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COMMON SUBSPACE-HYPERCYCLIC VECTORS

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Abstract: In this article, we show that hypercyclic operators have a dense invariant subspace, consisting except for zero, of subspace-hypercyclic vectors. Also, we prove that any finite set of hypercyclic operators has common subspace-hypercyclic vectors. Moreover, we show that for a countable family of hypercyclic operators, there exists a dense G_δ-set of X such that any of its members are subspace-hypercyclic vectors for any operator of this family.

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1. Introduction and Preliminaries

Let X be a separable, infinite dimensional Banach space over $\mathbb{C}$, the complex field, and let $B(X)$ be the algebra of operators $T : X \to X$, that are bounded and linear. An operator $T \in B(X)$ is called a hypercyclic operator, if there is a vector x in X whose orbit, $Orb(T, x) = \{x, Tx, T^2x, ...\}$, is dense in X. In this case, we say that x is a hypercyclic vector for T.

We denote the set of all hypercyclic vectors for T, by $HC(T)$.

The first example of hypercyclic operators was constructed by Rolewicz in 1969. Rolewicz in [11] proved that if B be the backward shift on $l^2(\mathbb{N})$, then for any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, λB is hypercyclic. An excellent book about hypercyclicity is the book of Grosse-Erdmann and Peris ([5]). For more interesting facts about hypercyclicity, one can read [2].

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Madore and Martinez-Avendano in 2011, defined subspace-hypercyclic operators as follows.

Definition 1.1. Let $T \in B(X)$. We say that T is subspace-hypercyclic with respect to a closed subspace M of X, if there exists a vector $x \in X$ such that $orb(T, x) \cap M$ is dense in M. In this case, x is called an M-hypercyclic vector for T.

One can read more about this concept in [9] and [10].

Madore and Martinez-Avendano asked this question in [7]:

"If T is hypercyclic, must there be a proper subspace M such that T is subspace-hypercyclic for M?"

Le answered positively to this question under some conditions in [6]. Martinez-Avendano and Zatarian-Vera in [8], showed that for every finite-codimensional subspace, hypercyclic coanalytic Toeplitz operators are subspace-hypercyclic. Finally in 2016, Bamerni, Kadets, and Kilicman answered yes to this question. Theorem1.2 is the main theorem of their paper.

Theorem 1.2. *([1]) If A is a dense subset of a Banach space X, then there is a nontrivial closed subspace M such that $A \cap M$ is dense in M.*

Let x be a hypercyclic vector for T. Then $orb(T, x)$ is dense in M. By Theorem1.1, we can find a closed and proper subspace M, such that $orb(T, x) \cap M$ is dense in M. That means x is a subspace-hypercyclic vector for T with respect to M. So we have the following corollary.

Corollary 1.3. *([1]) If T is a hypercyclic operator on a Banach space X, then there is a nontrivial closed subspace M of X such that T is M-hypercyclic.*

In this paper, we show that hypercyclic operators have a dense invariant subspace, consisting except for zero, of subspace-hypercyclic vectors. Also, we use Theorem1.2 and Corollary1.3 and prove that for every finite set of hypercyclic operators, there exists common subspace-hypercyclic vectors. Moreover, we show that if T is a hypercyclic operator, then common subspace-hypercyclic vectors for T^n's make a dense G_δ-set.

2. Main results and Examples

Let us recall Herrero-Bourdon theorem from [5].

Theorem 2.1. *If x is a hypercyclic vector for T, then*

$$\{P(T)x : P \quad is \quad a \quad polynomial\} \setminus \{0\}$$

is a dense set of hypercyclic vectors. Specially, any hypercyclic operator admits a dense invariant subspace, consisting except for zero, of hypercyclic vectors.

While T, must always contain a dense subspace, it not always contains a closed infinite dimensional one. For more information, one can see [12].

Now we extend Theorem2.1 for subspace-hypercyclic vectors as follows.

Theorem 2.2. *Let $T \in B(X)$ be a hypercyclic operator. Then T has a dense invariant subspace, that any of its nonzero members, is a subspace-hypercyclic vector for T.*

Proof. T is hypercyclic. So there exists $x \in X$ such that $orb(T, x)$ is dense in X. By Corollary1.3, there is a proper and closed subspace M of X such that T is M-hypercyclic. So x is a subspace-hypercyclic vector for T too. By Theorem2.1, with respect to T, we can find a dense invariant subspace, such that any of its nonzero members is a hypercyclic vectors for T. As we see, any of these vectors are subspace-hypercyclic vectors and this completes the proof. □

The following examples are two classic operators that satisfy in condition of Theorem2.2.

Example 2.3. (Birkhoff's operators) Let

$$T_a f(z) = f(z + a), \quad a \neq 0$$

be Birkhoff's operators on $H(\mathbb{C})$ of entire functions.

The Birkhoff's operators are hypercyclic ([5], p. 275). So by Theorem2.2, for any $a \in \mathbb{C}$, the operator T_a has a dense invariant subspace, such that any of its nonzero members is a subspace-hypercyclic vectors for T_a.

Example 2.4. (MacLane's operator) The operator of differentiation, $D : f \to f'$ on $H(\mathbb{C})$, is hypercyclic ([5], p. 275). So by Theorem2.1, it has a dense invariant subspace, such that any of its nonzero members is a subspace-hypercyclic vector for D.

One can find another example by setting $T = \lambda B$, where B is the backward shift on l^2, and λ is a complex number with $|\lambda| > 1$.
If T be a hypercyclic operator, then $HC(T)$ is a dense G_δ-set ([5], p. 39). If Λ is a countable family Λ of hypercyclic operators, Baire Category Theorem implies that $\bigcap_{T\in\Lambda} HC(T)$ is also a dense G_δ-set ([3]).

Theorem 2.5. *([5]) Let Λ be a countable set. Then the set of common hypercyclic vectors for $(T_\lambda)_{\lambda\in\Lambda}$ is a dense G_δ-set.*

Costaki and Sambarino in [4] and Shakrin in [12], have proved sufficient conditions for uncountable families of operators to have a dense G_δ-set of common hypercyclic vectors.

Theorem 2.6. *Let Λ be a countable set. Then the set of common subspace-hypercyclic vectors for $(T_\lambda)_{\lambda\in\Lambda}$, where $T_\lambda \in B(X)$ for any $\lambda \in \Lambda$, form a dense G_δ-set of X.*

Proof. By hypothesis T_λ is hypercyclic, for any $\lambda \in \Lambda$. Theorem2.5 asserts that there exists a dense G_δ-set of common hypercyclic vectors for T_λ's, where $\lambda \in \Lambda$. Let's put its name A.

T_λ's, where $\lambda \in \Lambda$, have a dense G_δ-set of common hypercyclic vectors like A.

Let x be a member of the A. Then by Corollary1.3, correspond to any natural number n, we can find a closed nontrivial subspace $M_{x,\lambda}$ such that T_λ be $M_{x,\lambda}$-hypercyclic. So each $x \in A$ is a subspace-hypercyclic vector for any T_λ. Hence the members of the A, are subspace-hypercyclic vectors.

Note that the members of A, may be subspace-hypercyclic vectors with respect to different subspaces. □

By Theorem2.6, we can assert a corollary about common vectors of T^n's, where T is hypercyclic as follows.

Corollary 2.7. *Let $T \in B(X)$ be hypercyclic. Then common subspace-hypercyclic vectors for $\{T^n : n \in \mathbb{N}\}$, make a dense G_δ-set of X.*

In 2016, Bamerni, Kadets and Kilicman in [1] asked the following question:

"if T^n is M_n-hypercyclic for all $n \geq 1$, is there any relation among all M_n?"

In what follows we try to answer this question. First, we show that for any $n \in N$, one can find a common subspace M such that T and T^n are both M-hypercyclic.

Theorem 2.8. *Let $T \in B(X)$ be a hypercyclic operator and let n be a positive integer. Then there exists a closed nontrivial subspace M such that T and T^n are both M-hypercyclic.*

Especially T and T^n have a dense subset of common M-hypercyclic vectors.

Proof. Let n be a positive integer. T is hypercyclic hence, T^n is also a hypercyclic operator. By Corollary1.3, corresponding to this n, we can find a nontrivial closed subspace M_n such that T^n is M_n-hypercyclic.

Let x be an M_n-hypercyclic vector for T^n. So $orb(T^n, x)$ is dense in M_n. But:

$$orb(T^n, x) \cap M_n \subseteq orb(T, x) \cap M_n. \tag{2.1}$$

Hence $orb(T, x) \cap M_n$ is also dense in M_n. That means x is a subspace-hypercyclic vector for T with respect to M_n. Therefore T is an M_n-hypercyclic operator. If we consider $M := M_n$, then both T and T^n are M-hypercyclic operators.

Also relation (2.1) show us that any M-hypercyclic vector for T^n is an M-hypercyclic vector for T. But the M-hypercyclic vectors for T^n or $HC(T^n, M)$ is dense in M ([7]). So we have a dense set of common M-hypercyclic vectors for T and T^n in M.

□

In the next theorem, we extend Theorem2.8 to finite number of operators as follows.

Theorem 2.9. *Let T be a hypercyclic operator. Let $\{n_1, n_2, ..., n_k\}$ be a finite sequence of natural numbers. Then there is a closed nontrivial subspace M such that T^{n_i} is M-hypercyclic for any $1 \leq i \leq k$. Especially the set of common M-hypercyclic vectors for T^{n_i}'s , is dense in M.*

Proof. T is a hypercyclic operator, hence T^{n_i}'s is also a hypercyclic operator. Now Corollary1.3, asserts that for any i, there exists a closed nontrivial subspace M_i, such that T^{n_i} is M_i-hypercyclic. Let $n_\alpha = max\{n_1, n_2, ..., n_k\}$. Let x be an M_{n_α}-hypercyclic vector for T^{n_α}. Then $orb(T^{n_\alpha}, x) \cap M_{n_\alpha}$ is dense in M_{n_α}. But $n_i \leq n_\alpha$ for any $1 \leq i \leq k$. Hence:

$$orb(T^{n_\alpha}, x) \cap M_{n_\alpha} \subseteq orb(T^{n_i}, x) \cap M_{n_\alpha}, \quad 1 \leq i \leq k. \tag{2.2}$$

So $orb(T^{n_i}, x) \cap M_{n_\alpha}$ is dense in M_{n_α}, for any $1 \leq i \leq k$. If we consider $M := M_{n_\alpha}$, then any T^{n_i} is M-hypercyclic.

Also by relation (2.2), we conclude that any M-hypercyclic vector for T^{n_α} is an M-hypercyclic vector for $T^{n_i}(1 \leq i \leq k)$. By (Madore and Martinez-Avendano (2011)), the set of M-hypercyclic vectors for T^{n_α} are dense in M. Therefore common M-hypercyclic vectors of T^{n_i}'s are dense in M.

□

If in Theorem2.9 we consider $n_1 = 1, n_2 = 2, ..., n_{k-1} = k - 1$ and $n_k = k$, then we have the following corollary.

Corollary 2.10. *Let $T \in B(X)$ be hypercyclic. Then there exists a closed nontrivial subspace M such that $T, T^2, ..., T^{k-1}$ and T^k are M-hypercyclic. Especially there is a dense subset of common M-hypercyclic vectors for $T, T^2, ..., T^{k-1}$ and T^k, in M.*

As a consequence of Corollary2.10, Question3 of Bamerni, Kadets and Kilicman in [1], is answered partially. In fact, we show that $T, T^2, ..., T^{n-1}$ and T^n can have a common nontrivial closed subspace for subspace-hypercyclicity with many common subspace-hypercyclic vectors.

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