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ON METACOMPACTNESS IN BITOPOLOGICAL SPACES

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Abstract: In this paper we define pairwise metacompact spaces and pairwise locally metacompact spaces, and study their properties and their relations with other topological spaces. Several examples are discussed and many will known theorems are generalized concerning metacompact spaces.

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1. Introduction

In 1963, Kelly [10] introduced the notion of a bitopological space, i.e. a triple (X, τ_1, τ_2) where X is a set and τ_1, τ_2 are two topologies on X. He also defined pairwise regular ($P-$regular), pairwise normal ($P-$normal), and obtained generalization of several standard results such as Urysohn′s lemma and Tietze extension theorem. Several authors have since considered the problem of defining compactness for such spaces, see Kim [12], Fletcher, Hoyle and Patty [5]. In 1969, Fletcher et. al, [5] gave the definitions of $\tau_1\tau_2-$open and $P-$open covers in bitopolgical spaces. A cover $\tilde{U}$ of the bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2-$open if $\tilde{U} \subset \tau_1 \cup \tau_2$, if in addition, $\tilde{U}$ contains at least one non-empty mem-

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ber of τ_1 and at least one non-empty member of τ_2, it is called P—open. Also they defined the concept of P—compact space as follows: A bitopological space (X, τ_1, τ_2) is called P—compact if every $P-$open cover of the space (X, τ_1, τ_2) has a finite subcover. While in 1972 Datta [3], defined S—compact space as follows: A bitopological space (X, τ_1, τ_2) is called S—compact if every $\tau_1\tau_2-$open cover of the space (X, τ_1, τ_2) has a finite subcover. In 1969 Birsan [1] gave the following definitions: A bitopological space (X, τ_1, τ_2) is called τ_1—compact with respect to τ_2 if for each τ_1—open cover of X, there is a finite τ_2—open subcover. A bitopological space (X, τ_1, τ_2) is called B—compact if it is τ_1—compact with respect to τ_2 and τ_2—compact with respect to τ_1. In 1975 Cooke and Reilly [2] discussed the relations between these definitions. In 1979 Hdieb [8] introduced important theorem to the theory of $[n\ ,\ m]$ $-$compact, paracompact and normal spaces. In 1983 Fora and Hdieb [6] introduced the definition of $P-$Lindelöf, $S-$Lindelöf, $B-$Lindelöf spaces in analogue manner. They also gave the definitions of certain types of functions as follows : A function $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is called P—continuous ($P-$open, $P-$closed, $P-$homeomorphism, respectively), if both functions $f_1 : (X, \tau_1) \longrightarrow (Y, \sigma_1)$ and $f_2 : (X, \tau_2) \longrightarrow (Y, \sigma_2)$ are continuous (open, closed, homeomorphism, respectively).

In this paper we introduce the notion of metacompact spaces in bitopological spaces and derive some related results. When (X, τ_1, τ_2) has the property Q this means that both (X, τ_1) and (X, τ_2) have this property. For instance a bitopological space (X, τ_1, τ_2) is called metacompact, if both (X, τ_1) and (X, τ_2) are metacompact spaces.

We will use the letters $P-$, $S-$ to denote the pairwise and semi, respectively, e.g. P—metacompact stands for pairwise metacompact, S—metacompact stands for semi metacompact.

τ_i-closure, τ_i-interior of a set A will be denoted by CL_iA, Int_iA respectively. The product of τ_1 and τ_2 will be denoted by $\tau_1 \times \tau_2$.

Let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ denote the set of all real numbers, integer numbers, natural numbers, and rational numbers respectively. Let τ_{dis}, τ_{ind}, τ_u, τ_s, τ_{coc}, τ_{cof}, τ_l, τ_r denote the discrete, the indiscrete usual, Sorgenfrey line, cocountable, cofinite, left-ray, and right-ray topologies respectively. Let ω_0 and ω_1 denote the cardinal numbers of $\mathbb{Z}$ and $\mathbb{R}$ respectively.

2. Pairwise Metacompact Spaces

In this section, we will introduce the concept of metacompactness in bitopological spaces, and introduce some of their properties, and relate it to other spaces.

Let us recall known definitions which will be used in the sequel.

Definition 2.1. [4] *A pairwise open cover $\tilde{V}$ of a bitopological space (X, τ_1, τ_2) is called parallel refinement of of pairwise open cover $\tilde{U}$ of X if each τ_i- open set of $\tilde{V}$ is contained in some τ_i- open set of $\tilde{U}$ $(i = 1, 2)$.*

Definition 2.2. *A pairwise open cover $\tilde{U}$ of the bitopological space (X, τ_1, τ_2) is called pairwise point finite if each $x \in X$ is contained in a finite number of of τ_1—open members of $\tilde{U}$, or it contained in a finite number of of τ_2—open members of $\tilde{U}$.*

Definition 2.3. *A bitopological space (X, τ_1, τ_2) is called P—metacompact if every P—open cover of the space (X, τ_1, τ_2) has a pairwise point finite parallel refinement.*

A bitopological space (X, τ_1, τ_2) is called S—metacompact if every $\tau_1\tau_2-$open cover of the space (X, τ_1, τ_2) has a pairwise point finite parallel refinement.

A bitopological space (X, τ_1, τ_2) is called τ_1—metacompact with respect to τ_2 if for each τ_1—open cover of X has a point finite τ_2—parallel refinement.

A bitopological space (X, τ_1, τ_2) is called B—metacompact, if it is τ_1—metacompact with respect to $\tau_2$and τ_2—metacompact with respect to τ_1.

If (X, τ_1) and (X, τ_2) have the same topological property P, then we simply say that the bitopological space (X, τ_1, τ_2) has property P, without referring to $\tau_1$or τ_2.

So, we will say that a bitopological space (X, τ_1, τ_2) is called metacompact, if both (X, τ_1) and (X, τ_2) are metacompact.

Example 2.4. *(1) The bitopological space $(\mathbb{R}, \tau_{cof}, \tau_{dis})$ is Hausdorff, since both $(\mathbb{R}, \tau_{cof})$ and $(\mathbb{R}, \tau_{dis})$ are Hausdorff.*

(2) The bitopological space $(\mathbb{R}, \tau_{cof}, \tau_u)$ is $P-$metacompact, $B-$metacompact, and metacompact space since both $(\mathbb{R}, \tau_{cof})$ and $(\mathbb{R}, \tau_u)$ are metacompact.

(3) The bitopological spaces $(\mathbb{R}, \tau_{cof}, \tau_{dis})$ is $P-$metacompact, $B-$metacompact, and metacompact space since both $(\mathbb{R}, \tau_{cof})$ and $(\mathbb{R}, \tau_{dis})$ are metacompact.

(4) The bitopological spaces $(\mathbb{N}, \tau_{dis}, \tau_{dis})$ is $P-$metacompact, $B-$metacompact, metacompact space since $(\mathbb{R}, \tau_{dis})$ is metacompact.

Theorem 2.5. *The bitopological space (X, τ_1, τ_2) is S—metacompact if and only if it is metacompact and P—metacompact.*

Proof. $\Longrightarrow$) Assume that (X,τ_1,τ_2) is S—metacompact. Let $\tilde{U}$ be a P—open cover of X, then $\tilde{U}$ is a $\tau_1\tau_2-$open cover of the space (X,τ_1,τ_2). Since (X,τ_1,τ_2) is S—metacompact,$\tilde{U}$ has a pairwise point finite parallel refinement. Hence (X,τ_1,τ_2) is P—metacompact. Also any τ_1-open or τ_2-open cover of (X,τ_1,τ_2) is a $\tau_1\tau_2-$open cove. Hence (X,τ_1) and (X,τ_2) are metacompact. So (X,τ_1,τ_2) is metacompact.

$\Longleftarrow$) Assume that (X,τ_1,τ_2) is metacompact and P—metacompact. Let $\tilde{U}$ be a $\tau_1\tau_2-$ open cover of (X,τ_1,τ_2). If $\tilde{U}$ is a P—open cover, then the result follows. If $\tilde{U}$ is not P—open cover, then it is τ_1-open or τ_2-open cover of (X,τ_1,τ_2). Since (X,τ_1,τ_2) is metacompact, $\tilde{U}$ has a pairwise point finite parallel refinement. So (X,τ_1,τ_2) is S—metacompact. □

Recall that a space has a hereditary property P, if every subspace of it has this property.

Theorem 2.6. *If a bitopological space (X,τ_1,τ_2) is hereditary metacompact, then it is S—metacompact.*

Proof. Let $\tilde{U}$ be a $\tau_1\tau_2-$ open cover of (X,τ_1,τ_2). Then $\tilde{U}=\{U_\alpha:\alpha\in\Delta\}\cup\{V_\beta:\beta\in\Gamma\}$, where $U_\alpha\in\tau_1$ for each $\alpha\in\Delta$ and $V_\beta\in\tau_2$ for each $\beta\in\Gamma$.

Since $U=\bigcup\{U_\alpha:\alpha\in\Delta\}$ is τ_1-metacompact, it has a point finite open parallel refinement say $\{U^*_\alpha:\alpha\in\Delta\}$ and $U=\bigcup\limits_{\alpha\in\Delta}U^*_\alpha$.

Now let $V=\bigcup\{V_\beta:\beta\in\Gamma\}$, since V is τ_2-metacompact, it has a point finite open parallel refinement say $\left\{V^*_\beta:\beta\in\Gamma\right\}$ and $V=\bigcup\limits_{\beta\in\Gamma}V^*_\beta$.

Hence $\{U^*_\alpha:\alpha\in\Delta\}\cup\left\{V^*_\beta:\beta\in\Gamma\right\}$ is a point finite $\tau_1\tau_2-$open parallel refinement of $\tilde{U}$. □

The following definitions can be found in [6].

Definition 2.7. *A bitopological space (X,τ_1,τ_2) is called P—Lindelöf if every $P-$open cover of the space (X,τ_1,τ_2) has a countable subcover.*

A bitopological space (X,τ_1,τ_2) is called S—Lindelöf if every $\tau_1\tau_2-$open cover of the space (X,τ_1,τ_2) has a countable subcover.

A bitopological space (X,τ_1,τ_2) is called τ_1—Lindelöf with respect to $\tau_2$if for each τ_1—open cover of X, there is a countable τ_2—open subcover.

A bitopological space (X,τ_1,τ_2) is called B—Lindelöf if it is τ_1—Lindelöf with respect to τ_2 and τ_2—Lindelöf with respect to τ_1.

Example 2.8. *Consider the two topologies $\tau_1$and τon $\mathbb{R}$defined by the basis:*

$\beta_1 = \{(-\infty, a) : a > 0\} \cup \{\{x\} : x > 0\}$
$\beta_2 = \{(a, \infty) : a < 0\} \cup \{\{x\} : x < 0\}$

Then X is $P-$metacompact but not a $B-$metacompact, since for the τ_1-open cover $(-\infty, 2) \cup \{\{x\} : x > 1\}$of $\mathbb{R}$ has no point finite τ_2-open refinement.

It is clear that both $(\mathbb{R}, \tau_1)$ and $(\mathbb{R}, \tau_2)$ are metacompact spaces, so $(\mathbb{R}, \tau_1, \tau_2)$ is metacompact. Observe that $(\mathbb{R}, \tau_1, \tau_2)$ is not $S-$metacompact. On the other hand we have

$(\mathbb{R}, \tau_1, \tau_2)$ is countably $P-$metacompact. Now we observe that $(\mathbb{R}, \tau_1, \tau_2)$ is $P-$Lindelöf, it is also clear that $(\mathbb{R}, \tau_1, \tau_2)$ is not $B-$Lindelöf.

Definition 2.9. *A subset D of a bitopological space (X, τ_1, τ_2) is called pairwise dense denoted by (P—dense) in X , if $CL_{\tau_1}D = CL_{\tau_2}D = X$.*

A bitopological space (X, τ_1, τ_2) is called P—separable, if it has a P—dense countable subset D.

It is clear that the bitopological space $(\mathbb{R}, \tau_{coc}, \tau_{cof})$ is $P-$separable.

Theorem 2.10. *A P—separable, P—metacompact space (X, τ_1, τ_2) is P—Lindelöf.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a P—open cover of X. Assume that $\tilde{U}$ has no countable subcover of X. Let $\tilde{V} = \{V_\beta : \beta \in \Gamma\}$ be a point finite parallel refinement of $\tilde{U}$. Let D be a countable dense subset of X. Then $V_\beta \cap D \neq \phi$ for each $\beta \in \Gamma$. Thus D is an uncountable set because $\tilde{V}$ is uncountable, which is a contradiction. Hence the result. □

Definition 2.11. *A bitopological space (X, τ_1, τ_2) is called countably P—metacompact if every countable P—open cover of the space (X, τ_1, τ_2) has a pairwise point finite parallel refinement.*

A bitopological space (X, τ_1, τ_2) is called countably S—metacompact if every countable $\tau_1\tau_2-$open cover of the space (X, τ_1, τ_2) has a pairwise point finite parallel refinement.

A bitopological space (X, τ_1, τ_2) is called countably τ_1—metacompact with respect to τ_2 if for each countable τ_1—open cover of X has a point finite τ_2—parallel refinement .

A bitopological space (X, τ_1, τ_2) is called countably B—metacompact, if it is countably τ_1—metacompact with respect to τ_2 and countably τ_2—metacompact with respect to τ_1.

A bitopological space (X, τ_1, τ_2) is called countably metacompact, if it is countably τ_1—metacompact and countably τ_2—metacompact.

Example 2.12. *(1) The bitopological space* $(\mathbb{N}, \tau_{dis}, \tau_{ind})$ *is* $P-$*metacompact,* $B-$*metacompact, metacompact space since* $(\mathbb{N}, \tau_{dis})$ *and* $(\mathbb{N}, \tau_{ind})$ *are metacompact. It is also countably* $P-$*metacompact, countably* $B-$*metacompact.*

(2) The bitopological space $(\mathbb{R}, \tau_{dis}, \tau_u)$ *is* $P-$*metacompact,* $B-$*metacompact, metacompact space since* $(\mathbb{R}, \tau_{dis})$ *and* $(\mathbb{R}, \tau_u)$ *are metacompact. It is also countably* $P-$*metacompact, countably* $B-$*metacompact. It is not* P*—separable. It is clear that* $(\mathbb{R}, \tau_{dis}, \tau_u)$ *is neither* $P-$*Lindelöf nor* $P-$*compact. Also it is not* $P-$*countably compact.*

Theorem 2.13. *Every* P*—Lindelöf countably* P*—metacompact space* (X, τ_1, τ_2) *is* P*—metacompact space.*

Proof. Let$\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a P—open cover of X. Since X is P—Lindelöf, $\tilde{U}$ has a countable subcover $\tilde{V} = \{V_{\alpha i}\}_{i=1}^{\infty}$ .Since X is countably P—metacompact,$\tilde{V}$ has a point finite parallel refinement$\tilde{W}$ of $\tilde{U}$. Hence (X, τ_1, τ_2) is P—metacompact. □

it is clear that the bitopological space $(\mathbb{Z}, \tau_{dis}, \tau_{ind})$ is $P-$metacompact, since it is countably $P-$metacompact and $P-$Lindelöf.

Definition 2.14. *A space* (X, τ_1, τ_2) *is said to be* $p-$*metalindelöf if every* $p-$*open cover of* (X, τ_1, τ_2) *has a pairwise point countable parallel refinement.*

Theorem 2.15. *Every* P*—metalindelöf countably* P*—metacompact space* (X, τ_1, τ_2) *is* P*—metacompact space.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a P—open cover of X. Since X is P—metaLindelöf, $\tilde{U}$ has a point countable parallel refinement$\tilde{V} = \{V_{\alpha i}\}_{i=1}^{\infty}$, which is also a $p-$open cover of (X, τ_1, τ_2). Since X is countably P—metacompact,$\tilde{V}$ has a point finite parallel refinement$\tilde{W}$ of $\tilde{U}$. Hence (X, τ_1, τ_2) is P—metacompact. □

The following theorem is easily proved.

Theorem 2.16. *Every metaLindelöf countably metacompact space* (X, τ_1, τ_2) *is metacompact space.*

In [4], Engelking defined a cover$\{A_s : s \in S\}$of a space X to be irreducible if $\bigcup\limits_{s \in S_0} A_s \neq X$ for every proper subset S_0 of the set S.

Theorem 2.17. [4] *Every point finite cover* $\{A_s : s \in S\}$ *of a space* X *has an irreducible subcover.*

The following definitions can be found in [1], [3] and [5].

Definition 2.18. *A bitopological space (X, τ_1, τ_2) is called P—compact if every $P-$open cover of the space (X, τ_1, τ_2) has a finite subcover.*

A bitopological space (X, τ_1, τ_2) is called S—compact if every $\tau_1\tau_2-$open cover of the space (X, τ_1, τ_2) has a finite subcover.

A bitopological space (X, τ_1, τ_2) is called τ_1—compact with respect to τ_2 if for each τ_1—open cover of X , there is a finite τ_2—open subcover.

A bitopological space (X, τ_1, τ_2) is called B—compact if it is τ_1—compact with respect to τ_2 and τ_2—compact with respect to τ_1.

Example 2.19. *Let $X = \mathbb{R}$, $ß_1 = \{X\ , \{x\} : x \in X - \{0\}\}$, $ß_2 = \{X, \{x\} : x \in X - \{1\}\}$. Let τ_1 and τ_2 be the topologies on X which are generated by the bases $ß_1$ and $ß_2$, respectively. Then $(\mathbb{R}, \tau_1, \tau_2)$ is $B-$metacompact, since any τ_1-open cover or any τ_2-open cover of X must contain X as a member. It is also $P-$metacompact and countably $P-$metacompact, countably $B-$metacompact. On the other hand, $(\mathbb{R}, \tau_1, \tau_2)$ is $B-$Lindelöf, However it is not $P-$Lindelöf, since the $P-$open cover $\{\{x\}\ : x \in X\}$ of X has no countable subcover. It is clear that $(\mathbb{R}, \tau_1, \tau_2)$ is not $P-$compact, since the $P-$open cover $\{\{x\}\ : x \in X\}$ of X has no finite subcover, it is not compact space, but it is $B-$compact, since any τ_1-open cover or any τ_2-open cover of X must contain X as a member and $\{X\}$ is a finite subcover of any τ_1-open cover or any τ_2-open cover of X.*

The following definitions can be found in [15] and [6].

Definition 2.20. *A bitopological space (X, τ_1, τ_2) is called P—countably compact if every countable $P-$open cover of the space (X, τ_1, τ_2) has a finite subcover.*

A bitopological space (X, τ_1, τ_2) is called $S-$countably compact if every countable $\tau_1\tau_2-$open cover of the space (X, τ_1, τ_2) has a finite subcover.

A bitopological space (X, τ_1, τ_2) is called τ_1—countably compact with respect to τ_2 if for each countable τ_1—open cover of X, there is a finite τ_2—open subcover.

A bitopological space (X, τ_1, τ_2) is called B—countably compact if it is countably τ_1—compact with respect to τ_2 and countably τ_2—compact with respect to τ_1.

Example 2.21. *(1) The bitopological space $(\mathbb{R}, \tau_{dis}, \tau_{coc})$ is $P-$metacompact, not $B-$metacompact, since the τ_1-open cover $\{\{x\} : x \in \mathbb{R}\}$of $\mathbb{R}$ has no point finite τ_2-open refinement. It is also countably $P-$metacompact. It is clear that $(\mathbb{R}, \tau_{dis}, \tau_{coc})$ is $P-$Lindelöf space which is neither $P-$countably compact, nor $P-$compact.*

(2) Let τ_s denotes the Sorgenfrey line topology on $\mathbb{R}$.Then the bitopological

space $(\mathbb{R}, \tau_s, \tau_u)$ *is* $S-$*Lindelöf, so it is* $P-$*Lindelöf and Lindelöf. Also it is* $P-$*separable, so* $(\mathbb{R}, \tau_s, \tau_u)$ *is* $P-$*metacompact. It is clear that* $(\mathbb{R}, \tau_s, \tau_u)$ *is not* $B-$*metacompact, since the* τ_u *open cover* $\{(n, n+1) : n \in \mathbb{Z}\}$ *of* $\mathbb{R}$ *has no point finite* τ_s-*open refinement, because* $\tau_s \subsetneq \tau_u$. *It is also clear that* $(\mathbb{R}, \tau_s, \tau_u)$ *is neither* $S-$*compact nor* $S-$*countably compact.*

Theorem 2.22. *Every countably compact metacompact bitopological space* (X, τ_1, τ_2) *is compact.*

Proof. It is sufficient to show that both topological spaces (X, τ_1) and (X, τ_2) are compact spaces. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of a countably compact metacompact space (X, τ_1). So there exists an irreducible point finite open refinement$\tilde{V} = \{V_\alpha : \alpha \in \Delta\}$ of the cover $\tilde{U}$. The cover$\tilde{V} = \{V_\alpha : \alpha \in \Delta\}$ being irreducible, for every $\alpha \in \Delta$, there exists a point $x_\alpha \in V_\alpha \setminus \bigcup\limits_{\alpha \neq \alpha^*} V_{\alpha^*}$. Since the sets V_α cover the space (X, τ_1), every point $x \in X$ has a neighborhood which contains exactly one point of the set $A = \{X_\alpha : \alpha \in \Delta\}$. Hence, the derived set of A dented by $A^d = \phi$. Since (X, τ_1) is countably compact, the set A is finite. Thus the set Δ is also finite and the open cover$\tilde{V}$ is a finite open refinement of $\tilde{U}$ for (X, τ_1). So (X, τ_1) is compact. Similarly, we show that (X, τ_2) is compact. Hence (X, τ_1, τ_2) is compact. □

Definition 2.23. [6] *A function* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *is called* P—*continuous (*P*— open,* P*—closed,* P*—homeomorphism, respectively) if the functions* $f : (X, \tau_1) \longrightarrow (Y, \sigma_1)$ *and* $f : (X, \tau_2) \longrightarrow (Y, \sigma_2)$ *are continuous (open, closed, homeomorphism, respectively).*

Definition 2.24. *A function* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *is called* P—*perfect, if the function* f *is* P—*continuous,* P—*closed, and for all* $y \in Y$, *the set* $f^{-1}(y)$ *is* P—*compact.*

Theorem 2.25. *Let* $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ *be a* P—*perfect function. Then* X *is* P—*metacompact space if* Y *is so.*

Proof. Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\} \cup \{V_\beta : \beta \in \Gamma\}$ be any P—open cover of X, where $\{U_\alpha : \alpha \in \Delta\}$ is a set of τ_1-open members of $\tilde{U}$ and $\{V_\beta : \beta \in \Gamma\}$ is a set of τ_2-open members of $\tilde{U}$.

Now, since f is P—perfect, for every $y \in Y$ we have $f^{-1}(y)$ is P—compact subset of X. So there exist finite subsets Δ_1 and Δ_2 of Δ and Γ respectively such that $f^{-1}(y) \subseteq \left\{ \bigcup\limits_{i=1}^{n} U_i : i \in \Delta_1 \right\} \bigcup \left\{ \bigcup\limits_{i=1}^{m} V_i : i \in \Delta_2 \right\}$. Now;

$O_{y_1} = Y - f\left(X - \bigcup\limits_{i=1}^{n} U_i : i \in \Delta_1\right)$ is a $\tau_1$open subset of Y and $f^{-1}(O_{y_1}) \subseteq \left\{\bigcup\limits_{i=1}^{n} U_i : i \in \Delta_1\right\}$.

$O_{y_2} = Y - f\left(X - \bigcup\limits_{i=1}^{m} V_i : i \in \Delta_2\right)$ is a τ_1 open subset of Y and $f^{-1}(O_{y_2}) \subseteq \left\{\bigcup\limits_{i=1}^{m} V_i : i \in \Delta_2\right\}$.

$y \in O_{y_1} \cup O_{y_2}$. So, $\tilde{O} = \left\{\tilde{O} = \{O_{y_1} : y \in Y\} \cup \{O_{y_2} : y \in Y\} : y \in Y\right\}$ is a $P-$open cover of Y. Since Y is $P-$metacompact, $\tilde{O}$ has a pairwise point finite parallel refinement

$\tilde{O}^* = \tilde{O} = \{O^*_{y_1} : y \in Y\} \cup \{O^*_{y_2} : y \in Y\}$.

Now, $O^*_{y_1}$ is a τ_{1-}open subset of X and $O^*_{y_2}$ is a τ_{2-}open subset of X.

Since f is perfect, the set $\{f^{-1}(O^*_{y_1}) : y \in Y\} \cup \{f^{-1}(O^*_{y_2}) : y \in Y\}$ is a pairwise point finite parallel refinement of X. So, X is P—metacompact. □

Definition 2.26. [6] *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be bitopological spaces. Then the Cartesian product of (X, τ_1, τ_2) and (Y, σ_1, σ_2) is the bitopological space $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$.*

Lemma 2.27. [4] *(Wallace lemma) If A is a compact subset of a topological space $(X\ ,\tau)$ and B is a compact subset of of a topological space $(Y\ ,\sigma)$ and $A \times B \subseteq W$; where W is open subset of $X \times Y$, then there exist open sets U and V in X and$Y$ respectively such that $A \times B \subseteq U \times V \subseteq W$.*

Theorem 2.28. *If X is P—compact, then the projection function $P : X \times Y \longrightarrow Y$ is P—closed, where (X, τ_1, τ_2) and (Y, σ_1, σ_2) are a bitopological spaces.*

Proof. To show that the projection function $P : X \times Y \longrightarrow Y$ is P—closed, we show that the projection functions $P_1 : (X \times Y, \tau_1 \times \sigma_1) \longrightarrow (Y, \sigma_1)$ and $P_2 : (X \times Y, \tau_2 \times \sigma_2) \longrightarrow (Y, \sigma_2)$ are closed. Let $y \in Y$ and let U be an open set in $(X \times Y, \tau_1 \times \sigma_1)$ such that $P_1^{-1}(\{y\}) \subseteq U$. So by (Wallace lemma), there exists a σ_1-open set in$Y$ say V_y such that $P_1^{-1}(\{y\}) = X \times \{y\} \subseteq X \times V_y \subseteq U$. So, $y \in V_y$ and $P_1^{-1}(\{V_y\}) = X \times \{V_y\} \subseteq U$. So, $P_1 : (X \times Y, \tau_1 \times \sigma_1) \longrightarrow (Y, \sigma_1)$ is closed function. Similarly, we have $P_2 : (X \times Y, \tau_2 \times \sigma_2) \longrightarrow (Y, \sigma_2)$ is closed function. Hence, the projection function $P : X \times Y \longrightarrow Y$ is P—closed. □

Theorem 2.29. *The product of a P—compact space X and a P—metacompact space Y is P—metacompact, where (X, τ_1, τ_2) and (Y, σ_1, σ_2) are bitopological spaces.*

Proof. Let $P : X \times Y \longrightarrow Y$ be the projection function such that $(x, y) \longrightarrow y$. Then $P : X \times Y \longrightarrow Y$ is P—perfect function. Since Y is P—metacompact, then $X \times Y$ is P—metacompact by Theorem 2.28. □

Example 2.30. *The bitopological space $(\mathbb{R}, \tau_{cof}, \tau_{dis})$ is $P-$compact so it is $P-$metacompact. Also it is $B-$metacompact, metacompact space since both $(\mathbb{R}, \tau_{cof})$ and $(\mathbb{R}, \tau_{dis})$ are metacompact. The space $\left(\mathbb{R}^2, \tau_{cof} \times \tau_{cof}, \tau_{dis} \times \tau_{dis}\right)$ is also $P-$metacompact, but not $P-$compact nor $P-$Lindelöf, since the $P-$open cover $\{\mathbb{R} \times (\mathbb{R} - \{0\})\} \cup \{(x\ , 0) : x \in \mathbb{R}\}$ for $\mathbb{R}^2$ has no countable subcover.*

Lemma 2.31. [4] *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a continuous, onto function. If $\tilde{A} = \{A_\alpha : \alpha \in \Delta\}$ is a point finite family subset of X, then $\{f (A_\alpha) : \alpha \in \Delta\}$ is a point finite family subset of Y.*

Theorem 2.32. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a P—continuous, P—closed, onto function. Then Y is P—metacompact, if X is so.*

Proof. Let $\tilde{V} = \{U_\alpha : \alpha \in \Delta\} \cup \{V_\beta : \beta \in \Gamma\}$ be any $P-$open cover of Y, where $\{U_\alpha : \alpha \in \Delta\}$ are σ_1-open members of $\tilde{V}$ and $\{V_\beta : \beta \in \Gamma\}$ are σ_2-open members of $\tilde{V}$. Since f is P—continuous, onto function, the set
$\tilde{U} = \left\{f^{-1} (U_\alpha) : \alpha \in \Delta\right\} \cup \left\{f^{-1} (V_\beta) : \beta \in \Gamma\right\}$
is a $P-$open cover of X. Since X is P—metacompact space, there exists a pairwise point finite open parallel refinement of $\tilde{U}$, say $\tilde{U}^* = \left\{f^{-1} (U^*_\alpha) : \alpha \in \Delta\right\} \cup \left\{f^{-1}\left(V^*_\beta\right) : \beta \in \Gamma\right\}$. Thus, $\tilde{V}^* = \{U^*_\alpha : \alpha \in \Delta\} \cup \left\{V^*_\beta : \beta \in \Gamma\right\}$ is a pairwise point finite open parallel refinement of $\tilde{V}$. So,Y is P—metacompact. □

Definition 2.33. *A pairwise open cover $\tilde{U}$ of a bitopological space (X, τ_1, τ_2) is called pairwise locally finite if for every $x \in X$, there exists a τ_1-open neighborhood of x which meets only a finite number of τ_1-open elements of $\tilde{U}$, or there exists a τ_2-open neighborhood of x which meets only a finite number of τ_2-open elements of $\tilde{U}$.*

A space (X, τ_1, τ_2) is said to be $P-$paracompact if every $P-$open cover of X has a pairwise open locally finite parallel refinement.

A space (X, τ_1, τ_2) is said to be $S-$paracompact if every $\tau_1\tau_2-$open cover of X has a pairwise open locally finite parallel refinement.

A space (X, τ_1, τ_2) is said to be paracompact if both (X, τ_1) and (X, τ_2) are paracompact spaces.

A subset F of a space $(X, \tau)$is said to be $\tau-$paracompact relative to X, if every open cover of F by members of τ has a locally finite parallel refinement in X by members of τ.

Corollary 2.34. *Every $P-$paracompact ($S-$paracompact) space is $P-$metacompact ($S-$metacompact).*

Theorem 2.35. *For the bitopological space (X, τ_1, τ_2), the following are equivalent:*

1. (X, τ_1, τ_2) is $P-$metacompact.

2. Each τ_1-closed subset of X is τ_2-metacompact relative to X, and each τ_2-closed subset of X is τ_1-metacompact relative to X.

Proof. (1) $\Longrightarrow$ (2): Let $K \neq \phi$ be a τ_1-closed subset of X and $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a τ_2-open cover of K. Then $\tilde{O} = \{X - K\} \cup \{U_\alpha : \alpha \in \Delta\}$ is a $P-$open cover of X. Since (X, τ_1, τ_2) is $P-$metacompact, $\tilde{O}$ has a pairwise point finite open parallel refinement, say $\{V_\beta : \beta \in \Gamma\} \cup \{U^*_\alpha : \alpha \in \Delta\}$, where V_β is a τ_1-open for each $\beta \in \Gamma$, and U^*_α is a τ_2-open for each $\alpha \in \Delta$. Thus $\{U^*_\alpha : \alpha \in \Delta\}$ is a point finite parallel refinement of $\tilde{U}$. Hence K is a τ_2-metacompact relative to X. The proof of other case is similar.

(2) $\Longrightarrow$ (1): Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\} \cup \{V_\beta : \beta \in \Gamma\}$ be a $P-$open cover of X, where $\{U_\alpha : \alpha \in \Delta\}$ are τ_1-open members of $\tilde{U}$, and $\{V_\beta : \beta \in \Gamma\}$ are τ_2-open members of $\tilde{U}$. We have tow cases:

(i) If $\bigcup \{V_\beta : \beta \in \Gamma\} = X$, then, choose an $\alpha_0 \in \Delta$ such that $U_{\alpha 0} \neq \phi$. Since $\{V_\beta : \beta \in \Gamma\}$ is a τ_2-open cover of the τ_1-closed proper subset $(X - U_{\alpha_0})$ has a τ_2-open refinement $\left\{V^*_\beta : \beta \in \Gamma\right\}$. Hence, $U_{\alpha 0} \cup \left\{V^*_\beta : \beta \in \Gamma\right\}$ is a pairwise point finite open parallel refinement of$\tilde{U}$. Hence (X, τ_1, τ_2) is $P-$metacompact.

(ii) If $\bigcup \{V_\beta : \beta \in \Gamma\} \neq X$, then the set $K = X - \bigcup_{\beta \in \Gamma} V_\beta$ is a τ_2-closed proper subset of X, and $K \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$. Hence there is a τ_1-point finite parallel refinement of $\{U_\alpha : \alpha \in \Delta\}$, say $\{U^*_\alpha : \alpha \in \Delta\}$. Now, if $\bigcup_{\alpha \in \Delta} U^*_\alpha = X$, there is nothing to prove. But if $\bigcup_{\alpha \in \Delta} U^*_\alpha \neq X$, then $X - \bigcup_{\alpha \in \Delta} U^*_\alpha$ is a τ_1-closed proper subset contained in $\bigcup \{V_\beta : \beta \in \Gamma\}$. So there is a τ_2-point finite parallel refinement $\left\{V^*_\beta : \beta \in \Gamma\right\}$ of $\{V_\beta : \beta \in \Gamma\}$. Therefore, $\{U^*_\alpha : \alpha \in \Delta\} \cup \left\{V^*_\beta : \beta \in \Gamma\right\}$ is a pairwise point finite parallel refinement of $\tilde{U}$. Hence, (X, τ_1, τ_2) is $P-$— metacompact. $\square$

3. Locally $p-$ Metacompact Spaces

In this section we introduce the concept of locally p-metacompactness in bitopological spaces, and prove several properties of these spaces. We will give some examples illustrating its relations with other spaces.

Definition 3.1. *[17] If (X, τ_1, τ_2) is a bitopological space, then τ_1 is said to be locally compact with respect to τ_2 if each point of X has a τ_1 open neighborhood whose τ_2 closure is pairwise compact.*

A bitopological space (X, τ_1, τ_2) is said to be pairwise locally compact if it is $\tau_1$locally compact with respect to $\tau_2$and $\tau_2$locally compact with respect to τ_1.

Note that every $p-$compact space is $p-$locally compact.

Definition 3.2. *If (X, τ_1, τ_2) is a bitopological space, then τ_1 is said to be locally metacompact with respect to τ_2 if each point of X has a τ_1 open neighborhood whose τ_2 closure is pairwise metacompact.*

A bitopological space (X, τ_1, τ_2) is said to be pairwise locally metacompact if it is $\tau_1$locally metacompact with respect to $\tau_2$and $\tau_2$locally metacompact with respect to τ_1.

Example 3.3. *The bitopological spaces $(\mathbb{R}, \tau_{ind}, \tau_{dis})$ and $(\mathbb{R}, \tau_{cof}, \tau_{dis})$ are $p-$locally metacompact.*

Theorem 3.4. *If a bitopological space (X, τ_1, τ_2) is $p-$metacompact and A is a subset of X which is τ_1 closed, then it is $p-$metacompact. If moreover A is a proper subset of X, then A is also τ_2 metacompact.*

Proof. Let $\tilde{U}$ be any $p-$open cover of the subspace $(A, \tau_1^*.\tau_2^*)$ where $\tau_1^* = \{U \cap A : U \in \tau_1\}$ and $\tau_2^* = \{V \cap A : U \in \tau_2\}$. Then $\tilde{U} \cup \{X - A\}$ is a $p-$open cover of the $p-$metacompact space (X, τ_1, τ_2) which has a pairwise point finite open parallel refinement for X and hence $\tilde{U}$ for A . □

Corollary 3.5. *Every $p-$metacompact space is $p-$locally metacompact.*

Proof. we show that τ_1 is locally metacompact with respect to τ_2. Let $x \in X$ and U be any τ_1-open neighborhood of x. Then CL_2U is a τ_2- closed proper subset of a $p-$metacompact space X . So CL_2U is $p-$metacompact. Hence the result. Similarly we show that τ_2 is locally metacompact with respect to τ_1. Thus (X, τ_1, τ_2) is $p-$locally metacompact. □

The following example shows that the converse of the above Theorem needs not be true.

Example 3.6. *The bitopological space $(\mathbb{R}, \tau_l, \tau_r)$ is $p-$locally metacompact, but not $p-$ metacompact.*

The following definitions can be found in [10] and [19].

Definition 3.7. *A space (X, τ_1, τ_2) is said to be $P-$Hausdorff if for every distinct points x and y, there is a τ_1-neighborhood U of x and a τ_2-neighborhood V of y such that $U \cap V = \phi$.*

A space (X, τ_1, τ_2), τ_1 is said to be regular with respect to τ_2 if for each point $x \in X$ and each τ_1- closed set F such that $x \notin F$, there is τ_1-open set U and a τ_2-open set V such that $x \in U$, $F \subset V$ and $U \cap V = \phi$.

A space (X, τ_1, τ_2) is said to be $p-$regular if, τ_1 is regular with respect to τ_2 and vice versa.

Theorem 3.8. *[17] A bitopological space (X, τ_1, τ_2) is $p-$regular if for each point $x \in X$ and each τ_1-open set U containing x, there exists a τ_1-open set V containing x, such that $x \in V \subset CL_2V \subset U$.*

Theorem 3.9. *If (X, τ_1, τ_2) is $p-$regular, then the following statements are equivalent:*

(a) τ_1 is locally metacompact with respect to τ_2.

(b) For every point $x \in X$ and each τ_1- open set U containing x, there is a τ_1- open set V such that $x \in V \subset CL_2V \subset U$ and CL_2V is $p-$metacompact.

Proof. We only need to show that $(a) \Longrightarrow (b)$. Let $x \in X$ and U be a τ_1- open set containing x. So there is a τ_1- open set W containing x such that $W^* = CL_2W$ is $p-$metacompact. The subspace $(W^*, \tau_1^*.\tau_2^*)$ where $\tau_1^* = \{U \cap W : U \in \tau_1\}$ and $\tau_2^* = \{V \cap W : U \in \tau_2\}$ is $p-$metacompact and $p-$regular, thus there is a τ_1^* open set G such that $x \in G \subset CL_2G \subset U \cap W$. Now $G = W^* \cap E$ for some τ_1-open subset E of X . Let $V = E \cap W$. Then V is a τ_1-open set containing x and $x \in V \subset CL_2V \subset (CL_2V) \cap W^* = CL_2^*V$. So, CL_2V is $p-$metacompact. Moreover, $x \in V \subset CL_2V \subset CL_2^*G \subset U$. □

Theorem 3.10. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be an onto, $p-$continuous, $p-$open function. If (X, τ_1, τ_2) is $p-$locally metacompact, then (Y, σ_1, σ_2) is so.*

Proof. First we show that σ_1 is locally metacompact with respect to σ_2. Let $y \in Y$. Then $f^{-1}(y) \in X$, since (X, τ_1, τ_2) is $p-$locally metacompact, there is a τ_1-open set U containing $f^{-1}(y)$ such that CL_2U is $p-$metacompact. Now, since $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is $p-$open, then $f(U)$ is a σ_1-open subset of Y and $y \in f(U)$. Since $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is onto $p-$continuous,

$f\,(CL_2U)$ is $p-$metacompact. Thus $y \in f\,(U) \subset CL_2 f\,(U) \subset f\,(CL_2U)$ and $f\,(CL_2U)$ is $p-$metacompact. Similarly we show that σ_2 is locally metacompact with respect to σ_1. So (Y, σ_1, σ_2) is $p-$locally metacompact. □

Theorem 3.11. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a $p-$perfect function. Then (X, τ_1, τ_2) is $p-$locally metacompact if (Y, σ_1, σ_2) is so.*

References

[1] T. Birsan, Compacite dans les espaces bitopologiques, An. st. univ. Iasi, s.I.a. Matematica,**15** (1969), 317-328.

[2] I. E. Cooke and I. L.Reilly, On bitopological compactness, J. London Math. Soc. **(2), 9** (1975), 518-522. (DOI: 10.1112/jlms/s2-9.4.518)

[3] Datta, M. C. Projective bitopological spaces, J. Austral Math. Soc.13 (1972), 327-334 . (doi.org/10.1017/S1446788700013744)

[4] R. Engleking, General Topology, Heldermann Verlag Berlin, (1989).

[5] P. Fletcher, H.B. Hoyle and C.W. Patty , The comparison of topologies, Duke Math. J. **36** (1969), 325-331.

[6] A. Fora and H. Hdeib, On pairwise Lindelöf spaces, Revista Colombiana de Matematicas, **XVII** (1983), 37-58.

[7] M. Ganster, and I. L. Reilly, On pairwise paracompactness, J. Austral, Math, Soc, (Series A) **53** (1992), 281-285 . (Doi:10.1017/S1446788700035850)

[8] H. Z. Hdeib, Contribution to the theory of [n , m]-compact, paracompact and normal spaces, Ph. D. Thesis State University of Buffalo (1979).

[9] H. Hdeib, A. and Fora, On pairwise paracompact spaces, Dirasat, **IX(2)** (1982), 21-29.

[10] J. C. Kelly, Bitopological spaces, Proc. London Math. Soc.,**13** (1963) 71-89. (doi.org/10.1112/plms/s3-13.1.71)

[11] A. Killiman and Z. Salleh, Product properties for pairwise Lindelöf spaces, Buil Malays, Math. Soc., **34 (2)** (2011), 216-231.

[12] Y. W. Kim, Pairwise compactness, Publ. Math. Debrecen,**15** (1968) 87-90.

[13] M. M. Kovar, A note on Raghavan-Reilly′s pairwise paracompactness, Internat. J. Math. Sci (2000), 139-143 . (S0161171200002921)

[14] E. P. Lan, Bitopological spaces and quasi-uniform spaces, Proc. London Math. **17 (3),** (1967), 241-256.

[15] D. H. Pahk, and B. D. Choi, Notes on pairwise compactness, Kyungpook Math. J. , **11** (1971), 45-52.

[16] T. G. Raghvan and I. L. Reilly, A new topological paracompactness, J. Austral. Math.Soc. (Series) **41** (1986), 268-274. (Doi: 10.1017/S144678870003367X)

[17] Reilly, I. Bitopological local compactness, Indagationes Mathematicae (proceedings) **75 (5),** (1972), 407 - 4011. (Doi: 10.1016/1385-7258(72)90037-6)

[18] A. Srivastava, and T. Bhatia, On pairwise R-compact bitopological spaces, Bull. Cal. Math. Soc. **98 (2),** (2006), 93-96.

[19] J. D. Weston, On the comparison of topologies, J. London Math. Soc. **32** (1957) 342-354. (Doi: 10.1112/jlms/s1-32.3.342)