International Journal of Pure and Applied Mathematics
Volume 119 No. 1 2018, 145-151
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v119i1.12

ijpam.eu

LEVEL GRAPHS OF A $S-$VALUED GRAPH

S. Jeyalakshmi[1], M. Chandramouleeswaran[2 §]

[1]Devanga Arts College
Aruppukottai, 626101, INDIA
[2]Saiva Bhanu Kshatriya College
Aruppukottai, 626101, INDIA

Abstract: In this paper, we introduce the notion of level graphs of a $S-$valued graph.

AMS Subject Classification: 05C76, 16Y60, 05C25
Key Words: graph, semiring, semiring - valued graphs

1. Introduction

In [3], we have introduced the concept of semiring-valued graphs ($S-$valued graphs). In a $S-$valued graph, the vertices are assigned values from a semiring. Using the canonical pre-order existing in a semiring we assigned the weights for the edges. In this way, we generalized both the theory of crisp graphs and fuzzy graphs. For each value $a \in S$, we can define a subgraph of the underlying crisp graph G called the level subgraphs of G^S. This paper discusses the notion of level subgraphs on $S-$valued graphs.

2. Preliminaries

Definition 2.1. [2] A semiring $(S, +, \cdot)$ is an algebraic system with a non-empty set S together with two binary operations $+$ and $\cdot$ such that

1. $(S, +, 0)$ is a monoid.

Received: December 6, 2017
Revised: December 13, 2017
Published: June 14, 2018

§Correspondence author

2. $(S, \cdot)$ is a semigroup.

3. For all $a, b, c \in S$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.

4. $0 \cdot x = x \cdot 0 = 0 \; \forall \; x \in S$.

Definition 2.2. Let $(S, +, \cdot)$ be a semiring. $\preceq$ is said to be a Canonical Pre-order if for $a, b \in S$, $a \preceq b$ if and only if, there exists $c \in S$ such that $a + c = b$.

Definition 2.3. [3] Let $G = (V, E \subset V \times V)$ be a given graph with $V, E \neq \phi$. For any semiring $(S, +, \cdot)$, a semiring-valued graph (or a $S-$valued graph) G^S, is defined to be the graph $G^S = (V, E, \sigma, \psi)$ where $\sigma : V \to S$ and $\psi : E \to S$ is defined to be

$$\psi(x, y) = \begin{cases} \min\{\sigma(x), \sigma(y)\}, & \text{if } \sigma(x) \preceq \sigma(y) \text{ or } \sigma(y) \preceq \sigma(x), \\ 0, & \text{otherwise.} \end{cases}$$

for every unordered pair (x, y) of $E \subset V \times V$. We call σ a $S-$vertex set and ψ an $S-$edge set of S-valued graph G^S.

3. Level Subgraphs on S-Valued Graphs

In this section, we discuss the basic definitions on level subgraphs of G^S and illustrate them with several examples.

Definition 3.1. [3] Let $G^S = (V, E, \sigma, \psi)$ be an $S-$valued graph where $(S, +, \cdot)$ is a semiring with a canonical pre-order $\preceq$. For any $t \in S, \sigma^t = \{v \in V \;/\; t \preceq \sigma(v)\}$ is called the upper level vertex set and

$$\psi^t = \left\{(v_i, v_j) \in E \;/\; t \preceq \psi(v_i, v_j) \text{ where } v_i, v_j \in \sigma^t\right\}$$

is called the upper level edge set of G^S.

Theorem 3.2. *Consider the $S-$valued graph $G^S = (V, E, \sigma, \psi)$ and $a \in S$. Then $(G^S)^a$ is a subgraph of $G = (V, E)$.*

Proof. Let $G^S = (V, E, \sigma, \psi)$ be the given $S-$valued graph.
For any $a \in S$, $\sigma^a = \{v \in V \;/\; a \preceq \sigma(v)\}$ and

$$\psi^a = \{(v_i, v_j) \in E \;/\; a \preceq \psi(v_i, v_j) \; where \; v_i, v_j \in \sigma^a\}.$$

Since, $\sigma^a \subseteq V$ and $\psi^a \subseteq E$ the graph $(G^S)^a = (\sigma^a, \psi^a)$ is a crisp graph corresponding to the $S-$value a of G^S, and is a subgraph of $G = (V, E)$.

Definition 3.3. For any $a \in S$, the subgraph $(G^S)^a = (\sigma^a, \psi^a)$ of $G = (V, E)$ where $\sigma^a = \{v \in V \;/\; a \preceq \sigma(v)\}$ and

$$\psi^a = \{(v_i, v_j) \in E/a \preceq \psi(v_i, v_j), v_i, v_j \in \sigma^a\}$$

is called the upper level subgraph of the crisp graph G.

Example 3.4. Let $(S = \{0, a, b, c\}, +, \cdot)$ be a semiring with the following Cayley tables:

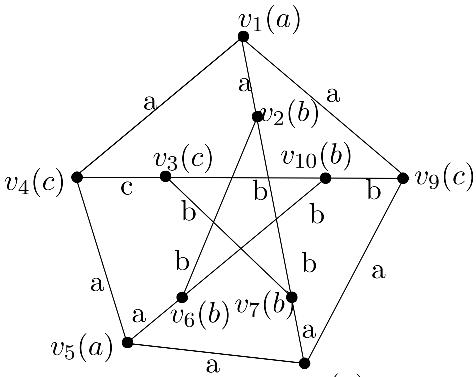
$+$	0	a	b	c
0	0	a	b	c
a	a	b	c	c
b	b	c	c	c
c	c	c	c	c

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	c	c
c	0	c	c	c

Let $\preceq$ be a canonical pre-order in S, given by

$$0 \preceq 0,\; 0 \preceq a,\; 0 \preceq b,\; 0 \preceq c,\; a \preceq a,\; a \preceq b,\; a \preceq c,\; b \preceq b,\; b \preceq c,\; c \preceq c.$$

Consider the $S-$valued graph G^S. For any $t \in S$,

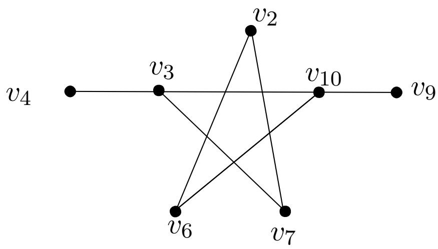


$\mathbf{G^S}$

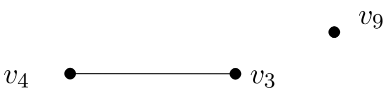
$$(G^S)^t = (V, E, \sigma^t = \{v \in V \;/\; t \preceq \sigma(v)\},\; \psi^t = \left\{(v_i, v_j) \in E \;/\; t \preceq \psi(v_i, v_j) \text{ where } v_i, v_j \in \sigma^t\right\}).$$

For $t = a$, $(G^S)^a = \langle V = \sigma^a, E = \psi^a\rangle = G$.
For $t = b$ and $t = c$ then $(G^S)^b$ and $(G^S)^c$ are as follows:

$(\mathbf{G^S})^{\mathbf{b}}$:



$(\mathbf{G^S})^{\mathbf{c}}$:



Definition 3.5. [3] Let G^S =(V, E, σ, ψ) be an $S-$valued graph where $(S, +, \cdot)$ is a semiring with a canonical pre-order $\preceq$. For any $t \in S, \sigma_t = \{v \in V \ / \ t \succeq \sigma(v)\}$ is called the lower level vertex set and

$$\psi_t = \{(v_i, v_j) \in E \ / \ t \succeq \psi(v_i, v_j) \ where \ v_i, v_j \in \sigma_t\})$$

is called the lower level edge set of G^S.

Theorem 3.6. Consider the $S-$valued graph $G^S = (V, E, \sigma, \psi)$ and $a \in S$. Then $(G^S)_a = (\sigma_a, \psi_a)$ is a subgraph of $G = (V, E)$.

Proof. Let G^S =(V, E, σ, ψ) be the given $S-$valued graph. For any $a \in S, \sigma_a = \{v \in V \ / \ a \succeq \sigma(v)\}$ and

$$\psi_a = \{(v_i, v_j) \in E \ / \ a \succeq \psi(v_i, v_j) \ where \ v_i, v_j \in \sigma_a\}).$$

Since, $\sigma_a \subseteq V$ and $\psi_a \subseteq E$ the graph $(G^S)_a = (\sigma_a, \psi_a)$ is a crisp graph corresponding to the $S-$value a of G^S, and is a subgraph of $G = (V, E)$.

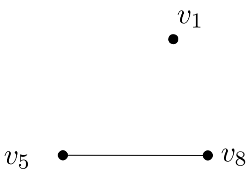
Definition 3.7. For any $a \in S$, the subgraph $(G^S)_a = (\sigma_a, \psi_a)$ of $G = (V, E)$ where $\sigma_a = \{v \in V \ / \ a \succeq \sigma(v)\}$ and

$$\psi_a = \{(v_i, v_j) \in E/a \succeq \psi(v_i, v_j), v_i, v_j \in \sigma_a\}$$

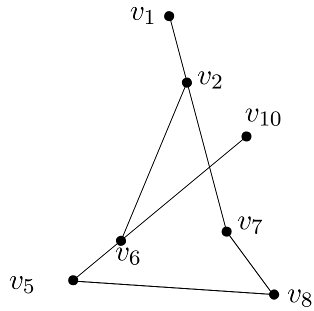
is called the lower level subgraph of the crisp graph G.

Example 3.8. Let $(S = \{0, a, b, c\}, +, \cdot)$ be a semiring with the Canonical pre-order given in 3.4. Consider the $S-$valued graph G^S in 3.4. For $t = c$, $(G^S)_c = G$, and for $t = a$ and $t = b(G^S)_a$ and $(G^S)_b$ are given as follows:

$(\mathbf{G^S})_\mathbf{a}$:



$(\mathbf{G^S})_\mathbf{b}$:



Theorem 3.9. Consider any $S-$valued graph $G^S = (V, E, \sigma, \psi)$ and $a, b \in S$ such that $a \preceq b$. Then the following are true in G^S.

1. The upper level subgraph $(G^S)^b$ is a subgraph of $(G^S)^a$.
2. The lower level subgraph $(G^S)_a$ is a subgraph of $(G^S)_b$.

Proof. The proof follows simply from the definition.

Definition 3.10. [3] Let $G^S = (V, E, \sigma, \psi)$ be an $S-$valued graph where $(S, +, \cdot)$ is a semiring with a canonical pre-order $\preceq$. For any $t \in S, \sigma(t) = \{v \in V \ / \ t = \sigma(v)\}$ is called the $t-$level vertex set and

$$\psi(t) = \{(v_i, v_j) \in E \ / \ t = \psi(v_i, v_j) \ where \ v_i, v_j \in \sigma(t)\}$$

is called the $t-$ level edge set of G^S.

Analogous to theroem 3.6, we obtain the following

Theorem 3.11. Consider the $S-$valued graph $G^S = (V, E, \sigma, \psi)$ and $a \in S$. Then $(G^S)(a) = (\sigma(a), \psi(a))$ is a subgraph of $G = (V, E)$.

Definition 3.12. For any $a \in S$, the subgraph $(G^S)(a) = (\sigma(a), \psi(a))$ of $G = (V, E)$ where $\sigma(a) = \{v \in V \ / \ a = \sigma(v)\}$ and

$$\psi(a) = \{(v_i, v_j) \in E / a = \psi(v_i, v_j), v_i, v_j \in \sigma(a)\}$$

is called the $t-$ level subgraph of the crisp graph G.

4. Properties of Level Graphs

In this section, we discuss some elementary properties of $t-$level subgraphs of $G^S, t \in S$.

Theorem 4.1. Let G^S be a $S-$valued graph. If a $S-$valued graph H^S is a $S-$ subgraph of G^S then $(H^S)_a$ is a subgraph of $(G^S)_a$, for any $a \in S$.

Proof. Let $G^S = (V, E, \sigma, \psi)$ be a given $S-$valued graph.
Let $H^S = (P, L, \tau, \gamma)$ be a $S-$valued subgraph of G^S.
$\therefore P \subset V$, $L \subset E$, $\tau \subset \sigma$ and $\gamma \subset \psi$, where $\tau \subset \sigma \Rightarrow \tau(v) \preceq \sigma(v)$, for all $v \in P$ and $\gamma \subset \psi \Rightarrow \gamma(v_i, v_j) \preceq \psi(v_i, v_j)$ for all $(v_i, v_j) \in L$. Let $a \in S$ and $H_a = (\tau_a, \gamma_a)$ where

$$\tau_a = \{v \in P \ / \ a \preceq \tau(v)\}\,; \quad and \ \gamma_a = \{(v_i, v_j) \in L \ / \ a \preceq \gamma(v_i, v_j)\}\,.$$

$G_a = (\sigma_a, \psi_a)$ where

$$\sigma_a = \{v \in V/a \preceq \sigma(v)\}\,; \quad and \ \psi_a = \{(v_i, v_j) \in E/a \preceq \psi(v_i, v_j)\}$$

For $v \in \tau_a, a \preceq \tau(v) \preceq \sigma(v) \Rightarrow a \preceq \sigma(v) \Rightarrow v \in \sigma_a$. Thus,

$$\tau_a \subset \sigma_a \tag{4.1}$$

For, $(v_i, v_j) \in \gamma_a, a \preceq \gamma(v_i, v_j) \preceq \psi(v_i, v_j) \Rightarrow (v_i, v_j) \in \psi_a$.

$$\therefore \gamma_a \subset \psi_a \tag{4.2}$$

From 4.1 and 4.2, we conclude that $(H^S)_a$ is a subgraph of $(G^S)_a$. Adapting the same method of proof as in theorem 4.1 we obtain the following results:

Theorem 4.2. Let G^S be a $S-$valued graph. If a $S-$valued graph H^S is a $S-$valued subgraph of G^S then $(H^S)^a$ is a subgraph of $(G^S)^a$, for any $a \in S$.

Theorem 4.3. Let G^S be a $S-$valued graph. If a $S-$valued graph H^S is a $S-$valued subgraph of G^S then $H^S(a)$ is a subgraph of $G^S(a)$, for any $a \in S$.

Remark 4.4. For any $S-$valued graph $G^S = (V, E, \sigma, \psi)$, we observe the following facts.

1. The intersection of two $t-$level subgraphs of G^S need not be a $t-$level subgraph of G.

2. The union of two $t-$level subgraphs of G^S need not be a $t-$level subgraph of G.

3. The sum of two $t-$level subgraphs of G^S need not be a $t-$level subgraph of G.

References

[1] J.A. Bondy, U.S.R. Murty, *Graph Theory with Applications*, North Holland, New York, 1982.

[2] Jonathan Golan, *Semirings and their Applications*, Kluwer Academic Publishers, London.

[3] M. Rajkumar, S. Jeyalakshmi, M. Chandramouleeswaran, Semiring-valued graphs, *International Journal of Math. Sci. and Engg. Appls.*, **9**(III) (2015), 141-152.