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GENERAL WEAK CONTRACTION OF CONTINUOUS AND DISCONTINUOUS FUNCTIONS

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Abstract: We prove general weak contraction theorem using three control functions in which one is continuous and other two are discontinuous.The result is verified with the help of suitable example.

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1. Introduction and Mathematical Preliminaries

Banach's fixed point theorem for contraction mapping is one of the pivotal results of functional analysis; and this type of contraction mapping implies continuity. A natural question is that whether we can find contractive condition which will imply existence of fixed point in complete metric space but will not imply continuity.

In 1968, Kannan [17], [18] established his result in which the above question has been answered in affirmative.

In 1997, the concept of weak contraction which is a generalization of Banach's contraction principle was introduced by Alber and Guerre-Delabriere [21]. Actually in [21], the authors defined such mapping for single valued maps

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on Hilbert spaces and proved the existence of fixed point. Rhoades [2] showed that most of the results of [21] are still true for any Banach space. There are a number of works in which weakly contractive mapping has been considered, some of these works are noted in [6], [9], [10], [14], [16].

Khan et. al. [13] initiated the use of control function in metric fixed point theory, which they called an Altering distance function. This function and its generalizations have been used in fixed and coincidence point problems in a large number of works; some of these works are noted in [4], [12], [20].

Some definitions and results are as follows:

Definition 1. [2] A mapping $T : X \to X$, where (X, d) is a metric space is said to be weakly contractive if x, y ∈ X

$$d(Tx, Ty) \leq d(x, y) - \phi\Big(d(x, y)\Big)$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

Theorem 2. *[2] Let (X, d) be a complete metric space, T a weakly contractive self-mapping of X. Then T has a unique fixed point p in X.*

Definition 3. [13](Altering distance function)
A function $\psi : [0, \infty) \to [0, \infty)$ is said to be altering distance function if the following properties are satisfied

(i) ψ is monotone nondecreasing and continuous

(ii) $\psi(t) = 0$ if and only if $t = 0$

Let by Θ we denote the set of all functions $\alpha : [0, \infty) \to [0, \infty)$ such that

(i_α) α is bounded on any bounded interval in $[0, \infty)$

(ii_α) α is continuous at 0 and $\alpha(0) = 0$.

Definition 4. [8] Let (X, d) be a metric space, T a self mapping of X, then we shall call T a generalized weak contractive mapping if for all $x, y \in X$

$$\psi\Big(d(Tx, Ty)\Big) \leq \psi\Big(m(x, y)\Big) - \phi\Big(max\big[d(x, y), d(y, Ty)\big]\Big) \tag{1}$$

where $m(x, y) = max\Big\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}\big[d(x, Ty) + d(y, Tx)\big]\Big\}$ and ψ is an altering distance function and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$.

A general weak contraction is more general than that satisfying

$$d(Tx, Ty) \leq km(x, y) \text{ for some } 0 < k < 1 \tag{2}$$

and is included in those mapping which satisfies

$$d(Tx, Ty) \leq m(x, y) \tag{3}$$

To verify that (1) is stronger than (2), using (2)

$$d(Tx, Ty) \leq m(x, y) - (1 - k)m(x, y)$$

which is the special case of (1) with ψ the identity function and ϕ defined by $\phi(t) = (1 - k)t$ and (3) is derived from (1) by taking ψ the identity function and $\phi(t) = 0$.

Theorem 5. *[8] Let (X, d) be a complete metric space, T a generalized weakly contractive self-mapping of X. Then T has a unique fixed point in X.*

2. Main Results

Theorem 6. *Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a self mapping, that satisfies*

$$\psi\Big(d(Tx, Ty)\Big) \leq \phi\Big(m(x, y)\Big) - \theta\Big(m(x, y)\Big) \tag{4}$$

where $m(x, y) = max\Big\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}\big[d(x, Ty) + d(y, Tx)\big]\Big\}$, and ψ is an altering distance function and $\phi, \theta \in \Theta$ and for all $x, y \in X$,

$$\psi(x) \leq \phi(y) \iff x \leq y \tag{5}$$

for any sequence $\{x_n\}$ in $[0, \infty)$ with $\{x_n\} \longrightarrow t > 0$

$$\psi(t) - \overline{\lim}\phi(x_n) + \underline{\lim}\theta(x_n) > 0 \tag{6}$$

Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$, we define a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ for all $n \geq 0$. If there exists a positive integer N such that $x_N = x_{N+1}$, then x_N is a fixed point of T, hence we shall assume that $x_N \neq x_{N+1}$ for all $n \geq 0$

Now substituting $x = x_{n+1}$ and $y = x_n$ in (4) we obtain

$$\psi\Big(d(x_{n+2}, x_{n+1})\Big) \leq \phi\Big(m(d(x_{n+1}, x_n))\Big) - \theta\Big(m(d(x_{n+1}, x_n))\Big) \tag{7}$$

Since

$$\frac{1}{2}\Big[d(x_{n+1},x_{n+2})+d(x_n,x_{n+2})\Big] \leq max\Big[d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\Big]$$

So

$$\psi\Big(d(x_{n+1},x_{n+2})\Big) \leq \phi\Big(max(d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}))\Big) - \theta\Big(max(d(x_n,x_{n+1}),d(x_{n+1},x_{n+2}))\Big) \quad (8)$$

Suppose $d(x_n,x_{n+1}) \leq d(x_{n+1},x_{n+2})$ for some positive integer n, then

$$\psi\Big(d(x_{n+1},x_{n+2})\Big) \leq \phi\Big(d(x_{n+1},x_{n+2})\Big) - \theta\Big(d(x_{n+1},x_{n+2})\Big) \quad (9)$$

From (5) it follows that $\theta\Big(d(x_{n+1},x_{n+2})\Big) \leq 0$, which implies that $d(x_{n+1},x_{n+2}) = 0$, contradicting our assumption that $x_N \neq x_{N+1}$ for each n.
Therefore $d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1})$ for all $n \geq 0$ and $\{d(x_n,x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers.
Let $G_n = d(x_n,x_{n+1})$.Then $\{G_n\}$ is also a monotone decreasing sequence of non-negative real numbers. Hence there exists an $r \geq 0$ such that

$$\lim_{n\to\infty} G_n = \lim_{n\to\infty} d(x_n,x_{n+1}) = r \quad (10)$$

In view of the above fact, from (8) we have for all $n \geq 0$,

$$\psi\Big(d(x_{n+1},x_{n+2})\Big) \leq \phi\Big(d(x_n,x_{n+1})\Big) - \theta\Big(d(x_n,x_{n+1})\Big) \quad (11)$$

i.e

$$\psi\Big(d(Ty,x_{n+2})\Big) \leq \phi\Big(d(y,Ty)\Big) - \theta\Big(d(y,Ty)\Big) \quad (12)$$

Taking limit supremum on both sides of (11) using (10) and the continuity property of ψ and (i_α) property of ϕ,θ we obtain

$$\psi(r) \leq \overline{\lim}\phi(G_n) + \overline{\lim}\Big(-\theta(G_n)\Big)$$

since $\overline{\lim}\Big(-\theta(G_n)\Big) = -\underline{\lim}\theta(G_n)$, it follows that

$$\psi(r) \leq \overline{\lim}\phi(G_n) - \underline{\lim}\theta(G_n)$$

That is

$$\psi(r) - \overline{\lim}\phi(G_n) + \underline{\lim}\theta(G_n) \leq 0$$

Which by (6) is a contraction unless $r = 0$, therefore

$$d(x_n, x_{n+1}) \longrightarrow 0 \text{ as } n \longrightarrow \infty \tag{13}$$

Next we show that $\{x_n\}$ is a cauchy sequence. Suppose that $\{x_n\}$ is not a cauchy sequence. Then there exits an $\epsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$, $\{n(k)\}$ such that for all positive integer k, $n(k) > m(k) > k$ and $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$. Assuming that $n(k)$ is the smallest such positive integer we get

$$\begin{aligned} &n(k) > m(k) > k, \\ &d(x_{m(k)}, x_{n(k)}) \geq \epsilon \text{ and} \\ &d(x_{m(k)}, x_{n(k)-1}) < \epsilon \end{aligned}$$

now

$$\epsilon \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)})$$

That is

$$\epsilon \leq d(x_{m(k)}, x_{n(k)}) \leq \epsilon + d(x_{n(k)-1}, x_{n(k)})$$

Letting $k \longrightarrow \infty$ in the above inequality and using (13) we have

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon \tag{14}$$

Again

$$d(x_{m(k)+1}, x_{n(k)+1}) \leq d(x_{m(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1})$$

Letting $k \longrightarrow \infty$ in the above inequality using (13) and (14) we have

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \epsilon \tag{15}$$

As $n(k) > m(k)$ so $d(x_{m(k)}, x_{n(k)}) > d(x_{m(k)+1}, x_{n(k)+1})$.
For $x = x_{n(k)}$, $y = x_{m(k)}$, and as $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers therefore we have

$$\begin{aligned} m(x, y) = max\Big[&d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1}), \\ &\frac{1}{2}\{d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)+1})\}\Big] \end{aligned}$$

$$= d(x_{m(k)}, x_{n(k)})$$

Putting these above result in(4) we get

$$\psi\Big(d(x_{n(k)+1}, x_{m(k)+1})\Big) \leq \phi\Big(d(x_{m(k)}, x_{n(k)})\Big) - \theta\Big(d(x_{m(k)}, x_{n(k)})\Big)$$

Taking limit supremum in both sides of the above inequality, using (15) and the continuity property of ψ and (i_α) property of ϕ, θ we get

$$\psi(\epsilon) \leq \overline{\lim}\phi\Big(d(x_{m(k)}, x_{n(k)})\Big) + \overline{\lim}\Big(- \theta(d(x_{m(k)}, x_{n(k)}))\Big)$$

since $\overline{\lim}\Big(- \theta(d(x_{m(k)}, x_{n(k)}))\Big) = -\underline{\lim}\theta\Big(d(x_{m(k)}, x_{n(k)})\Big)$, it follows that

$$\psi(\epsilon) \leq \overline{\lim}\phi\Big(d(x_{m(k)}, x_{n(k)})\Big) - \underline{\lim}\theta\Big(d(x_{m(k)}, x_{n(k)})\Big)$$

i.e

$$\psi(\epsilon) - \overline{\lim}\phi\Big(d(x_{m(k)}, x_{n(k)})\Big) + \underline{\lim}\theta\Big(d(x_{m(k)}, x_{n(k)})\Big) \leq 0$$

Which is a contradiction by (6). Therefore $\{x_n\}$ is a cauchy sequence in X
From the completeness of X, there exists $p \in X$ such that

$$x_n \longrightarrow p \text{ as } n \longrightarrow \infty$$

Now in (12) We put $y = p$, then we get

$$\psi\Big(d(Tp, x_{n+2})\Big) \leq \phi\Big(d(p, Tp)\Big) - \theta\Big(d(p, Tp)\Big)$$

Letting $n \longrightarrow \infty$, we get

$$\psi\Big(d(Tp, p)\Big) \leq \phi\Big(d(p, Tp)\Big) - \theta\Big(d(p, Tp)\Big)$$

from (5) it follows that $\theta\Big(d(p, Tp)\Big) \leq 0$ which implies that

$$d(p, Tp) = 0$$

i.e

$$Tp = p$$

Hence p is a fixed point of T.
Next we establish that the fixed point is unique. If there exists another fixed point $q \in X$ such that $Tq = q$. Then from (12) we get

$$\psi\Big(d(p, q)\Big) = \psi\Big(d(Tp, Tq)\Big) \leq \phi\Big(d(p, Tp)\Big) - \theta\Big(d(p, Tp)\Big)$$

Since both $d(p,Tp)=0$ and $d(q,Tq)=0$, so using (ii_α) property of ϕ,θ we get

$$\psi\Big(d(p,q)\Big)\leq 0$$

i.e

$$d(p,q)=0$$

Hence $p=q$.The proof is completed.

$\square$

3. Example

Let $X=[0,1,2,3,\cdots\cdots)$ and $d(x,y)=|x-y|$. Then (X,d) is a complete metric space.
Let $T:X\to X$ is given by $Tx=\frac{x}{2}$

and $\psi,\phi,\theta\in[0,\infty)\to[0,\infty)$ is given by

$$\psi(t)=\frac{t}{2},\quad \theta(t)=\frac{[t]}{6},\quad \phi(t)=\frac{[t]}{2}\ \text{ for all }\ t\in X$$
$$=[t],\quad \text{otherwise,}$$

Now we discuss three conditions of the theorem 6 for all x, y $\in$ X.
Condition (4) of the theorem 6:
Case-1: x$\neq$y.
(i) if x $>$ 2y and y $\neq$ 0, then

$$\psi\Big(d(Tx,Ty)\Big)=\psi\Big(d(\frac{x}{2},\frac{y}{2})\Big)=\psi\Big(\frac{x-y}{2}\Big)=\frac{x-y}{4}$$

and

$$\begin{aligned}m(x,y)&=max\Big(d(x,y),d(x,Tx),d(x,Ty),\frac{1}{2}\big[d(x,Ty)+d(y,Tx)\big]\Big)\\&=max\Big((x-y),\frac{x}{2},\frac{y}{2},\frac{3(x-y)}{4}\Big)\\&=x-y\end{aligned}$$

Then

$$\phi\Big(m(x,y)\Big)=\frac{[x-y]}{2}\quad\text{and}\quad\theta\Big(m(x,y)\Big)=\frac{[x-y]}{6}$$

and

$$\phi\Big(m(x,y)\Big) - \theta\Big(m(x,y)\Big) = \frac{[x-y]}{2} - \frac{[x-y]}{6} = \frac{[x-y]}{3}$$

Then

$$\psi\Big(d(Tx,Ty)\Big) \le \psi\Big(m(x,y)\Big) - \phi\Big(m(x,y)\Big).$$

(ii) $2y > x > y$ and $y \neq 0$, then

$$\psi\Big(d(Tx,Ty)\Big) = \psi\Big(d(\frac{x}{2},\frac{y}{2})\Big) = \psi\Big(\frac{x-y}{2}\Big) = \frac{x-y}{4}$$

and

$$\begin{aligned} m(x,y) &= max\Big(d(x,y), d(x,Tx), d(x,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)]\Big) \\ &= max\Big((x-y), \frac{x}{2}, \frac{y}{2}, \frac{3(x-y)}{4}\Big) \\ &= \frac{x}{2} \end{aligned}$$

Then

$$\phi\Big(m(x,y)\Big) = \frac{[\frac{x}{2}]}{2} \text{ and } \theta\Big(m(x,y)\Big) = \frac{[\frac{x}{2}]}{6}$$

and

$$\phi\Big(m(x,y)\Big) - \theta\Big(m(x,y)\Big) = \frac{[\frac{x}{2}]}{2} - \frac{[\frac{x}{2}]}{6} = \frac{[\frac{x}{2}]}{3}$$

Then

$$\psi\Big(d(Tx,Ty)\Big) \le \psi\Big(m(x,y)\Big) - \phi\Big(m(x,y)\Big).$$

(iii) If $y = 0$, and $x \neq 0$, then

$$\psi\Big(d(Tx,Ty)\Big) = \psi\Big(d(\frac{x}{2},0)\Big) = \psi\Big(\frac{x}{2}\Big) = \frac{x}{4}$$

and

$$\begin{aligned} m(x,y) &= max\Big(d(x,y), d(x,Tx), d(x,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)]\Big) \\ &= max\Big(x, \frac{x}{2}, \frac{3x}{4}\Big) \end{aligned}$$

$$= x$$

Then

$$\phi\Big(m(x,y)\Big) = \frac{[x]}{2} \text{ and } \theta\Big(m(x,y)\Big) = \frac{[x]}{6}$$

and

$$\phi\Big(m(x,y)\Big) - \theta\Big(m(x,y)\Big) = \frac{[x]}{2} - \frac{[x]}{6} = \frac{[x]}{3}$$

Then

$$\psi\Big(d(Tx,Ty)\Big) \leq \psi\Big(m(x,y)\Big) - \phi\Big(m(x,y)\Big).$$

(iv) If $x = 0$ and $y \neq 0$, then

$$\psi\Big(d(Tx,Ty)\Big) = \psi\Big(d(0,\frac{y}{2})\Big) = \psi\Big(\frac{y}{2}\Big) = \frac{y}{4}$$

and

$$\begin{aligned} m(x,y) &= max\Big(d(x,y), d(x,Tx), d(x,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\Big) \\ &= max\Big(y, \frac{y}{2}, \frac{3y}{4}\Big) \\ &= y \end{aligned}$$

Then

$$\phi\Big(m(x,y)\Big) = \frac{[y]}{2} \text{ and } \theta\Big(m(x,y)\Big) = \frac{[y]}{6}$$

and

$$\phi\Big(m(x,y)\Big) - \theta\Big(m(x,y)\Big) = \frac{[y]}{2} - \frac{[y]}{6} = \frac{[y]}{3}$$

Then

$$\psi\Big(d(Tx,Ty)\Big) \leq \psi\Big(m(x,y)\Big) - \phi\Big(m(x,y)\Big).$$

(v) If $x = y = 0$
Then trivially,

$$\psi\Big(d(Tx,Ty)\Big) \leq \psi\Big(m(x,y)\Big) - \phi\Big(m(x,y)\Big)$$

Case-2: $x = y \neq 0$
Then trivially,

$$\psi\Big(d(Tx, Ty)\Big) \leq \psi\Big(m(x, y)\Big) - \phi\Big(m(x, y)\Big)$$

Therefore

$$\psi\Big(d(Tx, Ty)\Big) \leq \psi\Big(m(x, y)\Big) - \phi\Big(m(x, y)\Big) \text{ for all } x, y \in X$$

Condition (5) is obviously satisfied for all x, y $\in$ X.
Now for condition (6) let us define a sequence $\{x_n\}$ in $[0, \infty)$ where $x_n = 3 + \frac{1}{n+1}$
Then $x_n \longrightarrow 3 > 0$ as $n \longrightarrow \infty$.
Therefore

$\psi(3) = \frac{3}{2}$ and $\phi(x_n) = \frac{[3+\frac{1}{n+1}]}{2} = \frac{3}{2}$ for all n$\in$ N and $\theta(x_n) = \frac{[3+\frac{1}{n+1}]}{6} = \frac{3}{6} = \frac{1}{2}$ all n$\in$ N.

Then obviously $\psi(t) - \overline{\lim}\phi(x_n) + \underline{\lim}\theta(x_n) > 0$ is satisfied.
Hence all the conditions of theorem 6 are satisfied.
Here it is seen that 0 is the unique fixed point of T.

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