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CHARACTERIZATIONS OF POINTWISE-$\mathcal{I}$-CONTINUOUS, POINTWISE-$\mathcal{I}$-OPEN AND POINTWISE-$\mathcal{I}$-CLOSED MAPS

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Abstract: We obtain new characterizations of pointwise-$\mathcal{I}$ open (closed, continuous) maps with respect to an ideal $\mathcal{I}$ by using the concept of local function and # operator. We also generalize some known results. In addition various characterizations of pointwise-$\mathcal{I}$-continuous onto maps in terms of saturated sets and pointwise-$\mathcal{I}$-continuous injective maps are given.

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1. Introduction

In [7], Noorie and Bala used the sets of the form $f^{\#}(E) = \{y \in Y : f^{-1}(y) \subseteq E\}$ to obtain new characterizations of open, closed or continuous mappings $f : X \to Y$. On the other hand the study of properties of maps with respect to an ideal in topology is already a well researched topic in the literature. Starting with Kuratowski[6] and Vaidyanathaswamy[10], ideals in topological spaces have been used to study topological properties, where an ideal $\mathcal{I}$ on a topological space (X, τ) is a collection of subsets of X which is closed under finite unions and closed downwards meaning that every subset of a member of $\mathcal{I}$ is in $\mathcal{I}$ i.e. if $B \in \mathcal{I}$ then $\wp(B) \subseteq \mathcal{I}$, where $\wp(B)$ means collection of all

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subsets of A. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X, a topology $\tau^*(\mathcal{I}, \tau)$ is given which is generally finer than the original topology τ and has its corresponding Kuratowski closure operator $cl^*(A) = A \cup A^*(\mathcal{I}, \tau)$[9], where $A^*(\mathcal{I}, \tau) = \{x \in X : U \cap A \notin \mathcal{I}$ for every open nhd. U of x in $X\}$. The set operator $(.)^* : \wp(X) \to \wp(X)$, called a local function[6] of A with respect to $\mathcal{I}$ and τ. We will write A^* for $A^*(\mathcal{I}, \tau)$ and $\tau^*(\mathcal{I})$ or τ^* for $\tau^*(\mathcal{I}, \tau)$. In [5], Kanicwski and Piotrowski introduced the concept of pointwise - $\mathcal{I}$-continuous maps and in [2] , Hamlett and Rose introduced pointwise - $\mathcal{I}$-open maps. In [8] , Sivaraz and Renuka Devi gave various characterizations and properties of these maps using the image map f and inverse image map f^{-1}.

In this paper, we use concepts of local function and the induced map $f^{\#} : \wp(X) \to \wp(X)$ for a map $f : X \to Y$, to obtain new characterizations of pointwise-$\mathcal{I}$ open maps (Theorem 3.1 below). This induces a generalization of the results of [8] which allows us to introduce and similarly characterize pointwise-$\mathcal{I}$ closed maps (Theorem 3.3 below). For the case of topologies compatible with the ideal, new characterization of pointwise-$\mathcal{I}$ closed maps is given directly in terms of f, rather than in terms of the induced map $f^{\#}$ (Theorem 3.4(a) and corollary 3.5 below). It is also shown that condition of compatibility can be dropped in case of $*$-dense in itself subsets (Theorem 3.5 below). We also give generalizations of some of the results of [7] and obtain new characterizations of pointwise-$\mathcal{I}$-continuous maps (Theorems 3.6 and 3.7 below). Finally pointwise-$\mathcal{I}$-continuous injections are characterized (Theorems 3.8 and Corollary 3.7 below). Examples are given throughout the paper to give counterexamples and illustrations.

2. Preliminaries

In this section we will give some basic definitions and the results that we need in section 3.

Given an ideal space $(X, \tau, \mathcal{I})$, $\mathcal{I}$ is said to be compatible with respect to τ[4], denoted by $\tau \sim \mathcal{I}$, if for every subset A of X which is locally in $\mathcal{I}$, is infact a member of $\mathcal{I}$. This means that if for a subset A of X, each point of A has a nhd. U such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$. Also $\mathcal{I}$ is said to be codense[1] if $\tau \cap \mathcal{I} = \emptyset$. A subset A of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$- dense if $A^* = X$[1] and $*$-dense in itself if $A \subseteq A^*$[3]. Also if $(X, \tau, \mathcal{I})$ is an ideal topological space and (Y, σ) be any topological space and $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be any map then $f(\mathcal{I}) = \{f(I) : I \in \mathcal{I}\}$ is an ideal on the topological space (Y, σ). In our results, we will denote $f(\mathcal{I})$ by $\mathcal{J}$.

Throughout this paper (X, τ) will denote topological space on which no seperation axioms are assumed. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal space. When there is no chance of confusion, by a open subset of X, we will mean open set in the topological space(X, τ). For a subset A of X, $cl(A)$ and $int(A)$ will denote the closure of A, interior of A in (X, τ), respectively, $cl^*(A)$ and $int^*(A)$ will denote the closure of A, interior of A in (X, τ^*), respectively, and A^C will denote the complement of A in X.

Definition 2.1. [5] A mapping $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be pointwise-$\mathcal{I}$-continuous if the inverse image of every open set in Y is $\tau^*(\mathcal{I})$-open. Equivalently, $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is pointwise-$\mathcal{I}$-continuous if and only if $f : (X, \tau^*(\mathcal{I})) \to (Y, \sigma)$ is continuous.

Definition 2.2. [2] A mapping $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be pointwise-$\mathcal{I}$-open if the image of every open set in X is $\sigma^*(\mathcal{J})$-open. Equivalently, $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is pointwise-$\mathcal{I}$-open if and only if $f : (X, \tau) \to (Y, \sigma^*(\mathcal{J}))$ is open.

We will also make use of the following results:

Lemma 2.1. *[7] For any sets X and Y, let $f : X \to Y$ be any map and E be any subset of X. Then:*

(a) $E^\# = \{f^{-1}(y) : y \in Y and f^{-1}(y) \subseteq E\} \subseteq E$.

(b) $f^\#(E^C) = (f(E))^C$ *and so* $f^\#(E) = (f(E^C))^C$ *and* $f(E) = (f^\#(E^C))^C$.

(c) *E is saturated if and only if* $E = E^\#$.

(d) *f is onto if and only if* $f^\#(A) = f(A^\#)$.

(e) $E^\# = f^{-1}(f^\#(E))$.

(f) $f^\#(X) = Y$.

Lemma 2.2. *[4] Let $(X, \tau, \mathcal{I})$ be an ideal space and A and B be subsets of X, then the following holds:*

(a) $A^* = cl(A^*) \subseteq cl(A)$ *and so* A^* *is a closed subset of* $cl(A)$.

(b) *A is $\tau^*(\mathcal{I})$-closed if and only if* $A^* \subseteq A$.

(c) $A^*(\mathcal{I}, \tau) = A^*(\mathcal{I}, \tau^*(\mathcal{I}))$.

(d) $\tau \sim \mathcal{I}$ *if and only if* $A - A^* \in \mathcal{I}$.

(e) $(A \cup B)^* = A^* \cup B^*$.

(f) $\tau \cap \mathcal{I} = \emptyset$ *if and only if* $X = X^*$.

Theorem 2.1. *[8] Let* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *with* $\mathcal{J} = f(\mathcal{I})$ *be a pointwise-$\mathcal{I}$-continuous injective map, then* $(f^{-1}(B))^* \subseteq f^{-1}(B^*)$ *for every subset* B *of* Y.

Theorem 2.2. *[8] Let* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *with* $\mathcal{J} = f(\mathcal{I})$ *be any map. If* $f(A^*) \subseteq (f(A))^*$ *for every subset* A *of* X *then* f *is pintwise-$\mathcal{I}$-continuous.*

3. Results

In [8], it is shown that a bijective map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is pointwise-$\mathcal{I}$-open if and only if $f^{-1}(B^*) \subseteq (f^{-1}(B))^*$ for every subset B of Y. In our first theorem below we show that the condition of bijection is not needed. Further we give new characterizations of pointwise-$\mathcal{I}$-open maps.

Theorem 3.1. *Let* $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ *be any map. Then the following conditions are equivalent:*

(a) f *is pointwise- $\mathcal{I}$-open.*

(b) $(f^{\#}(A))^* \subseteq f^{\#}(A^*)$ *for every subset* A *of* X.

(c) $f^{-1}(B^*) \subseteq (f^{-1}(B))^*$ *for every subset* B *of* Y.

(d) $f^{-1}(cl^*(B)) \subseteq cl(f^{-1}(B))$ *for every subset* B *of* Y.

(e) $cl^*(f^{\#}(A)) \subseteq f^{\#}(cl(A))$ *for every subset* A *of* X.

(f) For each closed subset F *of* X, $f^{\#}(F)$ *is* σ^* *- closed in* Y.

Proof. (a)$\Rightarrow$(b): Let A be a subset of X and $y \notin f^{\#}(A^*)$. This implies $f^{-1}(y) \nsubseteq A^*$. So there exists $x \in f^{-1}(y)$ such that $x \notin A^*$. By definition of A^*, there exists an open nhd. U of x in X such that $U \cap A \in \mathcal{I}$ which implies that $f(U \cap A) \in f(\mathcal{I})$. But $f(U) \cap f^{\#}(A) \subseteq f(U \cap A)$. Therefore, $f(U) \cap f^{\#}(A) \in f(\mathcal{I})$. Since f is pointwise-$\mathcal{I}$-open , $f(U) \in \sigma^*$and so $y \notin (f^{\#}(A))^*(\mathcal{I}, \tau^*)$. Therefore, by Lemma 2.2(c), $y \notin (f^{\#}(A))^*$. Hence $(f^{\#}(A))^* \subseteq f^{\#}(A^*)$ for every subset A of X.

(b) $\Rightarrow$(c): Let B be a subset of Y. $A = f^{-1}(B)$ in (b) implies $(f^{\#}(f^{-1}(B)))^* \subseteq f^{\#}((f^{-1}(B))^*)$. As $B \subseteq f^{\#}(f^{-1}(B))$, so $B^* \subseteq (f^{\#}(f^{-1}(B)))^* \subseteq f^{\#}((f^{-1}(B))^*)$ and so $f^{-1}(B^*) \subseteq$

$f^{-1}(f^{\#}((f^{-1}(B))^*)) \subseteq (f^{-1}(B))^*$. Therefore $f^{-1}(B^*) \subseteq (f^{-1}(B))^*$ for every subset B of Y.

(c)⇒(d): For any subset B of Y , $f^{-1}(cl^*(B)) = f^{-1}(B \cup B^*) = f^{-1}(B) \cup f^{-1}(B^*) \subseteq f^{-1}(B) \cup (f^{-1}(B))^* = cl^*(f^{-1}(B)) \subseteq cl(f^{-1}(B))$. This proves (d).

(d)⇒(e): Let A be a subset of Y. $B = f^{\#}(A)$ in (d) implies $f^{-1}(cl^*(f^{\#}(A))) \subseteq cl(f^{-1}(f^{\#}(A))) \subseteq cl(A)$. So $f^{\#}(f^{-1}(cl^*(f^{\#}(A)))\ ) \subseteq f^{\#}(cl(A))$. Since $A \subseteq f^{\#}(f^{-1}(A))$ for every subset A of X. So $cl^*(f^{\#}(A)) \subseteq f^{\#}(cl(A))$ which proves (e).

(e)⇒(f) : Let F be any closed subset of X. As $cl(F) = F$, so (e) implies that $cl^*(f^{\#}(F)) \subseteq f^{\#}(F)$. Hence $f^{\#}(F)$ is σ^* - closed in Y.

(f)⇒(a): Let U be any open subset of X. Then (f) implies $f^{\#}(U^C)$ is σ^*-closed in Y. But $f^{\#}(U^C) = (f(U))^C$ by Lemma 2.1(b). Therefore $f(U)$ is σ^*-open in Y. Hence f is pointwise-$\mathcal{I}$-open map. □

Corollary 3.1. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be onto. Then f is an pointwise-$\mathcal{I}$-open map if and only if for each closed subset F of X, $f(F^{\#})$ is σ^*-closed in Y.*

Proof. The proof follows immediately from the equivalence of (a) and (f) in the above theorem and Lemma 2.1(d). □

In the next corollary, we see that interestingly f is pointwise - $\mathcal{I}$ - open if and only if $f : (X, \tau^*(\mathcal{I})) \to (Y, \sigma^*(\mathcal{J}))$ is open.

Corollary 3.2. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be any map. Then the following are equivalent:*

(a) f is pointwise-$\mathcal{I}$-open map.

(b) Image of every τ^*- open subset of X is σ^*- open in Y.

(c) For each τ^*-closed subset F of X, $f^{\#}(F)$ is σ^*- closed in Y.

Proof. (a)⇒(b):Let U be any τ^*-open subset of X i.e U^C is τ^* - closed subset of X. So $(U^C)^* \subseteq U^C$ using Lemma 2.2(b). Now by using the equivalence of (a) and (b) of Theorem 3.1 it follows that $(f^{\#}(U^C))^* \subseteq f^{\#}((U^C)^*) \subseteq f^{\#}(U^C)$. But $f^{\#}(U^C) = (f(U))^C$ using Lemma 2.1(b). Therefore $(Y - f(U))^* \subseteq Y - f(U)$. Hence $Y - f(U)$ is σ^*- closed in Y and so $f(U)$ is σ^*- open in Y. This proves (b).

(b)⇒(c): follows using Lemma 2.1(b).

(c)⇒(a): follows using the equivalene of (a) and (f) of Theorem 3.1 and the fact that $\tau \subseteq \tau^*$. □

The following corollary drops the unnecessary assumption of bijection which was assumed in [8].

Corollary 3.3. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is a pointwise-$\mathcal{I}$-open map, then the following holds:*

(a) *The inverse image of every $\mathcal{J}$-dense subset of Y is $\mathcal{I}$-dense.*

(b) *The inverse image of every $*$ - dense in itself subset of Y under f is $*$-dense in itself in X.*

Now we give another proof of the following Theorem of [8]

Theorem 3.2. *[8] Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is a pointwise-$\mathcal{I}$-open map and $\sigma \cap \mathcal{J} = \emptyset$,then $\tau \cap \mathcal{I} = \emptyset$.*

Proof. For the subset $B = Y$, the equivalence of (a) and (c) of Theorem 3.1 implies that $f^{-1}(Y^*) \subseteq (f^{-1}(Y))^*$. Using Lemma 2.2 (f), $\sigma \cap \mathcal{J} = \emptyset$ implies that $f^{-1}(Y) \subseteq (f^{-1}(Y))^*$. So $X \subseteq X^*$. Therefore $X = X^*$. Hence $\tau \cap \mathcal{I} = \emptyset$ using Lemma 2.2(f). $\square$

For bijections the concepts of pointwise-$\mathcal{I}$-open and pointwise-$\mathcal{I}$-closed maps coincide. However since we have now generalized the theorem of [8] and dropped the unnecessary condition of bijection, therefore we introduce pointwise-$\mathcal{I}$-closed maps and give various characterizations of pointwise-$\mathcal{I}$-closed maps.

Definition 3.1. A mapping $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be pointwise-$\mathcal{I}$-closed if the image of every closed set is $\sigma^*(\mathcal{J})$ - closed. Equivalently, $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is pointwise-$\mathcal{I}$-closed if and only if $f : (X, \tau) \to (Y, \sigma^*(\mathcal{J}))$ is closed.

Clearly, every closed function is pointwise-$\mathcal{I}$-closed but the converse need not be true as shown by the following Example.

Example 3.1. Let $X = \{a, b, c\}$,$Y = \{0, 1\}$, $\tau = \{\emptyset, \{c\}, X\}$, $\mathcal{I} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma = \{\emptyset, Y\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = 0, f(b) = 0$ and $f(c) = 1$ with $\mathcal{J} = f(\mathcal{I}) = \wp(Y)$ and so $\sigma^*(\mathcal{J})$ is the discrete topology. Then f is not closed as $\{a, b\}$ is closed but $f\{a, b\} = \{0\}$ is not closed. But $\sigma^*(\mathcal{J})$ is the discrete topology. Therefore, f is pointwise-$\mathcal{I}$-closed vacuously.

The following theorem gives different characterizations of pointwise-$\mathcal{I}$-closed maps which is a generalization of Theorem 2.10 of [7] and in special case by taking $\mathcal{I} = \emptyset$, the following theorem coincides with the Theorem 2.10 of [7].

Theorem 3.3. *For any map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$, the following conditions are equivalent.*

(a) f is pointwise-$\mathcal{I}$-closed;

(b) for each subset A of X, $cl^*(f(A)) \subseteq f(cl(A))$;

(c) for each subset A of X, $f^{\#}(int(A)) \subseteq int^*(f^{\#}(A))$;

(d) for each open subset U of X, $f^{\#}(U)$ is σ^*-open in Y.

Proof. Proof is similar to that of Theorem 2.10 of [7] and hence is omitted. □

Remark 3.1. It follows immediately that if $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be any map. Then the image of τ^*-closed subset of X is σ^*-closed subset of Y if and only if $cl^*(f(A)) \subseteq f(cl^*(A))$ for any subset A of X.

Corollary 3.4. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be onto. Then f is a pointwise-$\mathcal{I}$-closed map if and only if for each open subset U of X, $f(U^{\#})$ is σ^*-open in Y.*

Proof. The proof follows immediately from the equivalence of (a) and (c) in the above theorem and Lemma 2.1(d). □

We now show that in case of compatible topology pointwise-$\mathcal{I}$-closed is equivalent to $(f(A))^* \subseteq f(A^*)$ for every subset A of X. In fact we prove a slightly stronger result in the following:

Theorem 3.4. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be any map then the following holds.*

(a) If f is pointwise-$\mathcal{I}$-closed and $\tau \sim \mathcal{I}$, then $(f(A))^* \subseteq f(A^*)$ for every subset A of X.

(b) If $(f(A))^* \subseteq f(A^*)$ for every subset A of X, then the image of every τ^*-closed set is σ^* - closed and hence in particular, f is pointwise- $\mathcal{I}$ - closed.

Proof. (a): For any subset A of X, as $A = (A - A^*) \cup (A \cap A^*)$. So using Lemma 2.2(e) $(f(A))^* = (f((A-A^*)\cup(A\cap A^*)))^* = (f(A-A^*))^*\cup(f(A\cap A^*))^*$ and so $(f(A))^* \subseteq (f(A - A^*))^* \cup (f(A^*))^*$. Since $\tau \sim \mathcal{I}$, so $A - A^* \in \mathcal{I}$ using Lemma 2.2(d) and so $f(A - A^*) \in f(\mathcal{I})$. Therefore, $(f(A - A^*))^* = \emptyset$. Also f is pointwise- $\mathcal{I}$ - closed and A^* is closed, so $f(A^*)$ is σ^* - closed. Therefore, $(f(A))^* \subseteq (f(A^*))^* \subseteq f(A^*)$ using Lemma 2.2(b). Hence (a) holds.

(b): For any subset A of X , $cl^*(f(A)) = f(A) \cup (f(A))^*$. So $cl^*(f(A)) \subseteq f(A) \cup f(A^*) = f(A \cup A^*) = f(cl^*(A)) \subseteq f(cl(A))$. Hence by the Remark 3.1 the image of every τ^*- closed set is σ^* - closed.
Also $f(cl^*(A)) \subseteq f(cl(A)$ implies f is pointwise - $\mathcal{I}$ - closed. □

Corollary 3.5. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be any map and $\tau \sim \mathcal{I}$, then the following are equivalent.*

(a) f is pointwise - $\mathcal{I}$ - closed.

(b) $(f(A))^* \subseteq f(A^*)$ for every subset A of X.

(c) The image of every τ^*- closed subset of X is σ^* - closed in Y.

We now show the condition of compatibility in Theorem 3.4(a) above can be dropped for $*$-dense in itself subsets of X.

Theorem 3.5. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be any map. If f is pointwise-$\mathcal{I}$-closed then $(f(A))^* \subseteq f(A^*)$ for every $*$- dense in itself subset A of X.*

Proof. Let A be any $*$-dense in iself subset of X. So $f(A) \subseteq f(A^*)$ and so $(f(A))^* \subseteq (f(A^*))^*$. Since f is pointwise-$\mathcal{I}$-closed and A^* is closed so by Lemma 2.2(b), $(f(A^*))^* \subseteq f(A^*)$. Therefore $(f(A))^* \subseteq f(A^*)$. □

Corollary 3.6. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a bijective map. Then the following are equivalent.*

(a) f is pointwise-$\mathcal{I}$-open.

(b) $(f(A))^* \subseteq f(A^*)$ for every subset A of X.

(c) f is pointwise-$\mathcal{I}$-closed for every subset A of X.

Proof. (a)⇒(b): follows from Theorem 3.1 and the fact that in case of bijective map $f^{\#}(A) = f(A)$ for any subset A of X.
(b)⇒(c): follows from Theorem 3.4(b).
(c)⇒(a): follows from the equivalence of (a) and (d) of Theorem 3.3 and the fact that in case of bijective map $f^{\#}(A) = f(A)$ for any subset A of X. □

The remaining results of this section give characterizations of pointwise - $\mathcal{I}$-continuous maps.
The following Theorems 3.6 and 3.7 give various characterizations of pointwise-$\mathcal{I}$-continuous maps which are generalization of Theorem 2.13 and Theorem 2.14 of [7].

Theorem 3.6. *A map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is pointwise-$\mathcal{I}$-continuous if and only if for each subset A of X, $int(f^{\#}(A)) \subseteq f^{\#}(int^*(A))$.*

Proof. Proof is similar to that of Theorem 2.13 of [7] and hence is omitted. $\square$

The following theorem characterizes pointwise-$\mathcal{I}$ - continuity for onto maps.

Theorem 3.7. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be any onto map. Then the following conditions are equivalent.*

(a) f is pointwise-$\mathcal{I}$-continuous.

(b) $int(f(A^{\#})) \subseteq f(int^*(A))^{\#})$ for all subsets A of X.

(c) $A^{\#}$ is τ^*-open in X whenever $f(A^{\#})$ is open in Y.

(d) for any saturated set E in X, E is τ^*-open in X whenever $f(E)$ is open in Y.

(e) for any saturated set E in X, E is τ^*-closed in X whenever $f(E)$ is closed in Y.

Proof. Proof is similar to that of Theorem 2.14 of [7] and hence is omitted. $\square$

The following theorem characterizes pointwise-$\mathcal{I}$ - continuity for injective maps.

Theorem 3.8. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a pointwise-$\mathcal{I}$-continuous injection, then the following holds.*

(a) $f(A^*) \subseteq (f(A))^*$ for every subset A of X.

(b) $f^{\#}(A^*) \subseteq (f^{\#}(A))^*$ for every subset A of X.

(c) The inverse image of every $\sigma^*(\mathcal{J})$ - open subset of Y is $\tau^*(\mathcal{I})$ - open in X.

Proof. (a): Let A be any subset of X. For the subset $B = f(A)$ using Theorem 2.1, we have $(f^{-1}(f(A)))^* \subseteq f^{-1}((f(A))^*)$. Now $A \subseteq f^{-1}(f(A))$, so $A^* \subseteq (f^{-1}(f(A)))^* \subseteq f^{-1}((f(A))^*)$ and so $f(A^*) \subseteq (f(A))^*$. Hence (a) holds.

(b): Since f is injective, so $f^{\#}(A) = f(A) \cup (f(X))^C$ for every subset A of X. So we will prove that $f(A^*) \cup (f(X))^C \subseteq (f(A))^* \cup ((f(X))^C)^*$. Further using part(a), we only need to check $(f(X))^C \subseteq ((f(X))^C)^*$. Let $y \notin ((f(X))^C)^*$. So there exists an open nhd. W of y in Y such that $W \cap (f(X))^C \in f(\mathcal{I})$ i.e.

$W \cap (f(X))^C = f(I)$ for some I in $\mathcal{I}$ and so $W \cap (f(X))^C = \emptyset$ or $W \subseteq f(X)$. Therefore, $y \in (f(X))$ or $y \notin (f(X))^C$. Hence $f^{\#}(A^*) \subseteq (f^{\#}(A))^*$ for every subset A of X.

(c): Let W be $\sigma^*(\mathcal{J})$ - open subset of Y. So W^C be $\sigma^*(\mathcal{J})$ - closed subset of Y and so $(W^C)^* \subseteq W^C$ using Lemma 2.2(b). Therefore, using Theorem 2.1 for the subset $B = W^C$, we have $(f^{-1}(W^C))^* \subseteq f^{-1}((W^C)^*)$ and so $(f^{-1}(W^C))^* \subseteq f^{-1}(W^C)$. Hence $((f^{-1}(W))^C)^* \subseteq (f^{-1}(W))^C$ implies $(f^{-1}(W))^C$ is τ^*-closed subset of X using Lemma 2.2(b) and so $f^{-1}(W)$ is τ^*-open subset of X which proves (c). □

Remark 3.2. The above theorem shows that the surjection condition in part (a) can be removed which was assumed in[8].

The following Example shows that the condition of injection cannot be dropped in the above Theorem 3.8.

Example 3.2. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{d\}, \{a, c\}, \{a, c, d\}, X\}$, $\mathcal{I} = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$, $Y = \{a, b, c\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Y\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = b, f(b) = c, f(c) = b$ and $f(d) = a$ with $\mathcal{J} = f(\mathcal{I}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then

$$\tau^*(\mathcal{I}) = \{\emptyset, \{a\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, X\}$$

and $\sigma^*(\mathcal{J}) = \wp(Y)$. It can be easily checked that f is pointwise - $\mathcal{I}$-continuous map which is not injective. Now for the subset $A = \{a\}$ of X, $A^* = \{a, b, c\}$. Also $f^{\#}(A^*) = f(A^*) = \{b, c\}$ and $(f^{\#}(A))^* = (f(A))^* = \emptyset$. Therefore, $f^{\#}(A^*) \nsubseteq (f^{\#}(A))^*$ and $f(A^*) \nsubseteq (f(A))^*$. Also $\{c\}$ is $\sigma^*(\mathcal{J})$ open in Y but $f^{-1}\{c\} = \{b\}$ is not $\tau^*(\mathcal{I})$ open in X.

Even though Theorem 3.8(a) above implies that f is pointwise-$\mathcal{I}$-continuous (Theorem 2.2 above). But the following Example shows that Theorem 3.8(b) does not imply that f is pointwise-$\mathcal{I}$-continuous.

Example 3.3. Let $X = \{a, b\}$, $\tau = \{\emptyset, \{a\}, X\}$, $\mathcal{I} = \{\emptyset\}$, $Y = \{0, 1, 2\}$ and $\sigma = \{\emptyset, \{0\}, \{0, 1\}, Y\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = 2, f(b) = 1$ with $\mathcal{J} = f(\mathcal{I}) = \{\emptyset\}$. Now $0 \in f^{\#}(A)$ and $\{0\}^* = Y$, so $(f^{\#}(A))^* = Y$ for every subset A of X. Hence $f^{\#}(A^*) \subseteq (f^{\#}(A))^*$ for every subset A of X. But f is not pointwise-$\mathcal{I}$-continuous since $\{0, 1\}$ is σ- open in Y but $f^{-1}\{0, 1\} = \{b\}$ is not $\tau^*(\mathcal{I})$ open in X.

Corollary 3.7. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be pointwise- $\mathcal{I}$-continuous injection, then the following holds.*

(a) *The # image of every $\mathcal{I}$-dense subset of X is $\mathcal{J}$-dense subset of Y.*

(b) *The # image of every $*$ - dense in itself subset of X is $*$ -dense in itself subset of Y.*

Proof. (a): Let A be $\mathcal{I}$-dense subset of X, so $A^* = X$. Therefore, using Theorem 3.8(b), it follows that $f^{\#}(X) \subseteq (f^{\#}(A))^*$. Since $f^{\#}(X) = Y$ using Lemma 2.1(f), so $(f^{\#}(A))^* = Y$. Hence $f^{\#}(A)$ is $\mathcal{J}$-dense subset of Y.

(b): follows from Theorem 3.8(b). □

Now we give another proof of the following Theorem of [8]

Theorem 3.9. *[8] Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is a pointwise-$\mathcal{I}$-continuous injection and $\tau \cap \mathcal{I} = \emptyset$ then $\sigma \cap \mathcal{J} = \emptyset$.*

Proof. For the subset $A = X$, Theorem 3.8(b) implies that $f^{\#}(X^*) \subseteq (f^{\#}(X))^*$. Using Lemma 2.2(f), $\tau \cap \mathcal{I} = \emptyset$ implies that $f^{\#}(X) \subseteq (f^{\#}(X))^*$. Since $f^{\#}(X) = Y$ using Lemma 2.1 (f), so $Y \subseteq Y^*$. Therefore $Y = Y^*$. Hence using Lemma 2.2(f), $\sigma \cap \mathcal{J} = \emptyset$. □

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