ON METACOMPACTNESS IN BITOPOLOGICAL SPACES

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Abstract: In this paper we define pairwise metacompact spaces and pairwise locally metacompact spaces, and study their properties and their relations with other topological spaces. Several examples are discussed and many will known theorems are generalized concerning metacompact spaces.

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1. Introduction

In 1963, Kelly [10] introduced the notion of a bitopological space, i.e. a triple \((X, \tau_1, \tau_2)\) where \(X\) is a set and \(\tau_1, \tau_2\) are two topologies on \(X\). He also defined pairwise regular \((P-regular)\), pairwise normal \((P-normal)\), and obtained generalization of several standard results such as Urysohn’s lemma and Tietze extension theorem. Several authors have since considered the problem of defining compactness for such spaces, see Kim [12], Fletcher, Hoyle and Patty [5]. In 1969, Fletcher et. al. [5] gave the definitions of \(\tau_1 \tau_2-\text{open}\) and \(P-\text{open}\) covers in bitopological spaces. A cover \(\hat{U}\) of the bitopological space \((X, \tau_1, \tau_2)\) is called \(\tau_1 \tau_2-\text{open}\) if \(\hat{U} \subset \tau_1 \cup \tau_2\), if in addition, \(\hat{U}\) contains at least one non-empty mem-
ber of \( \tau_1 \) and at least one non-empty member of \( \tau_2 \), it is called \( P \)-open. Also they defined the concept of \( P \)-compact space as follows: A bitopological space \((X, \tau_1, \tau_2)\) is called \( P \)-compact if every \( P \)-open cover of the space \((X, \tau_1, \tau_2)\) has a finite subcover. While in 1972 Datta [3], defined \( S \)-compact space as follows: A bitopological space \((X, \tau_1, \tau_2)\) is called \( S \)-compact if every \( \tau_1 \tau_2 \)-open cover of the space \((X, \tau_1, \tau_2)\) has a finite subcover. In 1969 Birsan [1] gave the following definitions: A bitopological space \((X, \tau_1, \tau_2)\) is called \( \tau_1 \)-compact with respect to \( \tau_2 \) if for each \( \tau_1 \)-open cover of \( X \), there is a finite \( \tau_2 \)-open subcover. A bitopological space \((X, \tau_1, \tau_2)\) is called \( B \)-compact if it is \( \tau_1 \)-compact with respect to \( \tau_2 \) and \( \tau_2 \)-compact with respect to \( \tau_1 \). In 1975 Cooke and Reilly [2] discussed the relations between these definitions. In 1979 Hdieb [8] introduced important theorem to the theory of \([n, m] \)-compact, paracompact and normal spaces. In 1983 Fora and Hdieb [6] introduced the definition of \( P \)-Lindelöf, \( S \)-Lindelöf, \( B \)-Lindelöf spaces in analogue manner. They also gave the definitions of certain types of functions as follows: A function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is called \( P \)-continuous (\( P \)-open, \( P \)-closed, \( P \)-homeomorphism, respectively), if both functions \( f_1 : (X, \tau_1) \rightarrow (Y, \sigma_1) \) and \( f_2 : (X, \tau_2) \rightarrow (Y, \sigma_2) \) are continuous (open, closed, homeomorphism, respectively).

In this paper we introduce the notion of metacompact spaces in bitopological spaces and derive some related results. When \((X, \tau_1, \tau_2)\) has the property \( Q \) this means that both \((X, \tau_1)\) and \((X, \tau_2)\) have this property. For instance a bitopological space \((X, \tau_1, \tau_2)\) is called metacompact, if both \((X, \tau_1)\) and \((X, \tau_2)\) are metacompact spaces.

We will use the letters \( P \)-, \( S \)- to denote the pairwise and semi, respectively, e.g. \( P \)-metacompact stands for pairwise metacompact, \( S \)-metacompact stands for semi metacompact.

\( \tau_i \)-closure, \( \tau_i \)-interior of a set \( A \) will be denoted by \( CL_iA \), \( Int_iA \) respectively. The product of \( \tau_1 \) and \( \tau_2 \) will be denoted by \( \tau_1 \times \tau_2 \).

Let \( \mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{Q} \) denote the set of all real numbers, integer numbers, natural numbers, and rational numbers respectively. Let \( \tau_{\text{dis}}, \tau_{\text{ind}}, \tau_{\text{us}}, \tau_{\text{s}}, \tau_{\text{coc}}, \tau_{\text{cof}}, \tau_l, \tau_r \) denote the discrete, the indiscrete usual, Sorgenfrey line, cocompact, cofinite, left-ray, and right-ray topologies respectively. Let \( \omega_0 \) and \( \omega_1 \) denote the cardinal numbers of \( \mathbb{Z} \) and \( \mathbb{R} \) respectively.
2. Pairwise Metacompact Spaces

In this section, we will introduce the concept of metacompactness in bitopological spaces, and introduce some of their properties, and relate it to other spaces.

Let us recall known definitions which will be used in the sequel.

**Definition 2.1.** [4] A pairwise open cover $\tilde{V}$ of a bitopological space $(X, \tau_1, \tau_2)$ is called parallel refinement of of pairwise open cover $\tilde{U}$ of $X$ if each $\tau_i$—open set of $\tilde{V}$ is contained in some $\tau_i$—open set of $\tilde{U}$ $(i = 1, 2)$.

**Definition 2.2.** A pairwise open cover $\tilde{U}$ of the bitopological space $(X, \tau_1, \tau_2)$ is called pairwise point finite if each $x \in X$ is contained in a finite number of of $\tau_1$—open members of $\tilde{U}$, or it contained in a finite number of of $\tau_2$—open members of $\tilde{U}$.

**Definition 2.3.** A bitopological space $(X, \tau_1, \tau_2)$ is called $P$—metacompact if every $P$—open cover of the space $(X, \tau_1, \tau_2)$ has a pairwise point finite parallel refinement.

A bitopological space $(X, \tau_1, \tau_2)$ is called $S$—metacompact if every $\tau_1 \tau_2$—open cover of the space $(X, \tau_1, \tau_2)$ has a pairwise point finite parallel refinement.

A bitopological space $(X, \tau_1, \tau_2)$ is called $\tau_1$—metacompact with respect to $\tau_2$ if for each $\tau_1$—open cover of $X$ has a point finite $\tau_2$—parallel refinement.

A bitopological space $(X, \tau_1, \tau_2)$ is called $B$—metacompact, if it is $\tau_1$—metacompact with respect to $\tau_2$ and $\tau_2$—metacompact with respect to $\tau_1$.

If $(X, \tau_1)$ and $(X, \tau_2)$ have the same topological property $P$, then we simply say that the bitopological space $(X, \tau_1, \tau_2)$ has property $P$, without referring to $\tau_1$or $\tau_2$.

So, we will say that a bitopological space $(X, \tau_1, \tau_2)$ is called metacompact, if both $(X, \tau_1)$ and $(X, \tau_2)$ are metacompact.

**Example 2.4.** (1) The bitopological space $(\mathbb{R}, \tau_{cof}, \tau_{dis})$ is Hausdorff, since both $(\mathbb{R}, \tau_{cof})$ and $(\mathbb{R}, \tau_{dis})$ are Hausdorff.

(2) The bitopological space $(\mathbb{R}, \tau_{cof}, \tau_u)$ is $P$—metacompact, $B$—metacompact, and metacompact space since both $(\mathbb{R}, \tau_{cof})$ and $(\mathbb{R}, \tau_u)$ are metacompact.

(3) The bitopological spaces $(\mathbb{R}, \tau_{cof}, \tau_{dis})$ is $P$—metacompact, $B$—metacompact, and metacompact space since both $(\mathbb{R}, \tau_{cof})$ and $(\mathbb{R}, \tau_{dis})$ are metacompact.

(4) The bitopological spaces $(\mathbb{N}, \tau_{dis}, \tau_{dis})$ is $P$—metacompact, $B$—metacompact, metacompact space since $(\mathbb{R}, \tau_{dis})$ is metacompact.

**Theorem 2.5.** The bitopological space $(X, \tau_1, \tau_2)$ is $S$—metacompact if and only if it is metacompact and $P$—metacompact.
Proof. \( \implies \) Assume that \( (X, \tau_1, \tau_2) \) is \( S \)-metacompact. Let \( \tilde{U} \) be a \( P \)-open cover of \( X \), then \( \tilde{U} \) is a \( \tau_1 \tau_2 \)-open cover of the space \( (X, \tau_1, \tau_2) \). Since \( (X, \tau_1, \tau_2) \) is \( S \)-metacompact, \( \tilde{U} \) has a pairwise point finite parallel refinement. Hence \( (X, \tau_1, \tau_2) \) is \( P \)-metacompact. Also any \( \tau_1 \)-open or \( \tau_2 \)-open cover of \( (X, \tau_1, \tau_2) \) is a \( \tau_1 \tau_2 \)-open cover. Hence \( (X, \tau_1) \) and \( (X, \tau_2) \) are metacompact. So \( (X, \tau_1, \tau_2) \) is metacompact.

\( \impliedby \) Assume that \( (X, \tau_1, \tau_2) \) is metacompact and \( P \)-metacompact. Let \( \tilde{U} \) be a \( \tau_1 \tau_2 \)-open cover of \( (X, \tau_1, \tau_2) \). If \( \tilde{U} \) is a \( P \)-open cover, then the result follows. If \( \tilde{U} \) is not \( P \)-open cover, then it is \( \tau_1 \)-open or \( \tau_2 \)-open cover of \( (X, \tau_1, \tau_2) \). Since \( (X, \tau_1, \tau_2) \) is metacompact, \( \tilde{U} \) has a pairwise point finite parallel refinement. So \( (X, \tau_1, \tau_2) \) is \( S \)-metacompact. \( \Box \)

Recall that a space has a hereditary property \( P \), if every subspace of it has this property.

**Theorem 2.6.** If a bitopological space \( (X, \tau_1, \tau_2) \) is hereditary metacompact, then it is \( S \)-metacompact.

Proof. Let \( \tilde{U} \) be a \( \tau_1 \tau_2 \)-open cover of \( (X, \tau_1, \tau_2) \). Then \( \tilde{U} = \{U_\alpha : \alpha \in \Delta\} \cup \{V_\beta : \beta \in \Gamma\} \), where \( U_\alpha \in \tau_1 \) for each \( \alpha \in \Delta \) and \( V_\beta \in \tau_2 \) for each \( \beta \in \Gamma \).

Since \( U = \bigcup \{U_\alpha : \alpha \in \Delta\} \) is \( \tau_1 \)-metacompact, it has a point finite open parallel refinement say \( \{U^*_\alpha : \alpha \in \Delta\} \) and \( U = \bigcup_{\alpha \in \Delta} U^*_\alpha \).

Now let \( V = \bigcup \{V_\beta : \beta \in \Gamma\} \), since \( V \) is \( \tau_2 \)-metacompact, it has a point finite open parallel refinement say \( \{V^*_\beta : \beta \in \Gamma\} \) and \( V = \bigcup_{\beta \in \Gamma} V^*_\beta \).

Hence \( \{U^*_\alpha : \alpha \in \Delta\} \cup \{V^*_\beta : \beta \in \Gamma\} \) is a point finite \( \tau_1 \tau_2 \)-open parallel refinement of \( \tilde{U} \). \( \Box \)

The following definitions can be found in [6].

**Definition 2.7.** A bitopological space \( (X, \tau_1, \tau_2) \) is called \( P \)-Lindelöf if every \( P \)-open cover of the space \( (X, \tau_1, \tau_2) \) has a countable subcover.

A bitopological space \( (X, \tau_1, \tau_2) \) is called \( S \)-Lindelöf if every \( \tau_1 \tau_2 \)-open cover of the space \( (X, \tau_1, \tau_2) \) has a countable subcover.

A bitopological space \( (X, \tau_1, \tau_2) \) is called \( \tau_1 \)-Lindelöf with respect to \( \tau_2 \) if for each \( \tau_1 \)-open cover of \( X \), there is a countable \( \tau_2 \)-open subcover.

A bitopological space \( (X, \tau_1, \tau_2) \) is called \( B \)-Lindelöf if it is \( \tau_1 \)-Lindelöf with respect to \( \tau_2 \) and \( \tau_2 \)-Lindelöf with respect to \( \tau_1 \).

**Example 2.8.** Consider the two topologies \( \tau_1 \) and \( \tau \) on \( \mathbb{R} \) defined by the basis:
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\[ \beta_1 = \{(-\infty, a) : a > 0\} \cup \{\{x\} : x > 0\} \]
\[ \beta_2 = \{(a, \infty) : a < 0\} \cup \{\{x\} : x < 0\} \]

Then \( X \) is \( P \)-metacompact but not a \( B \)-metacompact, since for the \( \tau_1 \)-open cover \((-\infty, 2) \cup \{\{x\} : x > 1\}\) of \( \mathbb{R} \) has no point finite \( \tau_2 \)-open refinement.

It is clear that both \((\mathbb{R}, \tau_1)\) and \((\mathbb{R}, \tau_2)\) are metacompact spaces, so \((\mathbb{R}, \tau_1, \tau_2)\) is metacompact. Observe that \((\mathbb{R}, \tau_1, \tau_2)\) is not \( S \)-metacompact.

On the other hand we have

\((\mathbb{R}, \tau_1, \tau_2)\) is countably \( P \)-metacompact. Now we observe that \((\mathbb{R}, \tau_1, \tau_2)\) is \( P \)-Lindelöf, it is also clear that \((\mathbb{R}, \tau_1, \tau_2)\) is not \( B \)-Lindelöf.

**Definition 2.9.** A subset \( D \) of a bitopological space \((X, \tau_1, \tau_2)\) is called pairwise dense denoted by \( \text{(P—dense)} \) in \( X \), if \( \text{CL}_{\tau_1} D = \text{CL}_{\tau_2} D = X \).

A bitopological space \((X, \tau_1, \tau_2)\) is called \( P \)-separable, if it has a \( P \)-dense countable subset \( D \).

It is clear that the bitopological space \((\mathbb{R}, \tau_{coc}, \tau_{cof})\) is \( P \)-separable.

**Theorem 2.10.** A \( P \)-separable, \( P \)-metacompact space \((X, \tau_1, \tau_2)\) is \( P \)-Lindelöf.

**Proof.** Let \( \tilde{U} = \{U_\alpha : \alpha \in \Delta\} \) be a \( P \)-open cover of \( X \). Assume that \( \tilde{U} \) has no countable subcover of \( X \). Let \( \tilde{V} = \{V_\beta : \beta \in \Gamma\} \) be a point finite parallel refinement of \( \tilde{U} \). Let \( D \) be a countable dense subset of \( X \). Then \( V_\beta \cap D \neq \emptyset \) for each \( \beta \in \Gamma \). Thus \( D \) is an uncountable set because \( \tilde{V} \) is uncountable, which is a contradiction. Hence the result.

**Definition 2.11.** A bitopological space \((X, \tau_1, \tau_2)\) is called countably \( P \)-metacompact if every countable \( P \)-open cover of the space \((X, \tau_1, \tau_2)\) has a pairwise point finite parallel refinement.

A bitopological space \((X, \tau_1, \tau_2)\) is called countably \( S \)-metacompact if every countable \( \tau_1 \tau_2 \)-open cover of the space \((X, \tau_1, \tau_2)\) has a pairwise point finite parallel refinement.

A bitopological space \((X, \tau_1, \tau_2)\) is called countably \( \tau_1 \)-metacompact with respect to \( \tau_2 \) if for each countable \( \tau_1 \)-open cover of \( X \) has a point finite \( \tau_2 \)-parallel refinement.

A bitopological space \((X, \tau_1, \tau_2)\) is called countably \( B \)-metacompact, if it is countably \( \tau_1 \)-metacompact with respect to \( \tau_2 \) and countably \( \tau_2 \)-metacompact with respect to \( \tau_1 \).

A bitopological space \((X, \tau_1, \tau_2)\) is called countably metacompact, if it is countably \( \tau_1 \)-metacompact and countably \( \tau_2 \)-metacompact.
**Example 2.12.** (1) The bitopological space \((\mathbb{N}, \tau_{\text{dis}}, \tau_{\text{ind}})\) is \(P\)–metacompact, \(B\)–metacompact, metacompact space since \((\mathbb{N}, \tau_{\text{dis}})\) and \((\mathbb{N}, \tau_{\text{ind}})\) are metacompact. It is also countably \(P\)–metacompact, countably \(B\)–metacompact. 

(2) The bitopological space \((\mathbb{R}, \tau_{\text{dis}}, \tau_u)\) is \(P\)–metacompact, \(B\)–metacompact, metacompact space since \((\mathbb{R}, \tau_{\text{dis}})\) and \((\mathbb{R}, \tau_u)\) are metacompact. It is also countably \(P\)–metacompact, countably \(B\)–metacompact. It is not \(P\)–separable.

It is clear that \((\mathbb{R}, \tau_{\text{dis}}, \tau_u)\) is neither \(P\)–Lindelöf nor \(P\)–compact.

**Theorem 2.13.** Every \(P\)–Lindelöf countably \(P\)–metacompact space \((X, \tau_1, \tau_2)\) is \(P\)–metacompact space.

**Proof.** Let \(\tilde{U} = \{U_\alpha : \alpha \in \Delta\}\) be a \(P\)–open cover of \(X\). Since \(X\) is \(P\)–Lindelöf, \(\tilde{U}\) has a countable subcover \(\tilde{V} = \{V_{a_i}\}_{i=1}^\infty\). Since \(X\) is countably \(P\)–metacompact, \(\tilde{V}\) has a point finite parallel refinement \(\tilde{W}\) of \(\tilde{U}\). Hence \((X, \tau_1, \tau_2)\) is \(P\)–metacompact.

it is clear that the bitopological space \((\mathbb{Z}, \tau_{\text{dis}}, \tau_{\text{ind}})\) is \(P\)–metacompact, since it is countably \(P\)–metacompact and \(P\)–Lindelöf.

**Definition 2.14.** A space \((X, \tau_1, \tau_2)\) is said to be \(p\)–metalindelöf if every \(p\)–open cover of \((X, \tau_1, \tau_2)\) has a pairwise point countable parallel refinement.

**Theorem 2.15.** Every \(P\)–metalindelöf countably \(P\)–metacompact space \((X, \tau_1, \tau_2)\) is \(P\)–metacompact space.

**Proof.** Let \(\tilde{U} = \{U_\alpha : \alpha \in \Delta\}\) be a \(P\)–open cover of \(X\). Since \(X\) is \(P\)–metalindelöf, \(\tilde{U}\) has a point countable parallel refinement \(\tilde{V} = \{V_{a_i}\}_{i=1}^\infty\), which is also a \(p\)–open cover of \((X, \tau_1, \tau_2)\). Since \(X\) is countably \(P\)–metacompact, \(\tilde{V}\) has a point finite parallel refinement \(\tilde{W}\) of \(\tilde{U}\). Hence \((X, \tau_1, \tau_2)\) is \(P\)–metacompact.

The following theorem is easily proved.

**Theorem 2.16.** Every metaLindelöf countably metacompact space \((X, \tau_1, \tau_2)\) is metacompact space.

In [4], Engelking defined a cover \(\{A_s : s \in S\}\) of a space \(X\) to be irreducible if \(\bigcup_{s \in S_0} A_s \neq X\) for every proper subset \(S_0\) of the set \(S\).

**Theorem 2.17.** [4] Every point finite cover \(\{A_s : s \in S\}\) of a space \(X\) has an irreducible subcover.

The following definitions can be found in [1], [3] and [5].
**Definition 2.18.** A bitopological space \((X, \tau_1, \tau_2)\) is called \(P\)–compact if every \(P\)–open cover of the space \((X, \tau_1, \tau_2)\) has a finite subcover.

A bitopological space \((X, \tau_1, \tau_2)\) is called \(S\)–compact if every \(\tau_1\tau_2\)–open cover of the space \((X, \tau_1, \tau_2)\) has a finite subcover.

A bitopological space \((X, \tau_1, \tau_2)\) is called \(\tau_1\)–compact with respect to \(\tau_2\) if for each \(\tau_1\)–open cover of \(X\), there is a finite \(\tau_2\)–open subcover.

A bitopological space \((X, \tau_1, \tau_2)\) is called \(B\)–compact if it is \(\tau_1\)–compact with respect to \(\tau_2\) and \(\tau_2\)–compact with respect to \(\tau_1\).

**Example 2.19.** Let \(X = \mathbb{R}, \beta_1 = \{X, \{x\} : x \in X - \{1\}\}, \beta_2 = \{X, \{x\} : x \in X - \{1\}\}\). Let \(\tau_1\) and \(\tau_2\) be the topologies on \(X\) which are generated by the bases \(\beta_1\) and \(\beta_2\), respectively. Then \((\mathbb{R}, \tau_1, \tau_2)\) is \(B\)–metacompact, since any \(\tau_1\)–open cover or any \(\tau_2\)–open cover of \(X\) must contain \(X\) as a member. It is also \(P\)–metacompact and countably \(P\)–metacompact, countably \(B\)–metacompact. On the other hand, \((\mathbb{R}, \tau_1, \tau_2)\) is \(B\)–Lindelöf, however it is not \(P\)–Lindelöf, since the \(P\)–open cover \(\{\{x\} : x \in X\}\) of \(X\) has no countable subcover. It is clear that \((\mathbb{R}, \tau_1, \tau_2)\) is not \(P\)–compact, since the \(P\)–open cover \(\{\{x\} : x \in X\}\) of \(X\) has no finite subcover, it is not compact space, but it is \(B\)–compact, since any \(\tau_1\)–open cover or any \(\tau_2\)–open cover of \(X\) must contain \(X\) as a member and \(\{x\}\) is a finite subcover of any \(\tau_1\)–open cover or any \(\tau_2\)–open cover of \(X\).

The following definitions can be found in [15] and [6].

**Definition 2.20.** A bitopological space \((X, \tau_1, \tau_2)\) is called \(P\)–countably compact if every countable \(P\)–open cover of the space \((X, \tau_1, \tau_2)\) has a finite subcover.

A bitopological space \((X, \tau_1, \tau_2)\) is called \(S\)–countably compact if every countable \(\tau_1\tau_2\)–open cover of the space \((X, \tau_1, \tau_2)\) has a finite subcover.

A bitopological space \((X, \tau_1, \tau_2)\) is called \(\tau_1\)–countably compact with respect to \(\tau_2\) if for each countable \(\tau_1\)–open cover of \(X\), there is a finite \(\tau_2\)–open subcover.

A bitopological space \((X, \tau_1, \tau_2)\) is called \(B\)–countably compact if it is countably \(\tau_1\)–compact with respect to \(\tau_2\) and countably \(\tau_2\)–compact with respect to \(\tau_1\).

**Example 2.21.** (1) The bitopological space \((\mathbb{R}, \tau_{\text{dis}}, \tau_{\text{coc}})\) is \(P\)–metacompact, not \(B\)–metacompact, since the \(\tau_1\)–open cover \(\{\{x\} : x \in \mathbb{R}\}\) of \(\mathbb{R}\) has no point finite \(\tau_2\)–open refinement. It is also countably \(P\)–metacompact. It is clear that \((\mathbb{R}, \tau_{\text{dis}}, \tau_{\text{coc}})\) is \(P\)–Lindelöf space which is neither \(P\)–countably compact, nor \(P\)–compact.

(2) Let \(\tau_s\) denotes the Sorgenfrey line topology on \(\mathbb{R}\). Then the bitopological
space \((\mathbb{R}, \tau_s, \tau_u)\) is \(S\)-Lindelöf, so it is \(P\)-Lindelöf and Lindelöf. Also it is \(P\)-separable, so \((\mathbb{R}, \tau_s, \tau_u)\) is \(P\)-metacompact. It is clear that \((\mathbb{R}, \tau_s, \tau_u)\) is not \(B\)-metacompact, since the \(\tau_u\) open cover \(\{(n, n+1) : n \in \mathbb{Z}\}\) of \(\mathbb{R}\) has no point finite \(\tau_s\)-open refinement, because \(\tau_s \not\subseteq \tau_u\). It is also clear that \((\mathbb{R}, \tau_s, \tau_u)\) is neither \(S\)-compact nor \(S\)-countably compact.

**Theorem 2.22.** Every countably compact metacompact bitopological space \((X, \tau_1, \tau_2)\) is compact.

**Proof.** It is sufficient to show that both topological spaces \((X, \tau_1)\) and \((X, \tau_2)\) are compact spaces. Let \(\hat{U} = \{U_\alpha : \alpha \in \Delta\}\) be an open cover of a countably compact metacompact space \((X, \tau_1)\). So there exists an irreducible point finite open refinement \(\hat{V} = \{V_\alpha : \alpha \in \Delta\}\) of the cover \(\hat{U}\). The cover \(\hat{V} = \{V_\alpha : \alpha \in \Delta\}\) being irreducible, for every \(\alpha \in \Delta\), there exists a point \(x_\alpha \in V_\alpha \setminus \bigcup_{\alpha \neq \alpha^*} V_{\alpha^*}\). Since the sets \(V_\alpha\) cover the space \((X, \tau_1)\), every point \(x \in X\) has a neighborhood which contains exactly one point of the set \(A = \{X_\alpha : \alpha \in \Delta\}\).

Hence, the derived set of \(A\) dented by \(A^d = \phi\). Since \((X, \tau_1)\) is countably compact, the set \(A\) is finite. Thus the set \(\Delta\) is also finite and the open cover \(\hat{V}\) is a finite open refinement of \(\hat{U}\) for \((X, \tau_1)\). So \((X, \tau_1)\) is compact. Similarly, we show that \((X, \tau_2)\) is compact. Hence \((X, \tau_1, \tau_2)\) is compact. \(\square\)

**Definition 2.23.** [6] A function \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is called \(P\)-continuous (\(P\)-open, \(P\)-closed, \(P\)-homeomorphism, respectively) if the functions \(f : (X, \tau_1) \to (Y, \sigma_1)\) and \(f : (X, \tau_2) \to (Y, \sigma_2)\) are continuous (open, closed, homeomorphism, respectively).

**Definition 2.24.** A function \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is called \(P\)-perfect, if the function \(f\) is \(P\)-continuous, \(P\)-closed, and for all \(y \in Y\), the set \(f^{-1}(y)\) is \(P\)-compact.

**Theorem 2.25.** Let \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) be a \(P\)-perfect function. Then \(X\) is \(P\)-metacompact space if \(Y\) is so.

**Proof.** Let \(\hat{U} = \{U_\alpha : \alpha \in \Delta\} \cup \{V_\beta : \beta \in \Gamma\}\) be any \(P\)-open cover of \(X\), where \(\{U_\alpha : \alpha \in \Delta\}\) is a set of \(\tau_1\)-open members of \(\hat{U}\) and \(\{V_\beta : \beta \in \Gamma\}\) is a set of \(\tau_2\)-open members of \(\hat{U}\).

Now, since \(f\) is \(P\)-perfect, for every \(y \in Y\) we have \(f^{-1}(y)\) is \(P\)-compact subset of \(X\). So there exist finite subsets \(\Delta_1\) and \(\Delta_2\) of \(\Delta\) and \(\Gamma\) respectively such that \(f^{-1}(y) \subseteq \bigcup_{i=1}^{n} U_i : i \in \Delta_1 \bigcup_{i=1}^{m} V_i : i \in \Delta_2\). Now;
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$$O_{y_1} = Y - f \left( X - \bigcup_{i=1}^{n} U_i : i \in \Delta_1 \right)$$ is a $\tau_1$ open subset of $Y$ and $f^{-1}(O_{y_1}) \subseteq \left\{ \bigcup_{i=1}^{n} U_i : i \in \Delta_1 \right\}$.

$$O_{y_2} = Y - f \left( X - \bigcup_{i=1}^{m} V_i : i \in \Delta_2 \right)$$ is a $\tau_1$ open subset of $Y$ and $f^{-1}(O_{y_2}) \subseteq \left\{ \bigcup_{i=1}^{m} V_i : i \in \Delta_2 \right\}$.

$y \in O_{y_1} \cup O_{y_2}$. So, $\tilde{O} = \left\{ \tilde{O} = \left\{ O_{y_1} : y \in Y \right\} \cup \left\{ O_{y_2} : y \in Y \right\} : y \in Y \right\}$ is a $P$ open cover of $Y$. Since $Y$ is $P$ metacompact, $\tilde{O}$ has a pairwise point finite parallel refinement

$$\tilde{O}^* = \tilde{O} = \left\{ O_{y_1}^* : y \in Y \right\} \cup \left\{ O_{y_2}^* : y \in Y \right\}.$$

Now, $O_{y_1}^*$ is a $\tau_1$ open subset of $X$ and $O_{y_2}^*$ is a $\tau_2$ open subset of $X$.

Since $f$ is perfect, the set $\left\{ f^{-1}(O_{y_1}^*) : y \in Y \right\} \cup \left\{ f^{-1}(O_{y_2}^*) : y \in Y \right\}$ is a pairwise point finite parallel refinement of $X$. So, $X$ is $P$ metacompact. \[\Box\]

**Definition 2.26.** [6] Let $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ be bitopological spaces. Then, the Cartesian product of $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ is the bitopological space $(X \times Y, \tau_1 \times \sigma_1, \tau_2 \times \sigma_2)$.

**Lemma 2.27.** [4] (Wallace lemma) If $A$ is a compact subset of a topological space $(X, \tau)$ and $B$ is a compact subset of a topological space $(Y, \sigma)$ and $A \times B \subseteq W$; where $W$ is open subset of $X \times Y$, then there exist open sets $U$ and $V$ in $X$ and $Y$ respectively such that $A \times B \subseteq U \times V \subseteq W$.

**Theorem 2.28.** If $X$ is $P$ compact, then the projection function $P : X \times Y \rightarrow Y$ is $P$ closed, where $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ are a bitopological spaces.

**Proof.** To show that the projection function $P : X \times Y \rightarrow Y$ is $P$ closed, we show that the projection functions $P_1 : (X \times Y, \tau_1 \times \sigma_1) \rightarrow (Y, \sigma_1)$ and $P_2 : (X \times Y, \tau_2 \times \sigma_2) \rightarrow (Y, \sigma_2)$ are closed. Let $y \in Y$ and let $U$ be an open set in $(X \times Y, \tau_1 \times \sigma_1)$ such that $P_1^{-1}(\{y\}) \subseteq U$. So by (Wallace lemma), there exists a $\sigma_1$ open set in $Y$ say $V_y$ such that $P_1^{-1}(\{y\}) = X \times \{y\} \subseteq X \times V_y \subseteq U$. So, $y \in V_y$ and $P_1^{-1}(\{V_y\}) = X \times \{V_y\} \subseteq U$. So, $P_1 : (X \times Y, \tau_1 \times \sigma_1) \rightarrow (Y, \sigma_1)$ is closed function. Similarly, we have $P_2 : (X \times Y, \tau_2 \times \sigma_2) \rightarrow (Y, \sigma_2)$ is closed function. Hence, the projection function $P : X \times Y \rightarrow Y$ is $P$ closed. \[\Box\]

**Theorem 2.29.** The product of a $P$ compact space $X$ and a $P$ metacompact space $Y$ is $P$ metacompact, where $(X, \tau_1, \tau_2)$ and $(Y, \sigma_1, \sigma_2)$ are bitopological spaces.
Proof. Let \( P : X \times Y \rightarrow Y \) be the projection function such that \( (x, y) \rightarrow y \). Then \( P : X \times Y \rightarrow Y \) is \( P \)-perfect function. Since \( Y \) is \( P \)-metacompact, then \( X \times Y \) is \( P \)-metacompact by Theorem 2.28.

**Example 2.30.** The bitopological space \((\mathbb{R}, \tau_{\text{cof}}, \tau_{\text{dis}})\) is \( P \)-compact so it is \( P \)-metacompact. Also it is \( B \)-metacompact, metacompact space since both \((\mathbb{R}, \tau_{\text{cof}})\) and \((\mathbb{R}, \tau_{\text{dis}})\) are metacompact. The space \((\mathbb{R}^2, \tau_{\text{cof}} \times \tau_{\text{cof}}, \tau_{\text{dis}} \times \tau_{\text{dis}})\) is also \( P \)-metacompact, but not \( P \)-compact nor \( P \)-Lindelöf, since the \( P \)-open cover \( \{ \mathbb{R} \times (\mathbb{R} - \{0\})\} \cup \{(x, 0) : x \in \mathbb{R}\}\) for \( \mathbb{R}^2 \) has no countable subcover.

**Lemma 2.31.** [4] Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a continuous, onto function. If \( A = \{A_\alpha : \alpha \in \Delta\} \) is a point finite family subset of \( X \), then \( \{f(A_\alpha) : \alpha \in \Delta\} \) is a point finite family subset of \( Y \).

**Theorem 2.32.** Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a \( P \)-continuous, \( P \)-closed, onto function. Then \( Y \) is \( P \)-metacompact, if \( X \) is so.

Proof. Let \( \tilde{V} = \{U_\alpha : \alpha \in \Delta\} \cup \{V_\beta : \beta \in \Gamma\} \) be any \( P \)-open cover of \( Y \), where \( \{U_\alpha : \alpha \in \Delta\} \) are \( \sigma_1 \)-open members of \( \tilde{V} \) and \( \{V_\beta : \beta \in \Gamma\} \) are \( \sigma_2 \)-open members of \( \tilde{V} \). Since \( f \) is \( P \)-continuous, onto function, the set
\[
\tilde{U} = \left\{ f^{-1}(U_\alpha) : \alpha \in \Delta\right\} \cup \left\{ f^{-1}(V_\beta) : \beta \in \Gamma\right\}
\]
is a \( P \)-open cover of \( X \). Since \( X \) is \( P \)-metacompact space, there exists a pairwise point finite open parallel refinement of \( \tilde{U} \), say \( \tilde{U}^* = \left\{ f^{-1}(U^*_\alpha) : \alpha \in \Delta\right\} \cup \left\{ f^{-1}(V^*_\beta) : \beta \in \Gamma\right\} \). Thus, \( \tilde{V}^* = \{U^*_\alpha : \alpha \in \Delta\} \cup \{V^*_\beta : \beta \in \Gamma\} \) is a pairwise point finite open parallel refinement of \( \tilde{V} \). So, \( Y \) is \( P \)-metacompact.

**Definition 2.33.** A pairwise open cover \( \tilde{U} \) of a bitopological space \((X, \tau_1, \tau_2)\) is called pairwise locally finite if for every \( x \in X \), there exists a \( \tau_1 \)-open neighborhood of \( x \) which meets only a finite number of \( \tau_1 \)-open elements of \( \tilde{U} \), or there exists a \( \tau_2 \)-open neighborhood of \( x \) which meets only a finite number of \( \tau_2 \)-open elements of \( \tilde{U} \).

A space \((X, \tau_1, \tau_2)\) is said to be \( P \)-paracompact if every \( P \)-open cover of \( X \) has a pairwise open locally finite parallel refinement.

A space \((X, \tau_1, \tau_2)\) is said to be \( S \)-paracompact if every \( \tau_1 \tau_2 \)-open cover of \( X \) has a pairwise open locally finite parallel refinement.

A space \((X, \tau_1, \tau_2)\) is said to be paracompact if both \((X, \tau_1)\) and \((X, \tau_2)\) are paracompact spaces.

A subset \( F \) of a space \((X, \tau)\) is said to be \( \tau \)-paracompact relative to \( X \), if every open cover of \( F \) by members of \( \tau \) has a locally finite parallel refinement in \( X \) by members of \( \tau \).
Corollary 2.34. Every $P$-paracompact ($S$-paracompact) space is $P$-metacompact ($S$-metacompact).

Theorem 2.35. For the bitopological space $(X, \tau_1, \tau_2)$, the following are equivalent:

1. $(X, \tau_1, \tau_2)$ is $P$-metacompact.

2. Each $\tau_1$-closed subset of $X$ is $\tau_2$-metacompact relative to $X$, and each $\tau_2$-closed subset of $X$ is $\tau_1$-metacompact relative to $X$.

Proof. (1) $\implies$ (2): Let $K \neq \emptyset$ be a $\tau_1$-closed subset of $X$ and $\tilde{U} = \{U_\alpha : \alpha \in \Delta\}$ be a $\tau_2$-open cover of $K$. Then $\tilde{\tau} = \{X - K\} \cup \{U_\alpha : \alpha \in \Delta\}$ is a $P$-open cover of $X$. Since $(X, \tau_1, \tau_2)$ is $P$-metacompact, $\tilde{\tau}$ has a pairwise point finite open parallel refinement, say $\{V_\beta : \beta \in \Gamma\} \cup \{U_\alpha^* : \alpha \in \Delta\}$, where $V_\beta$ is a $\tau_1$-open for each $\beta \in \Gamma$, and $U_\alpha^*$ is a $\tau_2$-open for each $\alpha \in \Delta$. Thus $\{U_\alpha^* : \alpha \in \Delta\}$ is a point finite parallel refinement of $\tilde{U}$. Hence $K$ is a $\tau_2$-metacompact relative to $X$. The proof of other case is similar.

(2) $\implies$ (1): Let $\tilde{U} = \{U_\alpha : \alpha \in \Delta\} \cup \{V_\beta : \beta \in \Gamma\}$ be a $P$-open cover of $X$, where $\{U_\alpha : \alpha \in \Delta\}$ are $\tau_1$-open members of $\tilde{U}$, and $\{V_\beta : \beta \in \Gamma\}$ are $\tau_2$-open members of $\tilde{U}$. We have two cases:

(i) If $\bigcup \{V_\beta : \beta \in \Gamma\} = X$, then, choose an $\alpha_0 \in \Delta$ such that $U_{\alpha_0} \neq \emptyset$. Since $\{V_\beta : \beta \in \Gamma\}$ is a $\tau_2$-open cover of the $\tau_1$-closed proper subset $(X - U_{\alpha_0})$ has a $\tau_2$-open refinement $\{V_\beta^* : \beta \in \Gamma\}$. Hence, $U_{\alpha_0} \cup \{V_\beta^* : \beta \in \Gamma\}$ is a pairwise point finite open parallel refinement of $\tilde{U}$. Hence $(X, \tau_1, \tau_2)$ is $P$-metacompact.

(ii) If $\bigcup \{V_\beta : \beta \in \Gamma\} \neq X$, then the set $K = X - \bigcup_{\beta \in \Gamma} V_\beta$ is a $\tau_2$-closed proper subset of $X$, and $K \subseteq \bigcup_{\alpha \in \Delta} U_\alpha$. Hence there is a $\tau_1$-point finite parallel refinement of $\{U_\alpha : \alpha \in \Delta\}$, say $\{U_\alpha^* : \alpha \in \Delta\}$. Now, if $\bigcup U_\alpha^* = X$, there is nothing to prove. But if $\bigcup U_\alpha^* \neq X$, then $X - \bigcup U_\alpha^*$ is a $\tau_1$-closed proper subset contained in $\bigcup \{V_\beta : \beta \in \Gamma\}$. So there is a $\tau_2$-point finite parallel refinement $\{V_\beta^* : \beta \in \Gamma\}$ of $\{V_\beta : \beta \in \Gamma\}$. Therefore, $\{U_\alpha^* : \alpha \in \Delta\} \cup \{V_\beta^* : \beta \in \Gamma\}$ is a pairwise point finite parallel refinement of $\tilde{U}$. Hence, $(X, \tau_1, \tau_2)$ is $P$-metacompact. \qed
3. Locally $p -$ Metacompact Spaces

In this section we introduce the concept of locally $p$-metacompactness in bitopological spaces, and prove several properties of these spaces. We will give some examples illustrating its relations with other spaces.

**Definition 3.1.** [17] If $(X, \tau_1, \tau_2)$ is a bitopological space, then $\tau_1$ is said to be locally compact with respect to $\tau_2$ if each point of $X$ has a $\tau_1$ open neighborhood whose $\tau_2$ closure is pairwise compact.

A bitopological space $(X, \tau_1, \tau_2)$ is said to be pairwise locally compact if it is $\tau_1$ locally compact with respect to $\tau_2$ and $\tau_2$ locally compact with respect to $\tau_1$.

Note that every $p-$ compact space is $p-$ locally compact.

**Definition 3.2.** If $(X, \tau_1, \tau_2)$ is a bitopological space, then $\tau_1$ is said to be locally metacompact with respect to $\tau_2$ if each point of $X$ has a $\tau_1$ open neighborhood whose $\tau_2$ closure is pairwise metacompact.

A bitopological space $(X, \tau_1, \tau_2)$ is said to be pairwise locally metacompact if it is $\tau_1$ locally metacompact with respect to $\tau_2$ and $\tau_2$ locally metacompact with respect to $\tau_1$.

**Example 3.3.** The bitopological spaces $(\mathbb{R}, \tau_{ind}, \tau_{dis})$ and $(\mathbb{R}, \tau_{cof}, \tau_{dis})$ are $p-$ locally metacompact.

**Theorem 3.4.** If a bitopological space $(X, \tau_1, \tau_2)$ is $p-$ metacompact and $A$ is a subset of $X$ which is $\tau_1$ closed, then it is $p-$ metacompact. If moreover $A$ is a proper subset of $X$, then $A$ is also $\tau_2$ metacompact.

**Proof.** Let $\tilde{U}$ be any $p-$ open cover of the subspace $(A, \tau_1^*, \tau_2^*)$ where $\tau_1^* = \{U \cap A : U \in \tau_1\}$ and $\tau_2^* = \{V \cap A : U \in \tau_2\}$. Then $\tilde{U} \cup \{X - A\}$ is a $p-$ open cover of the $p-$ metacompact space $(X, \tau_1, \tau_2)$ which has a pairwise point finite open parallel refinement for $X$ and hence $\tilde{U}$ for $A$.

**Corollary 3.5.** Every $p-$ metacompact space is $p-$ locally metacompact.

**Proof.** we show that $\tau_1$ is locally metacompact with respect to $\tau_2$. Let $x \in X$ and $U$ be any $\tau_1-$ open neighborhood of $x$. Then $CL_2U$ is a $\tau_2-$ closed proper subset of a $p-$ metacompact space $X$. So $CL_2U$ is $p-$ metacompact. Hence the result. Similarly we show that $\tau_2$ is locally metacompact with respect to $\tau_1$. Thus $(X, \tau_1, \tau_2)$ is $p-$ locally metacompact.

The following example shows that the converse of the above Theorem needs not be true.
Example 3.6. The bitopological space \((\mathbb{R}, \tau_1, \tau_f)\) is \(p\)–locally metacompact, but not \(p\)–metacompact.

The following definitions can be found in [10] and [19].

Definition 3.7. A space \((X, \tau_1, \tau_2)\) is said to be \(P\)–Hausdorff if for every distinct points \(x\) and \(y\), there is a \(\tau_1\)–neighborhood \(U\) of \(x\) and a \(\tau_2\)–neighborhood \(V\) of \(y\) such that \(U \cap V = \emptyset\).

A space \((X, \tau_1, \tau_2)\), \(\tau_1\) is said to be regular with respect to \(\tau_2\) if for each point \(x \in X\) and each \(\tau_1\)–closed set \(F\) such that \(x \notin F\), there is a \(\tau_1\)–open set \(U\) and a \(\tau_2\)–open set \(V\) such that \(x \in U\), \(F \subset V\) and \(U \cap V = \emptyset\).

A space \((X, \tau_1, \tau_2)\) is said to be \(p\)–regular if, \(\tau_1\) is regular with respect to \(\tau_2\) and vice versa.

Theorem 3.8. [17] A bitopological space \((X, \tau_1, \tau_2)\) is \(p\)–regular if for each point \(x \in X\) and each \(\tau_1\)–open set \(U\) containing \(x\), there exists a \(\tau_1\)–open set \(V\) containing \(x\), such that \(x \in V \subset CL_2V \subset U\).

Theorem 3.9. If \((X, \tau_1, \tau_2)\) is \(p\)–regular, then the following statements are equivalent:

(a) \(\tau_1\) is locally metacompact with respect to \(\tau_2\).

(b) For every point \(x \in X\) and each \(\tau_1\)–open set \(U\) containing \(x\), there is a \(\tau_1\)–open set \(V\) such that \(x \in V \subset CL_2V \subset U\) and \(CL_2V\) is \(p\)–metacompact.

Proof. We only need to show that \((a) \implies (b)\). Let \(x \in X\) and \(U\) be a \(\tau_1\)–open set containing \(x\). So there is a \(\tau_1\)–open set \(W\) containing \(x\) such that \(W^* = CL_2W\) is \(p\)–metacompact. The subspace \((W^*, \tau_1^*, \tau_2^*)\) where \(\tau_1^* = \{U \cap W : U \in \tau_1\}\) and \(\tau_2^* = \{V \cap W : U \in \tau_2\}\) is \(p\)–metacompact and \(p\)–regular, thus there is a \(\tau_1^*\)–open set \(G\) such that \(x \in G \subset CL_2G \subset U \cap W\).

Now \(G = W^* \cap E\) for some \(\tau_1\)–open subset \(E\) of \(X\). Let \(V = E \cap W\). Then \(V\) is a \(\tau_1\)–open set containing \(x\) and \(x \in V \subset CL_2V \subset (CL_2V) \cap W^* = CL_2^*V\).

So, \(CL_2V\) is \(p\)–metacompact. Moreover, \(x \in V \subset CL_2V \subset CL_2^*G \subset U\). \(\square\)

Theorem 3.10. Let \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) be an onto, \(p\)–continuous, \(p\)–open function. If \((X, \tau_1, \tau_2)\) is \(p\)–locally metacompact, then \((Y, \sigma_1, \sigma_2)\) is so.

Proof. First we show that \(\sigma_1\) is locally metacompact with respect to \(\sigma_2\). Let \(y \in Y\). Then \(f^{-1}(y) \in X\), since \((X, \tau_1, \tau_2)\) is \(p\)–locally metacompact, there is a \(\tau_1\)–open set \(U\) containing \(f^{-1}(y)\) such that \(CL_2U\) is \(p\)–metacompact. Now, since \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is \(p\)–open, then \(f(U)\) is a \(\sigma_1\)–open subset of \(Y\) and \(y \in f(U)\). Since \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is onto \(p\)–continuous,
$f(CL_2U)$ is $p-$metacompact. Thus $y \in f(U) \subset CL_2 f(U) \subset f(CL_2U)$ and $f(CL_2U)$ is $p-$metacompact. Similarly we show that $\sigma_2$ is locally metacompact with respect to $\sigma_1$. So $(Y, \sigma_1, \sigma_2)$ is $p-$locally metacompact. □

**Theorem 3.11.** Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a $p-$perfect function. Then $(X, \tau_1, \tau_2)$ is $p-$locally metacompact if $(Y, \sigma_1, \sigma_2)$ is so.

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**References**


