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ON NEW FUZZY METRIC ORLICZ SEQUENCE SPACE

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Abstract: The purpose of this work is to define a new class of sequences of fuzzy numbers using Orlicz functions and to derive several useful classes having rich structural properties. Further, a metric topological structure has been imposed, as well as, the topological behaviour has been investigated for this new class. Moreover, various inclusion relations among different classes of fuzzy sequences have been studied.

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1. Introduction

This work is an attempt to fuzzify the functional analytic study of Orlicz sequence space of real or complex number sequences. From functional analytic point of view, the Orlicz sequence spaces are the special cases of Orlicz spaces studied in [22]. Lindenstrauss and Tzafriri [26] first investigated Orlicz sequence spaces in detail.

An Orlicz function [24] is a function $M : [0,\infty) \longrightarrow [0,\infty)$, which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

An Orlicz function is said to satisfy Δ_2-condition for all values of x, if there exists a constant $K > 0$, such that $M(2x) \leq KM(x)$ for all $x \geq 0$. This condition is equivalent to $M(Lx) \leq K(L)M(x)$, for all values of $x \geq 0$ and for all $L \geq 1$. Also an Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for

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all λ with $0 < \lambda < 1$.

An Orlicz function M can always be represented (see [22]) in the integral form $M(x) = \int_0^x q(t)dt$, where q, known as the Kernel of M, is right differentiable for $t \geq 0$, q(0)=0, $q(t) > 0$ for $t > 0$, $q(t) \to \infty$ as $t \to \infty$.

If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x) + M(y)$, then the function reduces to a modulus function defined and discussed by Ruckle [38], Maddox [28], Srivastava *et al.*[42], Altin[2], Ghosh *et al.*[19] and many others.

Lindenstrauss and Tzafriri [26] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \Big\{ x \in \omega : \sum_{k=1}^{\infty} \Big(M(\frac{\mid x_k \mid}{\rho}) \Big) < \infty, \text{ for some } \rho > 0 \Big\}.$$

The space ℓ_M with the norm

$$\| x \| = \inf \Big\{ \rho > 0 : \sum_{k=1}^{\infty} \Big(M(\frac{\mid x_k \mid}{\rho}) \Big) \leq 1 \Big\}$$

becomes a Banach space which is called an Orlicz sequence space, where ω is the family of real or complex sequences. Subsequently Gungor *et al.* [20], Esi *et al.* [16], Nuray *et al.* [35], Bhardwaj *et al.* [7], Mursaleen *et al.* [31], Ahmad [1], Bektas *et al.* [6], Parashar *et al.* [36], Isik *et al.* [21], Dutta *et al.* [15], Karakaya *et al.* [25] and others used Orlicz function to construct several new sequence spaces. Later on Basu and Srivastava ([3], [4], [5]) extended the study of composite vector valued single and double sequences and their various convergence methods with the help of Orlicz function, modulus function, multiplier sequences etc.

The concept of fuzzy sets and fuzzy sets operations were first introduced by Zadeh [45] and since then a large number of research papers have appeared by using the concept of fuzzy set/numbers. Matloka [29] first introduced bounded and convergent sequences of fuzzy numbers and studied their properties. He showed that every convergent sequence of fuzzy numbers is bounded. Later on, applying the notion of fuzzy real numbers, fuzzy real-valued sequence space was first introduced and studied by Nanda [32]. He showed that the set of all convergent sequences of fuzzy numbers forms a complete metric space. Later on the study of theory of fuzzy sequence spaces has been continued by Tripathy and Nanda[33], Savas [40], Mursaleen et al. [30], Fang *et al.* [18], Talo *et al.* [43], Dutta *et al.*([13], [14]) Esi *et al.* [17] and many others.

Briefly, we recall some of the basic notations in the theory of fuzzy numbers. For more details, we refer Puri and Ralescu [37], Matloka [29] and Diamond and Kloeden [9].

Given an interval A, we denote its end points by $\underline{A}$ and $\overline{A}$. Let $A = \{A \subset R : \ A = [\underline{A}, \ \overline{A}]\}$. For $A, B \in D$, define $A \leq B$ iff $\underline{A} \leq \underline{B}$ and $\overline{A} \leq \overline{B}$, $d(A, B) = \max\{|\underline{A} - \underline{B}|, \ |\overline{A} - \overline{B}|\}$. Then d defines a metric on D and (D, d) is a complete metric space.

A fuzzy number is a mapping $X : \ R \to [0, \ 1]$ satisfying

(i) X is normal, i.e., $\exists$ an $x_0 \in R$ such that $X(x_0) = 1$;

(ii) X is fuzzy convex, i.e., for any $x, y \in R$ and $\lambda \in [0, 1]$, $X(\lambda x + (1 - \lambda)y) \geq \min(X(x), X(y))$.

(iii) X is upper semicontinuous, i.e., for each $\epsilon > 0$, $X^{-1}([0, a + \epsilon))$ is open in the usual topology of R, for all $a \in [0, 1]$.

(iv) The closure of $X^0 = \{x \in R : \ X(x) > 0\}$ is compact.

These properties imply that, for each $\alpha \in [0, 1]$, the $\alpha-$level set of a fuzzy real number X,

$$X^\alpha = \{x \in R : \ X(x) \geq \alpha\} = [{X_\alpha}^-, {X_\alpha}^+]$$

is a non-empty compact convex subset of R as the support $X^0 = \lim_{\alpha \to 0^+} X^\alpha$. Let $L(R)$ denote the set of all fuzzy real numbers.

Define a map $\overline{d} : \ L(R) \times L(R) \to R$ on $L(R)$ is defined by

$$\overline{d}(X, \ Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, \ Y^\alpha), \ X, \ Y \in L(R)$$

Then $(L(R), \bar{d})$ is a complete metric space (as proved by Puri and Ralescu [37]). For $X, \ Y \in L(R)$, define $X \leq Y$ iff $X^\alpha \leq Y^\alpha$ for any $\alpha \in [0, \ 1]$.

A metric on $L(R)$ is said to be a translation invariant if $\bar{d}(X + Z, Y + Z) = \bar{d}(X, Y)$ for $X, Y, Z \in L(R)$.

R can be embedded in $L(R)$ since each $r \in R$ can be regarded as a fuzzy number $\bar{r}$ defined by

$$\bar{r}(t) = \begin{cases} 1, & \text{if } t = r, \\ 0, & \text{if } t \neq r \end{cases}$$

The additive identity and multiplicative identity of $L(R)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

In our work, we will need the following definitions and Propositions [refer [37], [29], [30], [43], [44]].

A sequence $X = (X_n)$ of fuzzy numbers is said to be convergent to the fuzzy number X_0 if for every $\epsilon > 0$ there exists a positive integer n_0 such that $\bar{d}(X_n,\ X_0) < \epsilon$ for $n > n_0$ The set of all convergent sequences and null sequences of fuzzy numbers are denoted by $c(F)$ and $c_0(F)$ respectively.

A sequence $X = (X_n)$ of fuzzy numbers is said to be a Cauchy sequence if for every $\epsilon > 0$ there exists a positive integer n_0 such that $\bar{d}(X_n,\ X_m) < \epsilon$ for $n,\ m > n_0$.

A sequence $X = (X_n)$ of fuzzy real numbers is said to be bounded if the set $\{X_n :\ n \in N\}$ is bounded. The space of bounded sequences of fuzzy numbers is denoted by $\ell_\infty(F)$.

A sequence space λ of sequences fuzzy numbers is said to be solid (or normal) if $(Y_k) \in \lambda$ whenever $\bar{d}(Y_k,\ \bar{0}) \leq \bar{d}(X_k,\ \bar{0})\ \forall k \in N$ for some $(X_k) \in \lambda$.

A sequence space λ of sequences fuzzy numbers is said to be symmetric if $(X_{\pi(k)}) \in \lambda$ whenever $(X_k) \in \lambda$, where π is permutation of N.

A sequence space λ of sequences fuzzy numbers is said to be convergence free if $(X_k) \in \lambda$ when $(Y_k) \in \lambda$ and $Y_k = \bar{0}$ implies $X_k = \bar{0}$.

Proposition 1. *[43] Let $X,\ Y,\ Z,\ W \in L(R)$ and $k \in R$. Then,*

(i) $(L(R),\ \bar{d})$ *is a complete metric space [29];*

(ii) $\bar{d}(kX,\ kY) = |k|\bar{d}(X, Y)$;

(iii) $\bar{d}(X + Y, W + Y) = \bar{d}(X, W)$;

(iv) $\bar{d}(X + Y, W + Z) \leq \bar{d}(X, W) + \bar{d}(Y, Z)$;

(v) $|\bar{d}(X, \bar{0}) - \bar{d}(Y, \bar{0})| \leq \bar{d}(X, Y) \leq \bar{d}(X, \bar{0}) + \bar{d}(Y, \bar{0})$;

Proposition 2. *[44] Let $X,\ Y \in L(R)$ and $k \in N$. Then*

(i) $\bar{d}(XY,\ \bar{0}) \leq \bar{d}(X,\ \bar{0})\bar{d}(Y,\ \bar{0})$;

(ii) If $X_k \to X$ as $k \to \infty$, then $\bar{d}(X_k,\ \bar{0}) \to \bar{d}(X,\ \bar{0})$ as $k \to \infty$;

Proposition 3. *[44] If $\bar{d}$ is a translation invariant metric on $L(R)$ then*

(i) $\bar{d}(X + Y,\ \bar{0}) \leq \bar{d}(X,\ \bar{0}) + \bar{d}(Y,\ \bar{0})$;

(ii) If $\bar{d}(\lambda X,\ \bar{0}) \leq |\lambda|\bar{d}(X,\ \bar{0}),\ |\lambda| > 1$;

The following inequalities (see [27]) will be used throughout the paper.

Proposition 4. *Let (p_k) be a bounded sequence of strictly positive real numbers with $0 < p_k \le \sup p_k = H$, D=max$\Big(1, 2^{H-1}\Big)$ and $T = \max\Big(1,\ H\Big)$. Then*

1. $|a_k + b_k|^{p_k} \le D\Big\{|a_k|^{p_k} + |b_k|^{p_k}\Big\}$;
2. $|\lambda|^{p_k} \le \max(1, [\lambda]^H)$;

2. The new class $F(L(R), M, p, s)$

Let F be a normal sequence space of real numbers with paranorm g_F which satisfies the following properties:

(i) g_F is a monotone paranorm;

(ii) co-ordinatewise convergence implies convergence in paranorm g_F, which implies that for each $(X^n) = (X^n_k) \in F,\ n,\ k \in N$,

$$X^n_k \to 0 \text{ as } n \to \infty \text{ (for each } k) \Rightarrow g_F(X^n) \to 0 \text{ as } n \to \infty \tag{1}$$

Let M be a Orlicz function. We now define the new generalized class of fuzzy composite Orlicz sequence space as follows:

$$F(L(R), M, p, s) = \Big\{X = (X_k) : X_k \in L(R),$$
$$\Big(k^{-s}\Big[M\Big(\frac{\hat{d}(X_k, \bar{0})}{\rho}\Big)\Big]^{p_k}\Big) \in F,\ \text{for some } \rho > 0\Big\} \tag{2}$$

where $s \ge 0$ and $\{p_k\}$ is a bounded sequence of strictly +ve real numbers with $\inf p_k > 0$.

This generalized class gives rise the different classes of fuzzy composite Orlicz sequence spaces as follows:

$$F(L(R), M^r, p, s) = \Big\{X = (X_k) : X_k \in L(R),$$
$$\Big(k^{-s}\Big[M^r\Big(\frac{\hat{d}(X_k, \bar{0})}{\rho}\Big)\Big]^{p_k}\Big) \in F, \text{for some } \rho > 0\Big\}$$

where r is any positive integer.

$$F(L(R), M, s) = \Big\{X = (X_k) : X_k \in L(R),$$

$$\Big(k^{-s}\Big[M\Big(\frac{\hat{d}(X_k,\bar{0})}{\rho}\Big)\Big]\Big) \in F, \text{ for some } \rho > 0\Big\}$$

$$F(L(R),p,s) = \Big\{X = (X_k) : X_k \in L(R),$$
$$\Big(k^{-s}\Big[\frac{\hat{d}(X_k,\bar{0})}{\rho}\Big]^{p_k}\Big) \in F, \text{ for some } \rho > 0\Big\}$$

and so on.

We define a metric on $F(L(R),M,p,s)$ as follows:
For $X = (X_k),\ Y = (Y_k) \in F(L(R),M,p,s)$,

$$d(X,Y)$$
$$= \inf\Big\{\rho^{p_k/T} > 0 : \ \Big[g_F\Big(k^{-s}\Big[M\Big(\frac{\bar{d}(X_k,Y_k)}{\rho}\Big)\Big]^{p_k}\Big)\Big]^{1/T} \le 1\Big\} \tag{3}$$

where $T = \max(1,\ H),\ H = \sup_k p_k < \infty$ and $\inf p_k > 0$.

It is shown that this generalized class of sequences of fuzzy real numbers gives rise to many well known fuzzy sequence spaces on specifying the sequence space F, the Orlicz function M, the bounded sequence $\{p_k\}$ of positive real numbers and $s \ge 0$ as follows:

(i) If we take $F = c,\ \ell_p;\ s = 1;\ p_k = 1,$ for each $k \in N;\ M(x) = x,$ for all $x \in [0,\infty),\ F(L(R),M,p,s)$ reduces to the spaces of Nanda[32];

(ii) If we take $F = w(p),\ c_0(p);\ s = 1;\ M(x) = x,$ for all $x \in [0,\infty),\ F(L(R),M,p,s)$ reduces to the spaces of Mursaleen *et al.*[30];

(iii) If we take $F = \ell(p)$[30]; $s = 1;\ M(x) = x,$ for all $x \in [0,\infty),\ F(L(R),M,p,s)$ reduces to the spaces of Nurray and Savas [34];

(iv) If we take $F = \ell_\infty,\ c,\ c_0,\ \ell_p;\ M(x) = x,$ for all $x \in [0,\infty);\ s = 1;\ p_k = 1$ for each, $k \in N, F(L(R),M,p,s)$ gives rise to the space of Talo *et al.* [43].

3. Main results

Theorem 1. *$F(L(R),M,p,s)$ is a metric space under the metric d defined by (3).*

Proof. Let (X^i) be a Cauchy Sequence in $F(L(R), M, p, s)$. Then $d(X^i,\ X^j) \to 0$ as $i,\ j \to \infty$. For any given $\epsilon > 0$, let r and x_0 be such that $\frac{\epsilon}{rx_0} > 0$ and $M(\frac{rx_0}{2}) \geq \sup_{k\geq 1} k^{s/p_k}$.
Now, $d(X^i,\ X^j) \to 0$ as $i,\ j \to \infty$ implies that there exist $N_0 \in N$ such that

$$d(X^i,\ X^j) < \frac{\epsilon}{rx_0} \text{ for all } i, j \geq N_0$$

Then we have for $i, j \geq N_0$,

$$\inf\Big\{\rho^{p_k/T} > 0 :\ \Big[g_F\Big(k^{-s}\Big[M\Big(\frac{\bar{d}(X_k^i, X_k^j)}{\rho}\Big)\Big]^{p_k}\Big)\Big]^{1/T} \leq 1,\ k = 1, 2, \ldots\Big\} < \frac{\epsilon}{rx_0}$$

Since F is a $K-$space and $p_k \geq 0$,

$$k^{-s}\Big[M\Big(\frac{\bar{d}(X_k^i, X_k^j)}{\rho}\Big)\Big]^{p_k} \leq 1$$

for each k and $i, j \geq N_0$.
Now, since $d(X^i,\ X^j)$ is positive we can replace ρ by it and write

$$k^{-s}\Big[M\Big(\frac{\bar{d}(X_k^i, X_k^j)}{d(X^i,\ X^j)}\Big)\Big]^{p_k} \leq 1$$

Therefore,

$$M\Big(\frac{\bar{d}(X_k^i, X_k^j)}{d(X^i,\ X^j)}\Big) \leq k^{s/p_k} \leq M(\frac{rx_0}{2})$$

Thus we get

$$\bar{d}(X_k^i, X_k^j) < \frac{\epsilon}{rx_0}\frac{rx_0}{2} = \frac{\epsilon}{2}$$

for each k and $i, j \geq N_0$.
Therefore (X_k^i) becomes a Cauchy sequence in $L(R)$. Since $(L(R), \bar{d})$ is a complete metric space there exist $X = (X_k) \in L(R)$ such that $X_k^i \to X_k$ as $i \to \infty$ for each k in $L(R)$.
Since M is continuous it shows that

$$M\Big(\frac{\bar{d}(X_k, X_k^j)}{\rho}\Big) \to 0 \text{ as } i \to \infty$$

for each k and for some $\rho > 0$. Consequently,

$$k^{-s}\Big[M\Big(\frac{\bar{d}(X_k, X_k^j)}{\rho}\Big)\Big]^{p_k} \to 0 \text{ as } i \to \infty$$

for each k and for some $\rho > 0$.
Let

$$\alpha_k^j = k^{-s}\Big[M\Big(\frac{\bar{d}(X_k, X_k^j)}{\rho}\Big)\Big]^{p_k}$$

Then since M is non-decreasing, by suitable choice of δ (depending on j and k)

$$\alpha_k^j < \delta k^{-s}\Big[M\Big(\frac{\bar{d}(X_k^j, \bar{0})}{\rho}\Big)\Big]^{p_k}$$

where $0 < \delta < 1$. Since F is normal, it follows that $(\alpha^i) \in F$ for each i. Also $\alpha_k^i \to 0$ as $i \to \infty$ implies that $g_F(\alpha^i) \to 0$ as $i \to \infty$. Hence $X^i \overset{d}{\to} X$ as $i \to \infty$ in $F(L(R), M, p, s)$. Again

$$\begin{aligned} & k^{-s}\Big[M\Big(\frac{\bar{d}(X_k,\ \bar{0}}{\rho})\Big)\Big]^{p_k} \\ \le\ & Dk^{-s}\Big[M\Big(\frac{\bar{d}(X_k^i,\ \bar{0}}{\rho})\Big)\Big]^{p_k} + D\alpha_k^i \text{ (where } D = \max\{1, 2^{H-1}\}) \\ \le\ & D(1+\delta)k^{-s}\Big[M\Big(\frac{\bar{d}(X_k^i,\ \bar{0}}{\rho})\Big)\Big]^{p_k} \end{aligned}$$

Since $(X^i) \in F(L(R), M, p, s)$ and F is a normal space, it seems that $X = (X_k) \in F(L(R), M, p, s)$. hence it is complete. □

Theorem 2. *$F(L(R), M, p, s)$ is a K-space if F is a K-space.*

Proof. Let us define a mapping

$$P_n : F(L(R), M, p, s) \to L(R)$$

by $P_n(X) = X_n$, $\forall n \in N$. To show P_n is continuous.
Let (X^m) be a sequence in $F(L(R), M, p, s)$ such that $X^m \overset{d}{\to} 0$ as $m \to \infty$. Then for some suitable choice of $\rho > 0$,

$$\Big[g_F\Big(k^{-s}\Big[M\Big(\frac{\bar{d}(X_k^m, \bar{0})}{\rho}\Big)\Big]^{p_k}\Big)\Big]^{1/T} \to 0 \text{ as } m \to \infty$$

Since F is a K-space, this implies that for each k and as m tending to ∞,

$$k^{-s}\Big[M\Big(\frac{\bar{d}(X_k^m, \bar{0})}{\rho}\Big)\Big]^{p_k} \to 0$$

for some $\rho > 0$. Since M is an Orlicz function, it follows that

$$\bar{d}(X_k^m, \bar{0}) \to 0 \ \text{ as } m \to \infty.$$

Consequently, $X^m \to \bar{0}$ in $L(R)$. Hence the proof. □

4. Inclusion Relations

Theorem 3. *Let M_1 and M_2 be two Orlicz functions. Then*

$$F(L(R), M_1, p, s) \cap F(L(R), M_2, p, s) \subseteq F(L(R), M_1 + M_2, p, s)$$

if F is a normal sequence space.

Proof. Let $X = (X_k) \in F(L(R), M_1, p, s) \cap F(L(R, M_2, p, s)$.
Then there exists ρ_1 and ρ_2 such that

$$\left(k^{-s}\Big[M_1\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho_1}\Big)\Big]^{p_k}\right) \in F$$

and

$$\left(k^{-s}\Big[M_2\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho_2}\Big)\Big]^{p_k}\right) \in F$$

Let $\rho = \max(\rho_1, \rho_2)$. Then

$$\begin{aligned} & k^{-s}\Big[(M_1 + M_2)\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big)\Big]^{p_k} \\ \leq\ & k^{-s}D\Big\{\Big[M_1\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho_1}\Big)\Big]^{p_k} + \Big[M_2\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho_2}\Big)\Big]^{p_k}\Big\} \in F \end{aligned}$$

where $D = \max(1, 2^{H-1})$. Since F is normal sequence space the proof follows immediately. □

Theorem 4. *Let M be an Orlicz function. Then*

$$c_0(L(R), M, p, s) \subset c(L(R), M, p, s) \subset \ell_\infty(L(R), M, p, s)$$

Proof. Let $X = (X_k) \in c(L(R), M, p, s)$ and $X = (X_k) \to L = (\bar{L})$. Then for some $\rho > 0$, we have

$$\begin{aligned} & k^{-s}\Big[M\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big)\Big]^{p_k} \\ \leq\ & k^{-s}\Big[M\Big(\frac{\bar{d}(X_k, \bar{L}) + \bar{d}(\bar{L}, \bar{0})}{\rho}\Big)\Big]^{p_k} \\ & \quad \text{(as } M \text{ is non-decreasing and } d \text{ is translation invariant)} \\ \leq\ & k^{-s}D\Big[M\Big(\frac{\bar{d}(X_k, \bar{L})}{\rho}\Big)\Big]^{p_k} + k^{-s}D\Big[M\Big(\frac{\bar{d}(\bar{L}, \bar{0})}{\rho}\Big)\Big]^{p_k} \\ \leq\ & k^{-s}D\Big[M\Big(\frac{\bar{d}(X_k, \bar{L})}{\rho}\Big)\Big]^{p_k} + k^{-s}D\max\Big\{1, \Big[M\Big(\frac{\bar{d}(\bar{L}, \bar{0})}{\rho}\Big)\Big]^{H}\Big\} \end{aligned}$$

Thus $X = (X_k) \in \ell_\infty(L(R), M, p, s)$. The first inclusion is trivially true. □

Theorem 5. *Let M_1 and M_2 be Orlicz functions satisfying Δ_2-condition. Then the following inclusion relations hold:*

$$F(L(R), M_1, p, s) \subseteq F(L(R), M_2 \circ M_1, p, s) \text{ if } s > 1$$

Proof. Let $X = (X_k) \in F(L(R), M_1, p, s)$. Since M_2 is continuous from the right at 0, $\exists\, \eta > 0$ such that $0 < \eta < 1$ and for any arbitrary $\epsilon > 0$, $M_2(t) < \epsilon$ whenever $0 \leq t \leq \eta$.

Let us define the sets

$$N_1 = \Big\{k \in N : \; M_1\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big) \leq \eta\Big\}$$
$$N_2 = \Big\{k \in N : \; M_1\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big) > \eta\Big\}$$

for some $\rho > 0$.

If $k \in N_2$,

$$\begin{aligned} M_1\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big) &< \frac{1}{\eta} M_1\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big) \\ &< 1 + \Big[\frac{1}{\eta} M_1\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big)\Big] \end{aligned}$$

Since M_2 is non-decreasing and convex it follows that

$$\begin{aligned} M_2\Big[M_1\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big)\Big] &< M_2\Big[1 + \frac{1}{\eta} M_1\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big)\Big] \\ &< \frac{1}{2} M_2(2) + \frac{1}{2} M_2\Big[2 \frac{1}{\eta} M_1\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big)\Big] \end{aligned}$$

Again since M_2 satisfies Δ_2-condition, we have

$$\begin{aligned} M_2\Big[M_1\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big)\Big] &< \frac{1}{2} K \Big[\frac{1}{\eta} M_1\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big)\Big] M_2(2) \\ &+ \frac{1}{2} K \Big[\frac{1}{\eta} M_1\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big)\Big] M_2(2) \\ &= K \eta^{-1} M_2(2) M_1\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big) \end{aligned}$$

So,

$$k^{-s}\Big[M_2\Big(M_1\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big)\Big)\Big]^{p_k} \leq k^{-s} D_1 \Big[M_1\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big)\Big]^{p_k} \tag{4}$$

where $D_1 = \max\{1, [K\eta^{-1}M_2(2)]^H\}$.

For, $k \in N_1$,

$$M_1\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big) \leq \eta \Rightarrow M_2\Big[M_1\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big)\Big] < \epsilon$$

and therefore,

$$k^{-s}\Big[M_2\Big[M_1\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big)\Big]\Big]^{p_k} < k^{-s}[\epsilon]^H \tag{5}$$

Hence from (4) and (5) we have

$$\begin{aligned} & k^{-s}\Big[M_2\Big[M_1\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big)\Big]\Big]^{p_k} \\ \leq \; & k^{-s}[\epsilon]^H + k^{-s}D_1\Big[M_1\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big)\Big]^{p_k} \in F \end{aligned}$$

for all k. Then by the normality of F the proof follows. □

Theorem 6. *Let M be a Orlicz function. Then*

(i) If $0 < \inf p_k \leq p_k < 1$ then $c_0(L(R), M, s) \subset c_0(L(R), M, p, s)$;

(ii) If $1 \leq p_k \leq \sup p_k < \infty$, then $c_0(L(R), M, p, s) \subset c_0(L(R), M, s)$

Proof. (i) Let $X = (X_k) \in c_0(L(R), M, s)$. Since $0 < \inf p_k \leq p_k < 1$ the proof follows from the fact that

$$\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big)\Big]^{p_k} \leq \Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big)\Big].$$

□

Proof. (ii) Let $1 \leq p_k \leq \sup p_k < \infty$ and $X = (X_k) \in c_0(L(R), M, p, s)$. Then for each $0 < \epsilon < 1$ there exists a $N \in Z^+$

$$k^{-s}\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big)\Big]^{p_k} \leq \epsilon < 1,\ \forall k \geq N$$

Since $1 \leq p_k \leq \sup p_k < \infty$ the proof follows from the fact that

$$k^{-s}\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big)\Big] \leq k^{-s}\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big)\Big]^{p_k}$$

Thus $X \in c_0(L(R), M, s)$. □

Theorem 7. *Let M be a Orlicz function satisfying Δ_2-condition and $0 < A \leq M(t)/t \leq B$ for $t > 0$, where A and B are constants. Then*

$$F(L(R), M^r, p, s) = F(L(R), p, s)$$

Proof. Considering the left part of the inequality, i.e., $A \leq M(t)/t$ we get

$$t \leq \frac{1}{A} M(t)$$

Hence we have

$$t \leq \frac{1}{A} M(t) < (1 + [\frac{1}{A}]) M(t) \tag{6}$$

Since M satisfies Δ_2-condition,

$$M(t) < M\Big[(1 + [\frac{1}{A}]) M(t)\Big] \leq L(1 + [\frac{1}{A}]) M^2(t) \tag{7}$$

for some constant $L > 0$. From (6) and (7) we get

$$t < L(1 + [\frac{1}{A}])^2 M^2(t)$$

Similarly, after r steps we get

$$t < L(1 + [\frac{1}{A}])^r M^r(t) \tag{8}$$

Let $X = (X_k) \in F(L(R), M^r, p, s)$. Then $\left(k^{-s}\Big[M^r\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big)\Big]^{p_k}\right) \in F$.
Then putting $t = \bar{d}(\frac{\bar{X_k}, \bar{0}}{\rho})$ in the inequality (8) and multiplying both sides by k^{-s} we get

$$k^{-s}\Big[\bar{d}(\frac{\bar{X_k}, \bar{0}}{\rho})\Big]^{p_k} < \Big\{L(1 + [\frac{1}{A}])^r\Big\}^H k^{-s}\Big[M^r\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big)\Big]^{p_k}$$

Since F is a normal space, it follows that $\left(k^{-s}\Big[\frac{\bar{d}(\bar{X_k}, \bar{0})}{\rho}\Big]^{p_k}\right) \in F$. Consequently, $X = (X_k) \in F(L(R), p, s)$.
Similarly, considering the inequality $M(t)/t \leq B$ we get

$$M(t) \leq Bt$$

which implies

$$M^r(t) < (1 + [B])^r t \tag{9}$$

Let $X = (X_k) \in F(L(R), p, s)$. Then

$$\Big(k^{-s}\Big[\frac{\bar{d}(\bar{X_k}, \bar{0})}{\rho}\Big]^{p_k}\Big) \in F$$

From the inequality (9) and proceeding as before we have

$$k^{-s}\Big[M^r\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big)\Big]^{p_k} < [(1+[B])^r]^H k^{-s}\Big[\frac{\bar{d}(\bar{X_k}, \bar{0})}{\rho}\Big]^{p_k} \in F$$

for each $k \in N$. Since F is normal, $X = (X_k) \in F(L(R), M^r, p, s)$. □

Theorem 8. *Let M be a Orlicz function satisfying Δ_2-condition and $M(t)/t \leq B$ for $t \geq 0$, where B is a constant and $\rho > 0$. If $r,\ n \in N$ such that $r > n$ then*

$$F(L(R), M^n, p, s) \subseteq F(L(R), M^r, p, s)$$

Proof. Let $r - n = \ell > 0$. Now,

$$M(t) \leq B(t) \leq (1 + [B])t$$

Since M is non-decreasing,

$$M^2(t) \leq (1 + [B])M(t)$$

From the inequalities 10 and 10 we get

$$M^2(t) < (1 + [B])^2(t)$$

Repeating this step r times we have

$$M^r(t) < (1 + [B])^r t$$

Therefore

$$\begin{aligned} M^r(t) &= M^{n+\ell}(t) \\ &= M^n[M^\ell(t)] \\ &< M^n[(1+[B])^\ell t] \\ &\leq K(1+[B])^\ell M^n(t) \text{ (since } M \text{ satisfies } \Delta_2\text{-condition)} \end{aligned}$$

Let $X = (X_k) \in F(L(R), M^n, p, s)$. Then

$$\Big(k^{-s}\Big[M^n\Big(\frac{\bar{d}(X_k, \bar{0})}{\rho}\Big)\Big]^{p_k}\Big) \in F$$

Replacing t by $\bar{d}(X_k,\bar{0})/\rho$ on both sides of 10 we get

$$M^r\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big) < K(1+[B])^{\ell}M^n\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big)$$

This implies

$$k^{-s}\Big[M^r\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big)\Big]^{p_k} < \{K(1+[B])^{\ell}\}^H k^{-s}\Big[M^n\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big)\Big]^{p_k} \in F$$

Since F is normal it follows that $X=(X_k)\in F(L(R),M^r,p,s)$. Hence the proof. □

5. Further Properties

Theorem 9. *The space $F(L(R),M,p,s)$ is solid (normal) in general.*

Proof. Let $X=(X_k)\in F(L(R),M,p,s)$ and $Y=(Y_k)$ be such that

$$\bar{d}(Y_k,\bar{0})\le \bar{d}(X_k,\bar{0})$$

Since M is non-decreasing, we have

$$k^{-s}\Big[M\Big(\frac{\bar{d}(Y_k,\bar{0})}{\rho}\Big)\Big]^{p_k} \le k^{-s}\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big)\Big]^{p_k} \in F$$

for some $\rho>0$. Hence $Y=(Y_k)\in F(L(R),M,p,s)$, since F is normal. □

Theorem 10. *The space $F(L(R),M,p,s)$ is symmetric in general.*

Proof. Let $X=(X_k)\in F(L(R),M,p,s)$ and $Y=(Y_{m_k})$ be an arrangement of the sequence (X_k) such that $X_k=Y_{m_k}$ for each $k\in N$. Then

$$\Big(k^{-s}\Big[M\Big(\frac{\bar{d}(Y_{m_k},\bar{0})}{\rho}\Big)\Big]^{p_k}\Big) = \Big(k^{-s}\Big[M\Big(\frac{\bar{d}(X_k,\bar{0})}{\rho}\Big)\Big]^{p_k}\Big) \in F$$

Hence these spaces are symmetric in general.

However, it is easy to construct an example to see that the space $F(L(R),M,p,s)$ is not convergence free in general. Hence, we have the following result. □

Theorem 11. *The space $F(L(R),M,p,s)$ is not convergence free in general.*

Proof. Consider $F = \ell_\infty,\ s = 0,\ p_k = 1$, for each $k \in N$, $M(x) = x^2$, for all $x \in [0,\ \infty)$ and $(X_k) \in F(L(R), M, p, s)$ defined as follows:
For k is even,

$$X_k(t) = \begin{cases} 0, & \text{if } t \leq 10 \text{ or } t \geq 50 \\ \frac{t-10}{5}, & \text{if } 10 < t < 25 \\ 1, & \text{if } 25 \leq t \leq 35 \\ \frac{50-t}{5}, & \text{if } 35 < t < 50 \end{cases}$$

and for k is odd, $X_k = \bar{0}$.

Let us define a sequence (Y_k) as follows:
For k odd, $Y_k = \bar{0}$ and
for k even,

$$Y_k(t) = \begin{cases} 0, & \text{if } t \leq 35 \\ \frac{t-35}{5}, & \text{if } 35 < t < 50 \\ 1, & \text{if } t \geq 50 \end{cases}$$

Then $X_k = \bar{0}$ implies $Y_k = \bar{0}$ but $Y = (Y_k) \notin F(L(R), M, p, s)$. □

6. Conclusions:

1. This new generalized class of Orlicz fuzzy metric space unifies many fuzzy basic sequence spaces introduced by earlier authors;
2. The inclusion relations between various sets of sequences signify that some of the fuzzy sequence sets become identical or can be embedded to some other fuzzy sequence sets under certain conditions.
3. Some examples of fuzzy Orlicz sequence spaces are given which are exceptional in respect to some properties.

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