(X))$ be the distributive lattice of existential formulas with n free variables in T . We consider the formula φ up to equivalency with respect to the theory of T , $\varphi = \{\psi \in E_n(L) \mid T \vdash \varphi \leftrightarrow \psi\}$. Let $\varphi, \psi \in E_n(Fr(X))$ and $\varphi \cap \psi = 0$, where 0 is the minimum of the lattice $E_n(Fr(X))$. Then ψ is called the complement of φ , if $\varphi \cup \psi = 1$, where 1 is the lattice $E_n(Fr(X))$; ψ is a weak complement of φ , if for all $\mu \in E_n(Fr(X))$ we have $(\varphi \cup \psi) \cap \mu = 0 \Rightarrow \mu = 0$. φ is called weakly complemented, if φ has a weak complement. $E_n(Fr(X))$ is called weakly complemented if every $\varphi \in E_n(Fr(X))$ is weakly complemented.

We now recall some results which we will need later.

Theorem 9. [9] *Let T be complete for $\exists$ -sentences Jonsson theory. Then the following conditions are equivalent:*

- 1) T is perfect;
- 2) T^* is model-complete;
- 3) $E_n(T)$ is a Boolean algebra, where the completeness of the theory for $\exists$ -sentences means that any two models of this theory satisfy the same existential sentences.

We can refine some results of [4] and of classical Model theory to the frame of Jonsson theories.

Theorem 10. [9]

- 1) A theory T is a model complete if and only if every formula is preserved with respect to the submodels.
- 2) A theory T is model complete if and only if every formula is preserved under extensions of models.

Theorem 11. [7] *A theory T is model complete if and only if each $\varphi^T \in E_n(T)$ has an existential complement.*

The following result connects completeness and model completeness in the frame of Jonsson theories. It studies the properties of companions of the Jonsson

theories. The following result, that the property of model completeness and the property of completeness coincide for any Jonsson theory has a relation to the well-known theorem of P .

Theorem 12. [9] *Let T be a perfect Jonsson theory. The following conditions are equivalent:*

- 1) T is complete;
- 2) T is model complete.

Theorem 13. [4] *The existential formula φ is invariant in $\text{Mod}(\text{Th}_{\forall\exists})(E_T)$, where $E(T)$ is the class of existentially closed models of T , if and only if φ^T is weakly complemented in $E(T)$.*

We introduce the necessary definitions and state some known results which establish the relationship between model completeness, quantifier elimination, model completeness of the properties of the lattice theory and existential formulas $E_n(T)$.

Theorem 14. [2]

- 1) *Let T' be a model companion of a universal theory T . Then, T' is a model completion of T if and only if the theory T admits elimination of quantifiers.*
- 2) *Let T' be a model companion of T . Then, T' is a model completion of T if and only if the theory T has the amalgamation property.*

Theorem 15. [9] *A theory T has a model completion if and only if $E_n(T)$ is a Stone algebra.*

Theorem 16. [7] *A theory T has a model completion if and only if each $\varphi^T \in E_n(T)$ has a weak quantifier-free complement.*

3. Main results

Note that a theory which is complete for existential sentences satisfies the joint embedding property but the converse is not true. We see that the condition of existential-completeness in our theorems cannot be eliminated. Hereinafter all considered theories will be complete for the Σ -sentences.

Theorem 17. *Let $\text{Fr}(X)$ be a fragment of Jonsson set, $F^*(X)$ be the center of $\text{Fr}(X)$. Then*

- 1) $Fr^*(X)$ admits elimination of quantifiers if and only if every $\varphi \in E_n(Fr(X))$ has quantifier-free complement;
- 2) $Fr^*(X)$ is a model-complete if and only if every $\varphi \in E_n(Fr(X))$ has an existential complement.

Proof. We need to consider two cases.

Case a): Let $Fr(X)$ be a fragment of the Jonsson set X .

We prove the point 1). Since $Fr(X)$ is a fragment of the Jonsson set, M^* is a semantic model of $Fr(X)$, then as a center we consider $Fr^*(X) = Th(M)$, and one admits elimination of quantifiers. From this it follows that $Fr^*(X)$ is submodel complete. Then the theory $Fr^*(X)$ by definition is model complete, and so by Theorem 9, $\varphi \in E_n(Fr(X))$ is a Boolean algebra, i.e. every $\varphi \in E_n(Fr(X))$ has a complement. Since $E_n(Fr(X)) \subset E_n^*(Fr(X))$, then every $\varphi \in E_n(Fr(X))$ has a complement. By elimination of quantifiers $Fr^*(X)$, since $Fr^*(X)$ is the completion of the Jonsson set X , we have then with respect to the Jonsson set X that each $\varphi \in E_n(Fr(X))$ has a quantifier-free complement.

Conversely, suppose that each $\varphi \in E_n(Fr(X))$ has a quantifier-free complement. Then $\varphi \in E_n(Fr(X))$ is a Boolean algebra, therefore by Theorem 9 $Fr^*(X)$ is model-complete, and hence, in turn, by virtue of condition 2 of Theorem 10, we have that any formula with respect to $Fr^*(X)$ is equivalent to some existential formula, i.e. this formula belongs to the class $E_n^*(Fr(X))$. By Σ -completeness of the theory T we have that $E_n(Fr(X)) = E_n^*(Fr(X))$. Consequently, by virtue of the fact that each $\varphi \in E_n(Fr(X))$ has quantifier free complement, and $\varphi \in E_n(Fr(X))$ is a Boolean algebra, every formula in $\varphi \in E_n(Fr^*(X))$ is without quantifiers. Therefore, the theory of $Fr^*(X)$ admits elimination of quantifiers.

Now we prove the point 2). Let the theory $Fr^*(X)$ be a model-complete theory. Then by the definition of model completeness of $Fr^*(X)$, it is model-complete and for every existential formula φ there is an existential formula of ψ such that $Fr(X) \models -\varphi \leftrightarrow \psi$. By Theorem 9, $E_n(Fr^*(X))$ is a Boolean algebra, i.e. every $\varphi \in E_n(Fr(X))$ has an existential complement, and as for every existential formula ϕ there is an existential formula ψ such that $Fr^*(X) \models -\varphi \leftrightarrow \psi$, we find that each $\varphi \in E_n(Fr(X))$ has an existential complement. Now we prove the sufficiency of the point 2. Let every $\varphi \in E_n(Fr(X))$ has an existential complement. Then, by Theorem 11, the theory $Fr(X)$ is model-complete in the sense (recall the definition of A. Macintyre: Theory T is model-complete if T is model complete and every existential L is formula is equivalent in T to some existential L is formula). Then, by Theorem 12 we have that the theory T is complete, and as theory $Fr^*(X)$ is the center of the theory $Fr(X)$, we see

that $Fr(X) = Fr^*(X)$. Thus, $Fr^*(X)$ is model-complete. The proof of the case a) is complete.

Case b): $Fr(X)$ is not a Jonsson theory.

We prove the point 1). Since $Fr(X)$ is not a Jonsson theory, then as $Mod Fr(X)$ we consider E_n , and the center of $Fr(X)$ is the theory $Fr^*(X) = Th(U)$, where U is the universal domain of the language L , which is a model of the theory T . $Fr^*(X) = Th(U)$ and it admits elimination of quantifiers. From that, it follows that $Fr^*(X)$ is subset complete. Then, by Theorem 11 it follows that each $\varphi \in E_n(Fr^*(X))$ has a quantifier-free complement. Due to the fact that we work in E_T , $E_n(Fr(X)) = E_n^*(Fr(X))$. By Σ -completeness of the theory $Fr(X)$, $E_n(Fr(X)) = E_n^*(Fr(X))$, and hence every $\varphi \in E_n(Fr(X))$ has a quantifier-free complement.

Conversely, suppose that each $\varphi \in E_n(Fr(X))$ has a quantifier-free complement. Then $E_n(Fr(X))$ is a Boolean algebra. By Σ -completeness of the theory $Fr(X)$, we have that $E_n(Fr(X)) = E_n^*(Fr(X))$. Consequently, by virtue of the fact that each $\varphi \in E_n(Fr^*(X))$ has a quantifier-free complement, then by Theorem 11 it follows that $Fr^*(X)$ is submodel complete. Consequently, $Fr^*(X)$ admits elimination of quantifiers.

We now prove the point 2). Let the theory $Fr^*(X)$ be a model-complete. Then by the definition of model completeness of theory, $Fr^*(X)$ is model-complete and for every existential formula φ there is an existential formula ψ such that $Fr^*(X) \models -\varphi \leftrightarrow \psi$. But since $E_n(Fr(X)) \subset E_n(Fr^*(X))$, and $E_n^*(Fr(X))$ is a Boolean algebra, i.e. every $\varphi \in E_n(Fr^*(X))$ has an existential complement, and as for every existential formula φ there is an existential formula ψ such that $Fr(X) \models -\varphi \leftrightarrow \psi$, we can conclude that each $\varphi \in E_n(Fr^*(X))$ has an existential complement. But then every $\varphi \in E_n(Fr(X))$ has an existential complement. Thus, the necessity of the point 2 is proved.

We prove the sufficiency of the point 2. Let every $\varphi \in E_n(Fr(X))$ has an existential complement. Then, by Theorem 11 the theory $Fr(X)$ is model-complete, and therefore, by definition is model complete. But since $Mod Fr(X) = E_T$, and for any A of E_T , A is immersed in U , it follows that U is saturated in its power for $\exists$ -types. Thus, the theory $Fr(X)$ is perfect. Then $Fr^*(X)$ is model-complete.

□

Theorem 18. *Let $Fr(X)$ be a fragment of Jonsson set X . Then the following conditions are equivalent:*

- 1) $Fr(X)$ is a perfect;
- 2) $E_n(Fr(X))$ is weakly complemented;

3) $\varphi \in E_n(Fr(X))$ is a Stone lattice.

Proof. We consider two cases. a): Let $Fr(X)$ be a fragment of Jonsson set.

We will prove the implication from 1) to 2). Let the Jonsson theory $Fr(X)$ be perfect. Then $Fr(X)$ is perfect in the Jonssonien sense. Then the theory $Fr(X)$ has a model companion $Fr^M(X)$. It is known that in this case in such frame of conditions of our statement we have that $Fr^M(X) = Fr^0(X)$, where $Fr^0(X) = Th_{\forall\exists}(E_T)$ is the Kaisers hull of the Jonsson theory T . Since by definition the model companion $Fr^M(X)$ is model-complete, we have therefore that every formula of the language is persistent with respect to the submodels $Mod Fr^M(X)$. Consequently, every existential formula of the language is persistent with respect to the submodels $Mod Fr^M(X)$, while at the same time every existential formula of the language is persistent under extensions of models in the $Mod Fr^M(X)$, and therefore, this formula is invariant in $Mod Fr^M(X)$. Hence, by Theorem 13 it follows that every existential formula is weakly complemented. Thus, $E_n(Fr(X))$ is weakly complemented.

We will prove implication from 2) to 1). If $E_n(Fr(X))$ is weakly complemented, then the theory $Fr(X)$ has a model companion. Therefore the theory $Fr(X)$ is perfect, and its model companion is $Fr^*(X)$. Thus, the theory $Fr^*(X)$ is model-complete. So, 1) is equivalent to 2).

We will prove the implication from 1) to 3). Let $Fr(X)$ be perfect. Then the theory T is perfect in the Jonssonien sense and the theory $Fr(X)$ has a model companion. Note that by the point 2 of Theorem 14 the model companion of the Jonsson theory is its model completion. Then, by Theorem 15 it follows that $E_n(Fr(X))$ is a Stone lattice. Therefore, since $E_n(Fr(X)) \subset E_n(Fr^*(X))$ we have that $E_n(Fr(X))$ is a Stone lattice.

We will prove the implication from 3) to 1). If $E_n(Fr(X))$ is a Stone lattice, then by Theorem 15 the theory T has a model companion, and consequently, the theory T is perfect. Thus, the theory T is perfect.

The case b): $Fr(X)$ is not a Jonsson theory.

We will prove the implication from 1) to 2). Let the theory $Fr(X)$ be perfect. Since $Fr(X)$ is not a Jonsson theory, then as $Mod Fr(X)$ we consider E_T , and the center of $Fr(X)$ is the theory of $Fr^*(X) = Th(U)$, where U is a k -universal domain of the language L , which is a model of the theory of $Fr(X)$. The theory of $Fr^*(X)$ is complete. Then $Fr^*(X)$ is L model-complete and every formula of the language is persistent with respect to the submodels in $Mod Fr^*(X)$. Consequently, every existential formula of the language is persistent with respect to the submodels in $Mod Fr^*(X)$, at the same time, every existential formula of the language is persistent under extensions of models in the $Mod Fr^*(X)$, and therefore, by definition this formula is invariant in

$Mod Fr^*(X)$. Hence, by Theorem 13 it follows that every existential formula is weakly complemented. Thus, $E_n(Fr(X))$ is weakly complemented. Therefore, since $E_n(Fr(X)) \subset E_n(Fr^*(X))$, $E_n(Fr(X))$ is weakly complemented.

We will prove the implication from 2) to 1). If $E_n(Fr(X))$ is weakly complemented, then the theory T has a model companion $E_n(Fr(X))$. Then every $\varphi \in E_n(Fr(X))$ has an existential weak complement. Since $E_n(Fr(X)) \subset E_n(Fr^*(X))$, then every $\varphi \in E_n(Fr(X))$ has an existential complement. Then the theory T is model-complete, and therefore, by definition is model complete. But since $Mod Fr(X) = E_T$, and for any A from E_T , A is immersed into the U , it follows that U is saturated in its power for $\exists$ -types. Thus, the theory $Fr(X)$ is perfect.

We will prove the implication from 2) to 3). Since $E_n(Fr(X))$ is weakly complemented, then the theory $Fr(X)$ has a model companion which we denote by $Fr^M(X)$. By Theorem 18, $Fr^M(X)$ is a model completion. Then $E_n(Fr(X))$ is a Stone lattice.

We will prove the implication from 3) to 1). Let $E_n(Fr(X))$ be a Stone lattice. Since $E_n(Fr(X))$ is a Stone lattice, then the theory $Fr(X)$ has a model companion $Fr^M(x)$. Since $Mod Fr(X) = E_T$. Then it follows that $Fr^M(X) = Fr^0(X) = Th_{\forall\exists}(E_T)$. Hence, since $U \in E_T$, it follows that $Fr(X)$ is perfect. $\square$

Theorem 19. *Let $Fr(X)$ be a fragment of Jonsson set. Then the following conditions are equivalent:*

- 1) $Fr^*(X)$ is a theory;
- 2) each $\varphi \in E_n(Fr(X))$ has a weak quantifier-free complement.

To prove the necessity, we need the following statement: Fact (*)[11]: If the model companion of T^M is defined, then there is a model companion $(T_{\forall})^M$ and $T^M = (T_{\forall})^M$.

Proof. We consider two cases.

a): $Fr(X)$ is a Jonsson theory.

We will prove that 1) implies 2). Since $Fr(X)$ is a Jonsson theory, then as a center we consider the theory $Fr^*(X) = Th(C)$. If $Fr^*(X)$ is a Jonsson theory, it follows from [12] that the theory of $Fr(X)$ is perfect. Then the theory of $Fr(X)$ has a model companion equals to the theory of $Fr^*(X)$, which is the model completion of the theory of $Fr(X)$. By virtue of the mutual model consistency of the theory $Fr(X)$ and the theory of $Fr_{\forall}(X)$ (all universal consequences of the theory of $Fr(X)$ and the fact (*)) we have that the central

completion of the theory $Fr(X)$ is a model completion of the theory $Fr_{\forall}(X)$. Since $E_n(Fr(X)) \subset E_n(Fr^*(X))$. Then every $\varphi^T \in E_n(Fr(X))$ has a weak quantifier-free complement.

We will prove that 2) implies 1). Every $\varphi^T \in E_n(Fr(X))$ has a weak quantifier-free complement. Then every $\varphi^T \in E_n(Fr(X))$ has a weak complement, that is, $E_n(Fr(X))$ is weakly complemented. Then, by Theorem 10 the theory of $Fr(X)$ is perfect.

b): $Fr(X)$ is not a Jonsson theory.

We will prove that 1) implies 2). Let $Fr^*(X)$ be a theory, in this case we consider as above that $E_{Fr^*(X)} = Mod Fr^*(X)$. One can see that $E_{Fr^*(X)}$ contained in $Mod Fr^*(X)$. But we know that the Kaisers hull $Fr^0(X) = (E_{Fr(X)})$ is always a theory, it follows that $Mod Fr^0(X) = E_{Fr(X)} = Mod Fr(X)$. This means that the theory $Fr(X)$ is perfect. Then $Fr(X)$ has a model completion. Then $Fr_{\forall}(X)$ has a model completion. Hence, by Theorem 16 every $\varphi \in E_n(Fr(X))$ has a weak quantifier-free complement. Then every $\varphi \in E_n(Fr(X))$ has a weak quantifier-free complement.

We will prove that 2) implies 1). Suppose every $\varphi \in E_n(Fr(X))$ has a weak quantifier-free complement. Then every $\varphi \in E_n(Fr(X))$ has a weak complement, that is, $E_n(Fr(X))$ is weakly complemented. Then, by Theorem 10 the theory $Fr(X)$ is perfect. This means that $Fr^*(X)$ is a L theory $Fr(X)$. $\square$

4. Conclusion

This article refers to the “eastern” Model Theory, i.e. to the problems that A. Robinson defined in his time. In connection with the fact that the Jonsson theories are generally incomplete, and the main achievements of modern Model Theory are related to the problems of complete theories, it is important to develop new methods for studying the Jonsson theories. The concept of the Jonsson set is a generalization of the concept of a basis of a linear space. The new class of theories considered, namely the class of convex existentially prime ones in combination with the Jonsson sets, is an entirely new approach in the study of Jonsson theories. In this paper, the main results are Theorems 17,18,19. In these theorems, results are obtained within the framework of the above limitations, which are refinements of the corresponding results of [4] with respect to fragments of Jonsson sets.

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