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SIMPLE APPROACH TO GEGENBAUER POLYNOMIALS

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Abstract: Gegenbauer polynomials are obtained through well known linear algebra methods based on Sturm-Liouville theory. A matrix corresponding to the Gegenbauer differential operator is found and its eigenvalues are obtained. The elements of the eigenvectors obtained corresponds to the Gegenbauer polynomials.

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1. Introduction

Gegenbauer polynomials are solutions of an ordinary differential equation (ODE) which is an hypergeometric equation. In general an hypergeometric equation may be writen as:

$$s(x)F''(x) + t(x)F'(x) + \lambda F(x) = 0 \tag{1}$$

where $F(x)$ is a real function of a real variable $F : U \longrightarrow \mathbb{R}$, where $U \subset \mathbb{R}$ is an open subset of the real line, and $\lambda \in \mathbb{R}$ a corresponding eigenvalue, and the functions $s(x)$ and $t(x)$ are real plynomials of at most second order and first order, respectively.

The Sturm-Liouville Theory is covered in most advanced courses. In this context an eigenvalue equation sometimes takes the more general self-adjoint form: $\mathscr{L}u(x) + \lambda w(x)u(x) = 0$, where $\mathscr{L}$ is a differential operator; $\mathscr{L}u(x) = \frac{d}{dx}\left[p(x)\frac{du(x)}{dx}\right] + q(x)u(x)$, λ an eigenvalue, and $w(x)$ is known as a weight

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or density function. The analysis of this equation and its solutions is called Sturm-Liouville theory. Specific forms of $p(x)$, $q(x)$, λ and $w(x)$, are given for Legendre, Laguerre, Hermite and other well-known equations in the given references. There, it is also shown the close analogy of this theory with linear algebra concepts. For example, functions here take the role of vectors there, and linear operators here take that of matrices there. Finally, the diagonalization of a real symmetric matrix corresponds to the solution of an ordinary differential equation, defined by a self-adjoint operator $\mathscr{L}$, in terms of its eigenfunctions which are the "continuous" analog of the eigenvectors [4].

There are different cases obtained depending on the kind of the $s(x)$ function in eq.(1). When $s(x)$ is a constant, eq.(1) takes the form $F''(x) - 2\alpha x F'(x) + \lambda F(x) = 0$, and if $\alpha = 1$ one obtains the Hermite Polynomials. When $s(x)$ is a polynomial of first degree, eq.(1) takes the form $xF''(x) + (-\alpha x + \beta + 1)F'(x) + \lambda F(x) = 0$, and when $\alpha = 1$ and $\beta = 0$ one obtains the Laguerre Polynomials. There are three different cases when $s(x)$ is a polynomial of second degree. When the second degree polynomial has two different real roots, eq.(1) takes the form $(1 - x^2)F''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]\, F'(x) + \lambda F(x) = 0$ and that is the Jacobi equation, for different values of α and β one obtains particular cases of polynomials; Gegenbauer polynomials if $\alpha = \beta$; Chebyshev I and II if $\alpha = \beta = \pm\frac{1}{2}$; Legrendre polynomials if $\alpha = \beta = 0$. When the second degree polynomial has one double real root, eq.(1) takes the form $x^2F''(x) + [(\alpha + 2)x + \beta]\, F'(x) + \lambda F(x) = 0$, and when $\alpha = -1$ and $\beta = 0$ one obtains the Bessel Polynomials. At last, when the second degree polynomial has two complex roots, eq.(1) takes the form $(1+x)^2F''(x) + (2\beta x + \alpha)F'(x) + \lambda F(x) = 0$, and that is the Romanovski equation. These results are sumarized in Table 1 [2].

2. Gegenbauer Polynomials

The Gegenbauer Polynomials, denoted by $C_n^\lambda(x)$, are particular solutions of the Gegenbauer differential equation and at the same time are a special case of Jacobi polynomials. The Gegenbauer differential equation is given by:

$$(1 - x^2)y'' - (2\lambda - 1)xy' + n(n + 2\lambda)y = 0 \tag{12}$$

Gegenbauer Polynomials are called sometimes *Ultraspherical Polynomials* [3]. The first few polynomials are given by:

$$C_0^\lambda(x) = 1 \tag{13}$$

$s(x)$	Canonical form and weight function	Example
Constant	$F''(x) - 2\alpha x F'(x) + \lambda F(x) = 0 \quad (2)$ $w(x) = e^{-\alpha x^2} \quad (3)$	When $\alpha = 1$ one obtains the Hermite equation, then $F(x) = H(x)$; this produce the Hermite polynomials, denoted $\{H_n^{(\alpha)}\}$.
First degree	$xF''(x) + (-\alpha x + \beta + 1)F'(x) + \lambda F(x) = 0 \quad (4)$ $w(x) = x^{\beta} e^{-\alpha x} \quad (5)$	When $\alpha = 1$ and $\beta = 0$ one obtains the Laguerre equation, then $F(x) = L(x)$; this produce the Laguerre polynomials, denoted $\{L_n^{(\alpha,\beta)}\}$.
Second degree: with two different real roots	$(1 - x^2)F''(x) + [\beta - \alpha - (\alpha + \beta + 2)x]\, F'(x) + \lambda F(x) = 0 \quad (6)$ $w^{(\alpha,\beta)}(x) = (1 - x)^{\alpha}(1 + x)^{\beta} \quad (7)$	Eq.(6) is the Jacobi equation considering $F(x) = P(x)$, and for each pair (α, β) one obtains the Jacobi polynomials, denoted $\{P_n^{(\alpha,\beta)}\}$. Particular cases: Gegenbauer polynomials if $\alpha = \beta$, Chebyshev I and II if $\alpha = \beta = \pm\frac{1}{2}$, Legrendre polynomials if $\alpha = \beta = 0$.
Second degree: with one double real root	$x^2 F''(x) + [(\alpha + 2)x + \beta]\, F'(x) + \lambda F(x) = 0 \quad (8)$ $w^{(\alpha,\beta)}(x) = x^{\alpha} exp(-\frac{\beta}{x}) \quad (9)$	When $\alpha = -1$ and $\beta = 0$ one obtains the Bessel equation, then $F(x) = B(x)$; this produce the Bessel polynomials, denoted $\{B_n^{(\alpha,\beta)}\}$.
Second degree: with two complex roots	$(1 + x)^2 F''(x) + (2\beta x + \alpha)F'(x) + \lambda F(x) = 0 \quad (10)$ $w^{(\alpha,\beta)}(x) = (1 + x^2)^{\beta - 1} exp(-\alpha cot^{-1} x) \quad (11)$	Eq.(10) is the Romanovski equation considering $F(x) = R(x)$, then one obtains the Romanovski polynomials, denoted $\{R_n^{(\alpha,\beta)}\}$.

Table 1: Polynomials obtained depending on the $s(x)$ function of eq.(1)

$$C_1^\lambda(x) = 2\lambda x \tag{14}$$

$$C_2^\lambda(x) = -\lambda + 2\lambda(1+\lambda)x^2 \tag{15}$$

$$C_3^\lambda(x) = -2\lambda(1+\lambda)x + \frac{4}{3}\lambda(1+\lambda)(2+\lambda)x^3 \tag{16}$$

Note that if $\lambda = 1/2$ the equation (12) reduces to the Legendre equation, and the Gegenbauer Polynomials reduce to the Legendre Polynomials. And, if $\lambda = 1$ the equation (12) reduces to the Chebyshev differential equation, and the Gegenbauer Polynomials reduce to the Chebyshev Polynomials of the second kind. In the next section, the Gegenbauer polynomials are obtained not by solving the Gegenbauer differential equation (12), but using a linear algebra method as shown in reference [1] and [4].

3. Gegenbauer polynomials through matrix algebra

Representing the general algebraic polynomial of degree n, as:

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n \tag{17}$$

with $a_0, a_1, a_2, \ldots, a_n \in \Re$, by the vector:

$$A_n = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} \tag{18}$$

Also the first derivative of (17) can be represented as a vector, considering the polynomial:

$$\frac{d}{dx}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n\right) = \\ a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} \tag{19}$$

$$\frac{dA_n}{dx} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ na_n \\ 0 \end{bmatrix} \tag{20}$$

Now taking the second derivative of (17):

$$\begin{aligned} &\frac{d^2}{dx^2}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n\right) = \\ &\quad 2a_2 + 6a_3 x + 12a_4 x^2 + \ldots + n(n-1)a_n x^{n-2} \end{aligned} \tag{21}$$

and,

$$\frac{d^2 A_n}{dx^2} = \begin{bmatrix} 2a_2 \\ 6a_3 \\ \vdots \\ n(n-1)a_n \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

Using (20), equation (19) may be written as:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ na_n \\ 0 \end{bmatrix} \tag{23}$$

Therefore the first derivative operator of A_n may be written as:

$$\frac{d}{dx} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{24}$$

Doing the same for equation (21):

$$\begin{bmatrix} 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n(n-1) \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 2a_2 \\ 6a_3 \\ \vdots \\ n(n-1)a_n \\ 0 \\ 0 \end{bmatrix} \tag{25}$$

Therefore the second derivative operator of A_n may be written as:

$$\frac{d^2}{dx^2} \to \begin{bmatrix} 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n(n-1) \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{26}$$

The Gegenbauer differential operator is given by:

$$(1-x^2)\frac{d^2}{dx^2} - (2\lambda+1)x\frac{d}{dx} \tag{27}$$

substituing (19) and (21) into (27):

$$(1-x^2)[2a_2 + 6a_3x + 12a_4x^2 + \ldots + n(n-1)a_nx^{n-2}] +$$

$$-(2\lambda+1)x[a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1}] \tag{28}$$

$$= 2a_2 + [6a_3 - (2\lambda+1)a_1]x + [12a_4 - 4(\lambda+1)a_2]x^2 +$$

$$+[20a_5 - 3(2\lambda+3)a_3]x^3 + \cdots + (-n^2 - 2\lambda n)a_nx^n \tag{29}$$

Which may be written as:

$$\begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -(2\lambda+1) & 0 & 6 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -4(\lambda+1) & 0 & 12 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -3(2\lambda+3) & 0 & 20 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -n^2-2\lambda n \end{bmatrix}$$

$$\cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 2a_2 \\ 6a_3-(2\lambda+1)a_1 \\ 12a_4-4(\lambda+1)a_2 \\ 20a_5-3(2\lambda+3)a_3 \\ \vdots \\ (-n^2-2\lambda n)a_n \end{bmatrix} \quad (30)$$

For simplicity, we take the matrix 4x4 of the Gegenbauer differential operator:

$$(1-x^2)\frac{d^2}{dx^2}-(2\lambda+1)x\frac{d}{dx} \rightarrow$$

$$\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & -(2\lambda+1) & 0 & 6 \\ 0 & 0 & -4(\lambda+1) & 0 \\ 0 & 0 & 0 & -3(2\lambda+3) \end{bmatrix} \quad (31)$$

The eigenvalues of a matriz M are the values that satisfy the equation $Det(M-\lambda' I)=0$. However since the matrix (31) is a triangular matrix, the eigenvalues λ'_i of this matrix are the elements of the diagonal, namely: $\lambda'_1=0$, $\lambda'_2=-(2\lambda+1)$, $\lambda'_3=-4(\lambda+1)$, $\lambda'_4=-3(2\lambda+3)$. The corresponding eigenvectors are the solutions of the equation $(M-\lambda'_i I)\cdot v=0$, where the eigenvector $v=[a_0,a_1,a_2,a_3]^T$.

$$\begin{bmatrix} 0-\lambda'_i & 0 & 2 & 0 \\ 0 & -(2\lambda+1)-\lambda'_i & 0 & 6 \\ 0 & 0 & -4(\lambda+1)-\lambda'_i & 0 \\ 0 & 0 & 0 & -3(2\lambda+3)-\lambda'_i \end{bmatrix}$$

$$\cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{32}$$

Substituing the eigenvalue λ'_1 in the equation (32) we obtain the eigenvector v_1:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{33}$$

The elements of this eigenvector corresponds to the first Gegenbauer Polynomial, $C_0^\lambda(x) = 1$.

Substituing the eigenvalue λ'_2 in the equation (32) we obtain the eigenvector v_2:

$$v_2 = \begin{bmatrix} 0 \\ 2\lambda \\ 0 \\ 0 \end{bmatrix} \tag{34}$$

The elements of this eigenvector corresponds to the second Gegenabuer Polynomial, $C_1^\lambda(x) = 2\lambda x$.

Substituing the eigenvalue λ'_3 in the equation (32) we obtain the eigenvector v_3:

$$v_3 = \begin{bmatrix} -\lambda \\ 2\lambda(\lambda+1) \\ 0 \\ 0 \end{bmatrix} \tag{35}$$

The elements of this eigenvector corresponds to the third Gegenabuer Polynomial, $C_2^\lambda(x) = -\lambda + 2\lambda(1+\lambda)x^2$.

Substituing the eigenvalue λ'_4 in the equation (32) we obtain the eigenvector v_4:

$$v_3 = \begin{bmatrix} 0 \\ -2\lambda(1+\lambda) \\ 0 \\ \frac{4}{3}\lambda(1+\lambda)(2+\lambda) \end{bmatrix} \quad (36)$$

The elements of this eigenvector corresponds to the fourth Gegenabuer Polynomial, $C_3^{\lambda}(x) = -2\lambda(1+\lambda)x + \frac{4}{3}\lambda(1+\lambda)(2+\lambda)x^3$.

4. Conclusion

Gegenbauer polynomials are obtained using basic linear algebra concepts such the eigenvalue and eigenvector of a matrix. Once the corresponding matrix of the Gegenbauer differential operator is obtained, the eigenvalues of this matrix are found and the elements of its eigenvectors correspond to the coefficients of Gegenbauer Polynomials.

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