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# HYBRID ONE-STEP BLOCK FOURTH DERIVATIVE METHOD FOR THE DIRECT SOLUTION OF THIRD ORDER INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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**Abstract:** An efficient one step block method with generalized two-point-hybrid is developed for solving initial value problems of third order ordinary differential equations directly. In driving this algorithm, a power series approximate function is interpolated at $\{x_n, x_{n+r}, x_{n+s}\}$ while its third and fourth derivatives are collocated at all points $\{x_n, x_{n+r}, x_{n+s}, x_{n+1}\}$ in the given interval. The proposed method is then tested for initial value problems of third order ordinary differential equations solved previously by other methods. The numerical results confirm the superiority of the new method to the existing methods regarding accuracy.

**AMS Subject Classification:** 65L05, 65L06
**Key Words:** direct solution, third order ordinary differential equation, interpolation and collocation, hybrid block methods, fourth derivative

## 1. Introduction

In this article, we consider an approximate solution of general third order initial value problems (IVPs) of the form

$$y''' = f(x, y, y', y''), \quad y(a) = \alpha, y_0'(a) = \beta, \quad y_0''(a) = \gamma, x \in [a, b]. \tag{1}$$

where $f$ is continuously differentiable on the given interval. Seeking an approximate solution for equation (1) is of great importance due to the wide application

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of this kind of differential equations in science, engineering and other real life problems. Normally, those equations have no analytical solutions. Several numerical methods were developed on the hands of many scholars to approximate the solution of such problems such as [11] and [12].

In order to achieve better accuracy and reduce execution time, hybrid block methods were first introduced according to [2], hybrid methods were initially introduced to overcome zero stability barrier occurred in block methods mentioned by Dahlquists ([15]). Furthermore, this method have the ability to change step-size beside utilizing data off-step points which contribute to the accuracy of the methods.

To enhance the performance of the numerical methods furthermore, citeNgwane, proposed second derivative methods, while [13] proposed a Simpson's type second derivative method for the solution of a first order stiff system of IVPs. These scholars inspired us to develop a new generalized one-step fourth derivative method for solving third order ODEs directly using the approach of interpolation and collocation.

## 2. Development of the Method

To develop this method the following approximate solution is considered

$$p(x)=\sum_{j=0}^{2q+p-1} a_j\left(\frac{x-x_n}{h}\right)^j, \tag{2}$$

where the number of interpolation and collocation points are $p=3$ and $q=4$ respectively. The third and fourth derivatives of (1) are give by

$$p'''(x)=\sum_{j=3}^{2q+p-1}\frac{a_j j!}{h^3(j-3)!}\left(\frac{x-x_n}{h}\right)^{j-3}=g(x,y,y',y''), \tag{3}$$

$$p^{iv}(x)=\sum_{j=4}^{2q+p-1}\frac{a_j j!}{h^4(j-4)!}\left(\frac{x-x_n}{h}\right)^{j-4}=f(x,y,y'.y''). \tag{4}$$

Interpolating (1) at $x_{n+\hat{p}}=x_n+\hat{p}h$, $\hat{p}=\{0,r,s\}$ and collocating (3) and (4) at all points i.e at $x_{n+\hat{q}}=x_n+\hat{q}h$, $\hat{q}=\{0,r,s,1\}$ where $\{r,s\}\in(0,1)$, produce a system of equations in matrix form as below

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & r & r^2 & r^3 & r^4 & r^5 & \cdots & r^{\hat{u}} \\ 1 & s & s^2 & s^3 & s^4 & s^5 & \cdots & s^{\hat{u}} \\ 0 & 0 & 0 & \frac{3!}{0!h^3} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{3!}{0!h^3} & \frac{4!r}{1!h^3} & \frac{5!r^2}{2!h^3} & \cdots & \frac{\hat{u}!r^{(\hat{u}-3)}}{(\hat{u}-3)!h^3} \\ 0 & 0 & 0 & \frac{3!}{0!h^3} & \frac{4!s}{1!h^3} & \frac{5!s^2}{2!h^3} & \cdots & \frac{\hat{u}!s^{(\hat{u}-3)}}{(\hat{u}-3)!h^3} \\ 0 & 0 & 0 & \frac{3!}{0!h^3} & \frac{4!}{1!h^3} & \frac{5!}{2!h^3} & \cdots & \frac{\hat{u}!}{(\hat{u}-3)!h^3} \\ 0 & 0 & 0 & 0 & \frac{4!}{0!h^4} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \frac{4!}{0!h^4} & \frac{5!r}{1!h^4} & \cdots & \frac{(\hat{u})!r^{(\hat{u}-4)}}{(\hat{u}-4)!h^4} \\ 0 & 0 & 0 & 0 & \frac{4!}{0!h^4} & \frac{5!s}{1!h^4} & \cdots & \frac{(\hat{u})!s^{(\hat{u}-4)}}{(\hat{u}-4)!h^4} \\ 0 & 0 & 0 & 0 & \frac{4!}{0!h^4} & \frac{5!}{1!h^4} & \cdots & \frac{(\hat{u})!}{(\hat{u}-4)!h^4} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+r} \\ y_{n+s} \\ f_n \\ f_{n+r} \\ f_{n+s} \\ f_{n+1} \\ g_n \\ g_{n+r} \\ g_{n+s} \\ g_{n+1} \end{bmatrix} \tag{5}$$

where $\hat{u} = 2q+p-1$. Matrix manipulation is then employed to solve the resulting system (5) for the unknown coefficients $a_i's$, $i = \{0, 1, .., 10\}$. Substituting them back into (1) yields

$$p(x) = \sum_{i=0,r} \alpha_i y_{n+i} + h^3 \sum_{i=0,r,s,1} \beta_i f_{n+i} + h^4 \sum_{i=0,r,s,1} \gamma_i g_{n+i}, \tag{6}$$

where $n = 0, 1, 2, ..., N-1$, $h = x_n - x_{n-1}$ is the constant step size for the partition $\pi_N$ of the interval $[a, b]$ which is given by $\pi_N = [a = x_0 < x_1 < ... < x_{N-1} < x_N = b]$, $\alpha_i, \beta_i$ and $\gamma_i$ are undetermined constants listed in Appendix I, $f_{n+i} = f(x + ih)$ and

$$g_{n+i} = \frac{df(x_{n+i}, y_{n+i}, y'_{n+i}, y''_{n+i})}{dx}.$$

the first and second derivatives of (6) are

$$p'(x) = \frac{1}{h} \sum_{i=0,r} \alpha'_i y_{n+i} + h^2 \sum_{i=0,r,s,1} \beta'_i f_{n+i} + h^3 \sum_{i=0,r,s,1} \gamma'_i g_{n+i}, \tag{7}$$

$$p''(x) = \frac{1}{h^2} \sum_{i=0,r} \alpha''_i y_{n+i} + h \sum_{i=0,r,s,1} \beta''_i f_{n+i} + h^2 \sum_{i=0,r,s,1} \gamma''_i g_{n+i}, \tag{8}$$

Evaluating (6) at the non interpolating points $\{x_{n+1}\}$ and to equations (7) and (8) at all points $\{x_{n+i}\}$, $i = \{0, r, s, 1\}$ produces the following general equations in block form

$$A^{[0]}Y_m^{[1]} = A^{[1]}Y_m^{[0]} + \sum_{i=0}^{1} B^{[i]}F_m^{[i]} + \sum_{i=0}^{1} D^{[i]}G_m^{[i]}. \tag{9}$$

where $A^{[0]}$ is an identity matrix of order 9 and

$$A^{[1]} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & rh & 0 & 0 & \frac{r^2h^2}{2} \\ 0 & 0 & 1 & 0 & 0 & sh & 0 & 0 & \frac{s^2h^2}{2} \\ 0 & 0 & 1 & 0 & 0 & h & 0 & 0 & \frac{h^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & rh \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & sh \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & h \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B^{[0]} = \begin{bmatrix} 0 & \cdots & 0 & B_{19}^{[0]} \\ 0 & \cdots & 0 & B_{29}^{[0]} \\ 0 & \cdots & 0 & B_{39}^{[0]} \\ 0 & \cdots & 0 & B_{49}^{[0]} \\ 0 & \cdots & 0 & B_{59}^{[0]} \\ 0 & \cdots & 0 & B_{69}^{[0]} \\ 0 & \cdots & 0 & B_{79}^{[0]} \\ 0 & \cdots & 0 & B_{89}^{[0]} \\ 0 & \cdots & 0 & B_{99}^{[0]} \end{bmatrix},$$

$$B^{[1]} = \begin{bmatrix} B_{11}^{[1]} & B_{12}^{[1]} & B_{13}^{[1]} \\ B_{21}^{[1]} & B_{22}^{[1]} & B_{23}^{[1]} \\ B_{31}^{[1]} & B_{32}^{[1]} & B_{33}^{[1]} \\ B_{41}^{[1]} & B_{42}^{[1]} & B_{43}^{[1]} \\ B_{51}^{[1]} & B_{52}^{[1]} & B_{53}^{[1]} \\ B_{61}^{[1]} & B_{62}^{[1]} & B_{63}^{[1]} \\ B_{71}^{[1]} & B_{72}^{[1]} & B_{73}^{[1]} \\ B_{81}^{[1]} & B_{82}^{[1]} & B_{83}^{[1]} \\ B_{91}^{[1]} & B_{92}^{[1]} & B_{93}^{[1]} \end{bmatrix}, D^{[0]} = \begin{bmatrix} 0 & \cdots & 0 & D_{19}^{[0]} \\ 0 & \cdots & 0 & D_{29}^{[0]} \\ 0 & \cdots & 0 & D_{39}^{[0]} \\ 0 & \cdots & 0 & D_{49}^{[0]} \\ 0 & \cdots & 0 & D_{59}^{[0]} \\ 0 & \cdots & 0 & D_{69}^{[0]} \\ 0 & \cdots & 0 & D_{79}^{[0]} \\ 0 & \cdots & 0 & D_{89}^{[0]} \\ 0 & \cdots & 0 & D_{99}^{[0]} \end{bmatrix}, D^{[1]} = \begin{bmatrix} D_{11}^{[1]} & D_{12}^{[1]} & D_{13}^{[1]} \\ D_{21}^{[1]} & D_{22}^{[1]} & D_{23}^{[1]} \\ D_{31}^{[1]} & D_{32}^{[1]} & D_{33}^{[1]} \\ D_{41}^{[1]} & D_{42}^{[1]} & D_{43}^{[1]} \\ D_{51}^{[1]} & D_{52}^{[1]} & D_{53}^{[1]} \\ D_{61}^{[1]} & D_{62}^{[1]} & D_{63}^{[1]} \\ D_{71}^{[1]} & D_{72}^{[1]} & D_{73}^{[1]} \\ D_{81}^{[1]} & D_{82}^{[1]} & D_{83}^{[1]} \\ D_{91}^{[1]} & D_{92}^{[1]} & D_{93}^{[1]} \end{bmatrix}.$$

whose entries are listed in Appendix II, while the vectors $Y_m^{[0]}, Y_m^{[1]}, F_m^{[0]}, F_m^{[1]}, G_m^{[0]}, G_m^{[1]}$ are defined as follows

$Y_m^{[0]} = [y_{n-s}, y_{n-r}, y_n, y'_{n-s}, y'_{n-r}, y'_n, y''_{n-s}, y''_{n-r}, y''_n]^T$,

$Y_m^{[1]} = [y_{n+r}, y_{n+s}, y_{n+1}, y'_{n+r}, y'_{n+s}, y'_{n+1}, y''_{n+r}, y''_{n+s}, y''_{n+1}]^T$,

$F_m^{[0]} = [f_{n-8}, f_{n-7}, f_{n-6}, f_{n-5}, f_{n-4}, f_{n-3}, f_{n-2}, f_{n-1}, f_n]$,

$F_m^{[1]} = [f_{n+r}, f_{n+s}, f_{n+1}]$,

$G_m^{[0]} = [g_{n-8}, g_{n-7}, g_{n-6}, g_{n-5}, g_{n-4}, g_{n-3}, g_{n-2}, g_{n-1}, g_n]$,

$G_m^{[1]} = [g_{n+r}, g_{n+s}, g_{n+1}]$.

## 3. Analysis of the Method

### 3.1. Zero Stability

**Definition 1.** The hybrid block method formula (9) is said to be zero stable if no root $R_m$ of the first characteristic equation $\rho(R)$ has modulus greater than one i.e $| R_m | \leqslant 1$ and if $R_m = 1$ then the multiplicity of $R_m$ must not exceed three .

To demonstrate that the roots of the first characteristic equation satisfies the prior definition we assume that $\{r, s\} \in (0, 1)$ and hence

$$\rho(R) = det[RA^{[0]} - A^{[1]}]$$

$$= \begin{vmatrix} R & 0 & -1 & 0 & 0 & -rh & 0 & 0 & -\frac{r^2h^2}{2} \\ 0 & R & -1 & 0 & 0 & -sh & 0 & 0 & -\frac{s^2h^2}{2} \\ 0 & 0 & R-1 & 0 & 0 & -h & 0 & 0 & -\frac{h^2}{2} \\ 0 & 0 & 0 & R & 0 & -1 & 0 & 0 & -rh \\ 0 & 0 & 0 & 0 & R & -1 & 0 & 0 & -sh \\ 0 & 0 & 0 & 0 & 0 & R-1 & 0 & 0 & -h \\ 0 & 0 & 0 & 0 & 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R-1 \end{vmatrix} = 0$$

which implies that

$$R^6(R-1)^3 = 0 \Rightarrow R_i = \begin{cases} 0, & \text{if } i = 1(1)6 \\ 1, & \text{if } i = 7(1)9 \end{cases}$$

Thus, the developed method is zero stable.

### 3.2. Order of the method

The linear operator $\Theta$ associated with the hybrid block methods formula (9) is defined as

$$\Theta\{y(x); h\} = A^{[0]}Y_m^{[1]} - A^{[1]}Y_m^{[0]} - \sum_{i=0}^{1} B^{[i]}F_m^{[i]} - \sum_{i=0}^{1} D^{[i]}G_m^{[i]}$$

Expanding the above equation in Taylor series and combining like terms we wind up with

$$\begin{aligned} \Theta\{y(x); h\} &= \hat{C}_0h^0y(x) + \hat{C}_1h^1y'(x) + \hat{C}_2h^2y''(x) + \\ &\quad \cdots + \hat{C}_{p+3}h^{p+3}y^{(p+3)}(x) + ... \end{aligned} \tag{10}$$

According to [1] and [2], method (9) is said to be of order $p$ if

$$\hat{C}_0 = \hat{C}_1 = ... = \hat{C}_{p+2} = 0 \text{ and } \hat{C}_{p+3} \neq 0$$

The term $\hat{C}_{p+3}$ is called the error constant and the local truncation error is given by :

$$t_{n+k} = \hat{C}_{p+3} y^{(p+3)} h^{p+3} + O(h^{p+3}).$$

Comparing like terms of $y^{(i)}$ and $h^i$ in Equation (10) produces the coefficients $\hat{C}_0 = \hat{C}_1 = ... = \hat{C}_{10} = 0$ with vector of error constants

$$\hat{C}_{11} = \left[ \hat{C}_{11}^{[1]} \quad \hat{C}_{11}^{[2]} \quad \hat{C}_{11}^{[3]} \quad \hat{C}_{11}^{[4]} \quad \hat{C}_{11}^{[5]} \quad \hat{C}_{11}^{[6]} \quad \hat{C}_{11}^{[7]} \quad \hat{C}_{11}^{[8]} \quad \hat{C}_{11}^{[9]} \right]^T$$

where

$$\hat{C}_{11}^{[1]} = \frac{r^7}{1117670400}(6r^4 - 22r^3 s - 22r^3 + 22r^2 s^2 + 88r^2 s + 22r^2 - 99rs^2 - 99rs + 132s^2),$$

$$\hat{C}_{11}^{[2]} = \frac{s^7}{1117670400}(22r^2 s^2 - 99r^2 s + 132r^2 - 22rs^3 + 88rs^2 - 99rs + 6s^4 - 22s^3 + 22s^2),$$

$$\hat{C}_{11}^{[3]} = \frac{1}{1117670400}(132r^2 s^2 - 99r^2 s + 22r^2 - 99rs^2 + 88rs - 22r + 22s^2 - 22s + 6),$$

$$\hat{C}_{11}^{[4]} = \frac{r^6}{101606400}(3r^4 - 10r^3 s - 10r^3 + 9r^2 s^2 + 36r^2 s + 9r^2 - 36rs^2 - 36rs + 42s^2),$$

$$\hat{C}_{11}^{[5]} = \frac{s^6}{101606400}(9r^2 s^2 - 36r^2 s + 42r^2 - 10rs^3 + 36rs^2 - 36rs + 3s^4 - 10s^3 + 9s^2),$$

$$\hat{C}_{11}^{[6]} = \frac{1}{101606400}(42r^2 s^2 - 36r^2 s + 9r^2 - 36rs^2 + 36rs - 10r + 9s^2 - 10s + 3),$$

$$\hat{C}_{11}^{[7]} = \frac{r^5}{50803200}(5r^4 - 15r^3 s - 15r^3 + 12r^2 s^2 + 48r^2 s + 12r^2 - 42rs^2 - 42rs + 42s^2),$$

$$\hat{C}_{11}^{[8]} = \frac{s^5}{50803200}(12r^2 s^2 - 42r^2 s + 42r^2 - 15rs^3 + 48rs^2 - 42rs + 5s^4 - 15s^3 + 12s^2),$$

$$\hat{C}_{11}^{[9]} = \frac{1}{50803200}(42r^2 s^2 - 42r^2 s + 12r^2 - 42rs^2 + 48rs - 15r + 12s^2 - 15s + 5).$$

Hence, the order of the developed method is $p = 8$.

### 3.3. Consistency

**Definition 2.** A block method is said to be consistent if its order $p$ is greater than one.

Consistency property is achieved for the hybrid block method from the above analysis since the order $p = 8 \geq 1$.

### 3.4. Convergence

**Theorem 3.** *(Henrici,1962): Consistency and zero stability are sufficient conditions for a linear multi step method to be convergent*

The hybrid block method Equation (9) is convergent since it fulfills both the consistency and zero stability conditions.

## 4. Numerical Examples

In this section, the efficiency and the performance of the general two hybrid one-step implicit hybrid block method (9) with order $p = 8$ is tested on three problems from the literature. The first example is solved using $h = \frac{5}{100}$ step size, the second with $h = \frac{1}{100}$, and the third is non linear physical problem with $h = \frac{1}{10}$. It's worth mentioning that this method works even for large interval and different step size. The values mentioned in this article are chosen just for the comparison purposes with the existing methods only.
**Problem (1)** : $y''' + 2y'' - 9y' - 18y = -18x^2 - 18x + 22,\ \ y(0) = -2,\ \ y'(0) = -8,\ \ y''(0) = -12$
**Exact Solution** : $y = -2e^{3x} + e^{-2x} + x^2 - 1$ with $h = \frac{5}{100}$.
**Source** : [10].

**Table I** : Comparison of the proposed method (New Error ) with [10].

| X | Exact - Solution | Computed - Solution For $r = \frac{1}{3}$ $s = \frac{2}{3}$ | New Error | Error for [10] |
|---|---|---|---|---|
| 0.1 | -2.8709868620740249 | -2.8709868620740244 | 4.44E(-16) | 3.01E(-14) |
| 0.2 | -3.9339175547453786 | -3.9339175547453777 | 8.88E(-16) | 6.24E(-15) |
| 0.3 | -5.2803945862198720 | -5.2803945862198702 | 1.77E(-15) | 3.39E(-13) |
| 0.4 | -7.0309048813558723 | -7.0309048813558661 | 6.21E(-15) | 1.23E(-12) |
| 0.5 | -9.3454986995046845 | -9.3454986995046703 | 1.42E(-14) | 2.98E(-12) |
| 0.6 | -12.438100716913688 | -12.438100716913659 | 2.84E(-14) | 5.90E(-12) |
| 0.7 | -16.595742861193692 | -16.595742861193639 | 5.32E(-14) | 1.03E(-11) |
| 0.8 | -22.204456243288554 | -22.204456243288455 | 9.94E(-14) | 1.66E(-11) |
| 0.9 | -29.784164561524104 | -29.784164561523934 | 1.70E(-13) | 2.50E(-11) |
| 1.0 | -40.035738563138757 | -40.035738563138487 | 2.70E(-13) | 3.60E(-11) |

**Problem (2)** : $y''' = 3\sin(x);,\ \ y(0) = 1,\ \ y'(0) = 0,\ \ y''(0) = -2,\ \ 0 \leq x \leq 1$
**Exact Solution** : $y = 3\cos(x) + \frac{x^2}{2} - 2$ with $h = \frac{1}{10}$.
**Source** : [9].

**Table II** : Comparison of the proposed method (New Error ) with [9].

| X | Exact-Solution | Computed - Solution For $r = \frac{1}{5}$ $s = \frac{3}{5}$ | New Error | Error for [9] |
|---|---|---|---|---|
| 0.1 | 0.99001249583407702 | 0.99001249583407724 | 2.22E(-16) | 2.53E(-14) |
| 0.2 | 0.96019973352372512 | 0.96019973352372490 | 2.22E(-16) | 1.61E(-13) |
| 0.3 | 0.91100946737681809 | 0.91100946737681809 | 0.00E(00) | 4.02E(-13) |
| 0.4 | 0.84318298200865538 | 0.84318298200865527 | 1.11E(-16) | 7.53E(-13) |
| 0.5 | 0.75774768567111828 | 0.75774768567111817 | 1.11E(-16) | 1.21E(-12) |
| 0.6 | 0.65600684472903525 | 0.65600684472903492 | 3.33E(-16) | 1.78E(-12) |
| 0.7 | 0.53952656185346548 | 0.53952656185346526 | 2.22E(-16) | 2.45E(-12) |
| 0.8 | 0.41012012804149611 | 0.41012012804149622 | 1.11E(-16) | 2.21E(-11) |
| 0.9 | 0.26982990481199343 | 0.26982990481199326 | 1.66E(-16) | 5.23E(-11) |
| 1.0 | 0.12090691760441930 | 0.12090691760441896 | 3.33E(-16) | 8.86E(-11) |

**Problem(3)**:**An Application Problem ( Thin Film Flow)**
In this example, the derived method is implemented to solve the thin film flow of a liquid. This fluid dynamics problem was transformed into third order ODE. The solution of this type of ODE was discussed by several authors such as [5],[6] and[7]. The discussion included the motion of the fluid on a plane surface in which the motion along the plane is in the same direction of the flow.

$$y''' = f(y)$$

In [8], the numerical method have been employed to solve special third order ODEs regarding the problem in thin film flow.

$$y''' = y^{-k}, \quad y(\rho) = \alpha, \quad y'(\rho) = \beta, \quad y''(\rho) = \gamma.$$

where $\rho = 0$ and $\alpha = \beta = \gamma = 1$

**Table III** : Comparison of the New Error for $k = 2$, $h = \frac{1}{10}$ with [6].

| X | Exact - Solution | Computed - Solution For $r = \frac{1}{4}$ $s = \frac{3}{4}$ | New Error | Error for [6] |
|---|---|---|---|---|
| 0.2 | 1.221211030 | 1.2212100045272658 | 1.0255E(-6) | 1.07E(-6) |
| 0.4 | 1.488834893 | 1.4888347798581165 | 1.1314E(-7) | 4.13E(-7) |
| 0.6 | 1.807361404 | 1.8073613976612986 | 6.3387E(-9) | 8.51E(-7) |
| 0.8 | 2.179819234 | 2.1798192338890714 | 1.1093E(-10) | 1.71E(-6) |
| 1.0 | 2.608275822 | 2.6082748675495666 | 9.5445E(-7) | 3.86E(-6) |

## 5. Conclusion

A general two-hybrid one-step block method of order 8 has been established successfully for the direct solution of general third order IVP of ODEs. The derived method is used to solve three different problems from different resources. The numerical analysis shows that the developed method is consistent and zero stable as a result convergence. The computed results are then compared with the results of existing methods in terms of error by considering different values of $r$ and $s$ . The new method is found to have superiority over them as shown in Tables I-III.

## References


[1] S. Fatunla,*Numerical Methods for Fatunla Initial Value Problems in Ordinary Differential Equations,* Academic Press, USA(1988).

[2] J. Lambert, *Computational Methods in Ordinary Differential Equations,* John Wiley & Sons, USA(1973).

[3] S. Jator, Block third derivative method based on trigonometric polynomials for periodic initial-value problems, *Afrika Matematika*, **27**, No.3-4(2016), 365-377.

[4] W. Enright, Second derivative multistep methods for stiff ordinary differential equations, *SIAM Journal on Numerical Analysis*, **11**, No.2(1974), 321-331.

[5] L. Yap, ,F. Ismail, , & N. Senu, An accurate block hybrid collocation method for third order ordinary differential equations, *Journal of Applied Mathematics*, **2014**,2014,1-9.

[6] M. Mechee, , N. Senu, F. Ismail, B. Nikouravan, & Z. Siri, A three-stage fourth-order Runge-Kutta method for directly solving special third-order differential equation with application to thin film problem. Mathematical Problems in Engineering, **2013**,2013,1-7.

[7] T . Myers, Thin films with high surface tension, *SIAM Review*, **40**, No. 3(1998), 441-462.

[8] E. Momoniat, Symmetries first integrals and phase planes of a third-order ordinary differential equation from thin film flow, *Mathematical and Computer Modelling*, **49**,No. 1-2(2009), 215-225.

[9] A. James, A. Adesanya, S. Joshua, Continuous block method for the solution of second order initial value problems of ordinary differential equation,*International Journal of Pure and Applied Mathematics,* **83**, No.3(2013), 405-416.

[10] J. Kuboye and Z. Omar. Numerical solution of third order ordinary differential equations using a seven-step block method, *International Journal of Mathematical Analysis*,**9**, No.15(2015), 743-754.

[11] A. Adesanya, M. Udoh, A. Alkali, A new block-predictor corrector algorithm for the solution of y"'= f (x, y, y', y"),*American J. of Computational Mathematics*, **2**, (2012), 341-344.

[12] N. Waeleh, Z. A Majid, F. Ismail, A new algoriths for solving higher order IVPs of ODEs, *Applied Mathematical Sciences*, **5**, No. 56 (2011). 2795-2805.

[13] R. Sahi, S. Jator, N. Khan, A Simpson's-type second derivative method for stiff systems, *International journal of pure and applied mathematics,* **81**, No. 4(2012), 619-633.

[14] F. Ngwane, S. Jator, Block hybrid-second derivative method for stiff systems, *International Journal of Pure and Applied Mathematics* , **80**, No. 4(2012), 543-559.

[15] G. Dahlquist, Convergence and stability in the numerical integration of ordinary differential equations,*Mathematica Scandinavica*, (1956), 33-53.

## Appendix I

$\alpha_0 = \frac{(x_n-x+hr)(x_n-x+hs)}{h^2rs}, \quad \alpha_r = \frac{-(x-x_n)(x_n-x+hs)}{h^2r(r-s)},$
$\alpha_s = \frac{(x-x_n)(x_n-x+hr)}{h^2s(r-s)},$

$\beta_0 = -\frac{(x-x_n)(x_n-x+hr)(x_n-x+hs)}{2520h^7r^3s^3}(6h^7r^7s^2 - 2h^7r^8 - 2h^7r^8s + 10h^7r^7s + 8h^7r^7 - 3h^7r^6s^3 -$
$18h^7r^6s^2 - 20h^7r^6s - 9h^7r^6 - 3h^7r^5s^4 + 9h^7r^5s^3 - 8h^7r^5s^2 + 18h^7r^5s - 3h^7r^4s^5 +$
$9h^7r^4s^4 + 40h^7r^4s^3 + 66h^7r^4s^2 - 3h^7r^3s^6 + 9h^7r^3s^5 + 40h^7r^3s^4 - 270h^7r^3s^3 + 6h^7r^2s^7 -$
$18h^7r^2s^6 - 8h^7r^2s^5 + 66h^7r^2s^4 - 2h^7rs^8 + 10h^7rs^7 - 20h^7rs^6 + 18h^7rs^5 - 2h^7s^8 +$
$8h^7s^7 - 9h^7s^6 - 2h^6r^7sx + 2h^6r^7sx_n - 2h^6r^7x + 2h^6r^7x_n + 6h^6r^6s^2x - 6h^6r^6s^2x_n +$
$10h^6r^6sx - 10h^6r^6sx_n + 8h^6r^6x - 8h^6r^6x_n - 3h^6r^5s^3x + 3h^6r^5s^3x_n - 18h^6r^5s^2x +$
$18h^6r^5s^2x_n - 20h^6r^5sx + 20h^6r^5sx_n - 9h^6r^5x + 9h^6r^5x_n - 3h^6r^4s^4x + 3h^6r^4s^4x_n +$
$9h^6r^4s^3x - 9h^6r^4s^3x_n - 8h^6r^4s^2x + 8h^6r^4s^2x_n + 18h^6r^4sx - 18h^6r^4sx_n - 3h^6r^3s^5x +$
$3h^6r^3s^5x_n + 9h^6r^3s^4x - 9h^6r^3s^4x_n + 40h^6r^3s^3x - 40h^6r^3s^3x_n + 66h^6r^3s^2x - 66h^6r^3s^2x_n +$
$6h^6r^2s^6x - 6h^6r^2s^6x_n - 18h^6r^2s^5x + 18h^6r^2s^5x_n - 8h^6r^2s^4x + 8h^6r^2s^4x_n + 66h^6r^2s^3x -$
$66h^6r^2s^3x_n - 2h^6rs^7x + 2h^6rs^7x_n + 10h^6rs^6x - 10h^6rs^6x_n - 20h^6rs^5x + 20h^6rs^5x_n +$
$18h^6rs^4x - 18h^6rs^4x_n - 2h^6s^7x + 2h^6s^7x_n + 8h^6s^6x - 8h^6s^6x_n - 9h^6s^5x + 9h^6s^5x_n -$
$2h^5r^6sx^2 + 4h^5r^6sxx_n - 2h^5r^6sx_n^2 - 2h^5r^6x^2 + 4h^5r^6xx_n - 2h^5r^6x_n^2 + 6h^5r^5s^2x^2 -$
$12h^5r^5s^2xx_n + 6h^5r^5s^2x_n^2 + 10h^5r^5sx^2 - 20h^5r^5sxx_n + 10h^5r^5$
$sx_n^2 + 8h^5r^5x^2 - 16h^5r^5xx_n + 8h^5r^5x_n^2 - 3h^5r^4s^3x^2 + 6h^5r^4s^3xx_n - 3h^5r^4s^3$
$x_n^2 - 18h^5r^4s^2x^2 + 36h^5r^4s^2xx_n - 18h^5r^4s^2x_n^2 - 20h^5r^4sx^2 + 40h^5r^4sxx_n - 20h^5r^4sx_n^2 -$
$9h^5r^4x^2 + 18h^5r^4xx_n - 9h^5r^4x_n^2 - 3h^5r^3s^4x^2 + 6h^5r^3s^4xx_n - 3h^5r^3s^4x_n^2 + 9h^5r^3s^3x^2 -$
$18h^5r^3s^3xx_n + 9h^5r^3s^3x_n^2 - 8h^5r^3s^2x^2 + 16h^5r^3s^2$
$xx_n - 8h^5r^3s^2x_n^2 + 18h^5r^3sx^2 - 36h^5r^3sxx_n + 18h^5r^3sx_n^2 + 6h^5r^2s^5x^2 - 12h^5r^2s^5xx_n +$
$6h^5r^2s^5x_n^2 - 18h^5r^2s^4x^2 + 36h^5r^2s^4xx_n - 18h^5r^2s^4x_n^2 - 8h^5r^2s^3x^2 + 16h^5r^2s^3xx_n -$
$8h^5r^2s^3x_n^2 - 18h^5r^2s^2x^2 + 36h^5r^2s^2xx_n - 18h^5r^2s^2x_n^2 - 2h^5rs^6x^2 + 4h^5rs^6xx_n - 2h^5rs^6x_n^2 +$
$10h^5rs^5x^2 - 20h^5rs^5xx_n + 10h^5rs^5x_n^2 - 20h^5rs^4x^2 + 40h^5rs^4xx_n - 20h^5rs^4x_n^2 + 18h^5rs^3x^2 -$
$36h^5rs^3x$
$x_n + 18h^5rs^3x_n^2 - 2h^5s^6x^2 + 4h^5s^6xx_n - 2h^5s^6x_n^2 + 8h^5s^5x^2 - 16h^5s^5xx_n + 8h^5s^5x_n^2 -$
$9h^5s^4x^2 + 18h^5s^4xx_n - 9h^5s^4x_n^2 - 2h^4r^5sx^3 + 6h^4r^5sx^2x_n - 6h^4r^5sxx_n^2 + 2h^4r^5sx_n^3 -$
$2h^4r^5x^3 + 6h^4r^5x^2x_n - 6h^4r^5xx_n^2 + 2h^4r^5x_n^3 + 6h^4r^4s^2x^3 - 18h^4r^4s^2x^2x_n + 18h^4r^4s^2xx_n^2 -$
$6h^4r^4s^2x_n^3 + 10h^4r^4sx^3 - 30h^4r^4sx^2x_n + 30h^4r^4sxx_n^2 - 10h^4r^4sx_n^3 + 8h^4r^4x^3 - 24h^4r^4x^2x_n +$
$24h^4r^4x$
$x_n^2 - 8h^4r^4x_n^3 - 3h^4r^3s^3x^3 + 9h^4r^3s^3x^2x_n - 9h^4r^3s^3xx_n^2 + 3h^4r^3s^3x_n^3 - 18h^4r^3s^2x^3 +$
$54h^4r^3s^2x^2x_n - 54h^4r^3s^2xx_n^2 + 18h^4r^3s^2x_n^3 - 20h^4r^3sx^3 + 60h^4r^3sx^2x_n - 60h^4r^3sxx_n^2 +$
$20h^4r^3sx_n^3 - 9h^4r^3x^3 + 27h^4r^3x^2x_n - 27h^4r^3x$
$x_n^2 + 9h^4r^3x_n^3 + 6h^4r^2s^4x^3 - 18h^4r^2s^4x^2x_n + 18h^4r^2s^4xx_n^2 - 6h^4r^2s^4x_n^3 - 18h^4r^2s^3x^3 +$

$54h^4r^2s^3x^2x_n-54h^4r^2s^3xx_n^2+18h^4r^2s^3x_n^3-56h^4r^2s^2x^3+168h^4r^2s^2x^2x_n-168h^4r^2s^2xx_n^2+$
$56h^4r^2s^2x_n^3-66h^4r^2sx^3+198h^4r^2sx^2x_n-198h^4r^2sxx_n^2+66h^4r^2sx_n^3-2h^4rs^5x^3+$
$6h^4rs^5x^2x_n-6h^4rs^5xx_n^2+2h^4rs^5x_n^3$
$+10h^4rs^4x^3-30h^4rs^4x^2x_n+30h^4rs^4xx_n^2-10h^4rs^4x_n^3-20h^4rs^3x^3+60h^4rs^3x^2x_n-$
$60h^4rs^3xx_n^2+20h^4rs^3x_n^3-66h^4rs^2x^3+198h^4rs^2x^2x_n-198h^4rs^2xx_n^2+66h^4rs^2x_n^3-$
$2h^4s^5x^3+6h^4s^5x^2x_n-6h^4s^5xx_n^2+2h^4s^5x_n^3+8h^4s^4x^3-24h^4s^4x^2x_n+24h^4s^4xx_n^2-$
$8h^4s^4x_n^3-9h^4s^3x^3+27h^4s^3x^2x_n-27h^4s^3xx_n^2+9h^4s^3x_n^3-2h^3r^4sx^4+8h^3r^4sx^3x_n-$
$12h^3r^4sx^2x_n^2+8h^3r^4sxx_n^3-2h^3r^4sx_n^4-2h^3r^4x^4+8h^3r^4x^3x_n-12h^3r^4x^2x_n^2+8h^3r^4xx_n^3-$
$2h^3r^4x_n^4+6h^3r^3s^2x^4-24h^3r^3s^2x^3x_n+36h^3r^3s^2x^2x_n^2-24h^3r^3s^2xx_n^3+6h^3r^3s^2x_n^4+$
$10h^3r^3sx^4-40h^3r^3sx^3x_n+60h^3r^3sx^2x_n^2-40h^3r^3sxx_n^3+10h^3r^3sx_n^4+8h^3r^3x^4-32h^3r^3x^3x_n+$
$48h^3r^3x^2x_n^2-32h^3r^3xx_n^3+8h^3r^3x_n^4+6h^3r^2s^3x^4-24h^3r^2s^3x^3x_n+36h^3r^2s^3x^2x_n^2-$
$24h^3r^2s^3xx_n^3+6h^3r^2s^3x_n^4+81h^3r^2s^2x^4-324h^3r^2s^2x^3x_n+486h^3r^2s^2x^2x_n^2-324h^3r^2s^2xx_n^3+$
$81h^3r^2s^2x_n^4+100h^3r^2$
$sx^4-400h^3r^2sx^3x_n+600h^3r^2sx^2x_n^2-400h^3r^2sxx_n^3+100h^3r^2sx_n^4+33h^3r^2$
$x^4-132h^3r^2x^3x_n+198h^3r^2x^2x_n^2-132h^3r^2xx_n^3+33h^3r^2x_n^4-2h^3rs^4x^4+8h^3rs^4x^3x_n-$
$12h^3rs^4x^2x_n^2+8h^3rs^4xx_n^3-2h^3rs^4x_n^4+10h^3rs^3x^4-40h^3rs^3$
$x^3x_n+60h^3rs^3x^2x_n^2-40h^3rs^3xx_n^3+10h^3rs^3x_n^4+100h^3rs^2x^4-400h^3rs^2x^3$
$x_n+600h^3rs^2x^2x_n^2-400h^3rs^2xx_n^3+100h^3rs^2x_n^4+54h^3rsx^4-216h^3rsx^3x_n+324h^3rsx^2x_n^2-$
$216h^3rsxx_n^3+54h^3rsx_n^4-2h^3s^4x^4+8h^3s^4x^3x_n-12h^3s^4x^2x_n^2$
$+8h^3s^4xx_n^3-2h^3s^4x_n^4+8h^3s^3x^4-32h^3s^3x^3x_n+48h^3s^3x^2x_n^2-32h^3s^3xx_n^3+8h^3s^3x_n^4+$
$33h^3s^2x^4-132h^3s^2x^3x_n+198h^3s^2x^2x_n^2-132h^3s^2xx_n^3+33h^3s^2x_n^4-2h^2r^3sx^5+10h^2r^3sx^4x_n-$
$20h^2r^3sx^3x_n^2+20h^2r^3sx^2x_n^3-10h^2r^3sxx_n^4+2h^2r^3sx_n^5-2h^2r^3x^5+10h^2r^3x^4x_n-20h^2r^3x^3x_n^2+$
$20h^2r^3x^2x_n^3-10h^2r^3xx_n^4+2h^2r^3x_n^5-27h^2r^2s^2x^5+135h^2r^2s^2x^4x_n-270h^2r^2s^2x^3x_n^2+$
$270h^2r^2s^2x^2x_n^3-135h^2r^2s^2xx_n^4+27h^2r^2s^2x_n^5-59h^2r^2sx^5+295h^2r^2sx^4x_n-590h^2r^2sx^3x_n^2+$
$590h^2r^2sx^2x_n^3-295h^2r^2sxx_n^4+59h^2r^2sx_n^5-40h^2r^2x^5+200h^2r^2x^4x_n-400h^2r^2x^3x_n^2+$
$400h^2r^2x^2x_n^3-200h^2r^2xx_n^4+40h^2r^2x_n^5-2h^2rs^3x^5+10h^2rs^3$
$x^4x_n-20h^2rs^3x^3x_n^2+20h^2rs^3x^2x_n^3-10h^2rs^3xx_n^4+2h^2rs^3x_n^5-59h^2rs^2x^5+295h^2rs^2x^4x_n-$
$590h^2rs^2x^3x_n^2+590h^2rs^2x^2x_n^3-295h^2rs^2xx_n^4+59h^2rs^2x_n^5-80h^2rsx^5+400h^2rsx^4x_n-$
$800h^2rsx^3x_n^2+800h^2rsx^2x_n^3-400h^2rsxx_n^4+80h^2rsx_n^5-15h^2rx^5+75h^2rx^4x_n-150h^2rx^3x_n^2+$
$150h^2rx^2x_n^3-75h^2rxx_n^4+15h^2rx_n^5-2h^2s^3x^5+10h^2s^3x^4x_n-20h^2s^3x^3x_n^2+20h^2s^3x^2x_n^3-$
$10h^2s^3xx_n^4+2h^2s^3x_n^5-40h^2s^2x^5+200h^2s^2x^4x_n-400h^2s^2x^3x_n^2+400h^2s^2x^2x_n^3-200h^2s^2$
$xx_n^4+40h^2s^2x_n^5-15h^2sx^5+75h^2sx^4x_n-150h^2sx^3x_n^2+150h^2sx^2x_n^3-75h^2sxx_n^4+$
$15h^2sx_n^5+13hr^2sx^6-78hr^2sx^5x_n+195hr^2sx^4x_n^2-260hr^2sx^3$
$x_n^3+195hr^2sx^2x_n^4-78hr^2sxx_n^5+13hr^2sx_n^6+13hr^2x^6-78hr^2x^5x_n+195hr^2x^4x_n^2-$
$260hr^2x^3x_n^3+195hr^2x^2x_n^4-78hr^2xx_n^5+13hr^2x_n^6+13hrs^2x^6-78hrs^2x^5x_n+195hrs^2x^4x_n^2-$
$260hrs^2x^3x_n^3+195hrs^2x^2x_n^4-78hrs^2xx_n^5+13hrs^2x_n^6+41hrsx^6-246hrsx^5x_n+615hrsx^4x_n^2-$
$820hrsx^3x_n^3+615hrsx^2$
$x_n^4-246hrsxx_n^5+41hrsx_n^6+20hrx^6-120hrx^5x_n+300hrx^4x_n^2-400hrx^3x_n^3$
$+300hrx^2x_n^4-120hrxx_n^5+20hrx_n^6+13hs^2x^6-78hs^2x^5x_n+195hs^2x^4x_n^2-260hs^2x^3x_n^3+$

$195hs^2x^2x_n^4 - 78hs^2xx_n^5 + 13hs^2x_n^6 + 20hsx^6 - 120hsx^5x_n + 300hsx^4x_n^2 - 400hsx^3x_n^3 + 300hsx^2x_n^4 - 120hsxx_n^5 + 20hsx_n^6 - 7rsx^7 + 49rsx^6x_n - 147rsx^5x_n^2 + 245rsx^4x_n^3 - 245rsx^3x_n^4 + 147rsx^2x_n^5 - 49rsxx_n^6 + 7rsx_n^7 - 7rx^7 + 49rx^6x_n - 147rx^5x_n^2 + 245rx^4x_n^3 - 245rx^3x_n^4 + 147rx^2x_n^5 - 49rxx_n^6 + 7rx_n^7 - 7sx^7 + 49sx^6x_n - 147sx^5x_n^2 + 245sx^4x_n^3 - 245sx^3x_n^4 + 147sx^2x_n^5 - 49sxx_n^6 + 7sx_n^7)$,

$\beta_r = \frac{(x-x_n)^9}{504h^6r^3(r-s)^3(r-1)^3}(7r^3+7r^2s+7r^2-8rs^2-13rs-8r+4s^2+4s) - \frac{(x-x_n)^7}{210h^4r^3(r-s)^3(r-1)^3}(5r^2s^3 - 28r^3s - 7r^3 - 7r^3s^2 + 13r^2s^2 + 13r^2s + 5r^2 + 5rs^3 + 4rs^2 + 5rs - 4s^3 - 4s^2) + \frac{(x-x_n)^{10}}{360h^7r^3(r-s)^3(r-1)^3}(2r - s + 2rs - 3r^2) + \frac{(x-x_n)^8}{168h^5r^3(r-s)^3(r-1)^3}(2r^2s^2 - 7r^3 - 7r^3s - 2r^2s + 2r^2 + 2rs^3 + 7rs^2 + 7rs + 2r - s^3 - 4s^2 - s) + \frac{h(x-x_n)^2}{2520r^3(r-s)^3(r-1)^3}(42r^9s - 14r^10 + 56r^9 - 32r^8s^2 - 178r^8s - 74r^8 - 2r^7s^3 + 147r^7s^2 + 251r^7s + 30r^7 - 2r^6s^4 + 12r^6s^3 - 229r^6s^2 - 105r^6s - 2r^5s^5 + 12r^5s^4 - 25r^5s^3 + 99r^5s^2 - 2r^4s^6 + 12r^4s^5 - 25r^4s^4 + 15r^4s^3 - 2r^3s^7 + 12r^3s^6 - 25r^3s^5 + 15r^3s^4 + 12r^2s^8 - 51r^2s^7 + 59r^2s^6 + 15r^2s^5 - 4rs^9 + 8rs^8 + 17rs^7 - 45rs^6 + 2s^9 - 8s^8 + 9s^7) - \frac{h^2s(x-x_n)}{2520r^2(r-s)^3(r-1)^3}(42r^8s - 14r^9 + 56r^8 - 32r^7s^2 - 178r^7s - 74r^7 - 2r^6s^3 + 147r^6s^2 + 251r^6s + 30r^6 - 2r^5s^4 + 12r^5s^3 - 229r^5s^2 - 105r^5s - 2r^4s^5 + 12r^4s^4 - 25r^4s^3 + 99r^4s^2 - 2r^3s^6 + 12r^3s^5 - 25r^3s^4 + 15r^3s^3 + 12r^2s^7 - 51r^2s^6 + 59r^2s^5 + 15r^2s^4 - 4rs^8 + 8rs^7 + 17rs^6 - 45rs^5 + 2s^8 - 8s^7 + 9s^6) - \frac{s(x-x_n)^6}{60h^3r^3(r-s)^3(r-1)^3}(7r^3s + 7r^3 - 5r^2s^2 - 7r^2s - 5r^2 + rs^2 + rs + s^2) - \frac{s^2(x-x_n)^5}{60h^2r^2(r-s)^3(r-1)^3}(5r - 3s + 5rs - 7r^2)$,

$\beta_s = \frac{(x-x_n)^8}{168h^5s^3(r-s)^3(s-1)^3}(r^3 - 2r^3s - 2r^2s^2 - 7r^2s + 4r^2 + 7rs^3 + 2rs^2 - 7rs + r + 7s^3 - 2s^2 - 2s) + \frac{(x-x_n)^7}{210h^4s^3(r-s)^3(s-1)^3}(5r^3s^2 + 5r^3s - 4r^3 - 7r^2s^3 + 13r^2s^2 + 4r^2s - 4r^2 - 28rs^3 + 13rs^2 + 5rs - 7s^3 + 5s^2) + \frac{(x-x_n)^{10}}{360h^7s^3(r-s)^3(s-1)^3}(r - 2s - 2rs + 3s^2) - \frac{(x-x_n)^9}{504h^6s^3(r-s)^3(s-1)^3}(4r^2 - 8r^2s + 7rs^2 - 13rs + 4r + 7s^3 + 7s^2 - 8s) - \frac{h(x-x_n)^2}{2520s^3(r-s)^3(s-1)^3}(2r^9 - 4r^9s + 12r^8s^2 + 8r^8s - 8r^8 - 2r^7s^3 - 51r^7s^2 + 17r^7s + 9r^7 - 2r^6s^4 + 12r^6s^3 + 59r^6s^2 - 45r^6s - 2r^5s^5 + 12r^5s^4 - 25r^5s^3 + 15r^5s^2 - 2r^4s^6 + 12r^4s^5 - 25r^4s^4 + 15r^4s^3 - 2r^3s^7 + 12r^3s^6 - 25r^3s^5 + 15r^3s^4 - 32r^2s^8 + 147r^2s^7 - 229r^2s^6 + 99r^2s^5 + 42rs^9 - 178rs^8 + 251rs^7 - 105rs^6 - 14s^{10} + 56s^9 - 74s^8 + 30s^7) + \frac{h^2r(x-x_n)}{2520s^2(r-s)^3(s-1)^3}(2r^8 - 4r^8s + 12r^7s^2 + 8r^7s - 8r^7 - 2r^6s^3 - 51r^6s^2 + 17r^6s + 9r^6 - 2r^5s^4 + 12r^5s^3 + 59r^5s^2 - 45r^5s - 2r^4s^5 + 12r^4s^4 - 25r^4s^3 + 15r^4s^2 - 2r^3s^6 + 12r^3s^5 - 25r^3s^4 + 15r^3s^3 - 32r^2s^7 + 147r^2s^6 - 229r^2s^5 + 99r^2s^4 + 42rs^8 - 178rs^7 + 251rs^6 - 105rs^5 - 14s^9 + 56s^8 - 74s^7 + 30s^6) + \frac{r(x-x_n)^6}{60h^3s^3(r-s)^3(s-1)^3}(r^2s - 5r^2s^2 + r^2 + 7rs^3 - 7rs^2 + rs + 7s^3 - 5s^2) - \frac{r^2(x-x_n)^5}{60h^2s^2(r-s)^3(s-1)^3}(3r - 5s - 5rs + 7s^2)$,

$\beta_1 = \frac{(x-x_n)^9}{504h^6(r-1)^3(s-1)^3}(4r^2s - 8r^2 + 4rs^2 - 13rs + 7r - 8s^2 + 7s + 7) - \frac{(x-x_n)^8}{168h^5(r-1)^3(s-1)^3}(r^3s - 2r^3 + 4r^2s^2 - 7r^2s - 2r^2 + rs^3 - 7rs^2 + 2rs + 7r - 2s^3 - 2s^2 + 7s) - \frac{h(x-x_n)^2}{2520(r-1)^3(s-1)^3}(4r^9 - 2r^9s + 6r^8s^2 - 2r^8s - 16r^8 - 3r^7s^3 - 27r^7s^2 + 43r^7s + 14r^7 - 3r^6s^4 + 27r^6s^3 + r^6s^2 - 49r^6s - 3r^5s^5 + 27r^5s^4 - 59r^5s^3 + 35r^5s^2 - 3r^4s^6 + 27r^4s^5 - 59r^4s^4 + 35r^4s^3 - 3r^3s^7 + 27r^3s^6 - 59r^3s^5 + 35r^3s^4 + 6r^2s^8 - 27r^2s^7 + r^2s^6 + 35r^2s^5 - 2rs^9 - 2rs^8 + 43rs^7 - 49rs^6 + 4s^9 - 16s^8 + 14s^7) - \frac{(x-x_n)^7}{210h^4(r-1)^3(s-1)^3}(5r^3s - 4r^3s^2 + 5r^3 - 4r^2s^3 + 4r^2s^2 + 13r^2s - 7r^2 + 5rs^3 + 13rs^2 - 28rs + 5s^3 - 7s^2) + \frac{(x-x_n)^{10}}{360h^7(r-1)^3(s-1)^3}(2r + 2s - rs - 3) + \frac{h^2rs(x-x_n)}{2520(r-1)^3(s-1)^3}(4r^8 -$

$2r^8s+6r^7s^2-2r^7s-16r^7-3r^6s^3-27r^6s^2+43r^6s+14r^6-3r^5s^4+27r^5s^3+r^5s^2-49r^5s-3r^4s^5+27r^4s^4-59r^4s^3+35r^4s^2-3r^3s^6+27r^3s^5-59r^3s^4+35r^3s^3+6r^2s^7-27r^2s^6+r^2s^5+35r^2s^4-2rs^8-2rs^7+43rs^6-49rs^5+4s^8-16s^7+14s^6)-\frac{rs(x-x_n)^6}{60h^3(r-1)^3(s-1)^3}(r^2s^2+r^2s-5r^2+rs^2-7rs+7r-5s^2+7s)-\frac{r^2s^2(x-x_n)^5}{60h^2(r-1)^3(s-1)^3}(5r+5s-3rs-7),$

$\gamma_0=-\frac{(x-x_n)(x_n-x+hr)(x_n-x+hs)}{5040h^6r^2s^2}(6h^7r^6s-2h^7r^7+8h^7r^6-3h^7r^5s^2-28h^7r^5s-9h^7r^5-3h^7r^4s^3+20h^7r^4s^2+39h^7r^4s-3h^7r^3s^4+20h^7r^3s^3-45h^7r^3s^2-3h^7r^2s^5+20h^7r^2s^4-45h^7r^2s^3+6h^7rs^6-28h^7rs^5+39h^7rs^4-2h^7s^7+8h^7s^6-9h^7s^5-2h^6r^6x+2h^6r^6x_n+6h^6r^5sx-6h^6r^5sx_n+8h^6r^5x-8h^6r^5x_n-3h^6r^4s^2x+3h^6r^4s^2x_n-28h^6r^4sx+28h^6r^4sx_n-9h^6r^4x+9h^6r^4x_n-3h^6r^3s^3x+3h^6r^3s^3x_n+20h^6r^3s^2x-20h^6r^3s^2x_n+39h^6r^3sx-39h^6r^3sx_n-3h^6r^2s^4x+3h^6r^2s^4x_n+20h^6r^2s^3x-20h^6r^2s^3x_n-45h^6r^2s^2x+45h^6r^2s^2x_n+6h^6rs^5x-6h^6rs^5x_n-28h^6rs^4x+28h^6rs^4x_n+39h^6rs^3x-39h^6rs^3x_n-2h^6s^6x+2h^6s^6x_n+8h^6s^5x-8h^6s^5x_n-9h^6s^4x+9h^6s^4x_n-2h^5r^5x^2+4h^5r^5xx_n-2h^5r^5x_n^2+6h^5r^4sx^2-12h^5r^4sxx_n+6h^5r^4sx_n^2+8h^5r^4x^2-16h^5r^4xx_n+8h^5r^4x_n^2-3h^5r^3s^2x^2+6h^5r^3s^2xx_n-3h^5r^3s^2x_n^2-28h^5r^3sx^2+56h^5r^3sxx_n-28h^5r^3sx_n^2-9h^5r^3x^2+18h^5r^3xx_n-9h^5r^3x_n^2-3h^5r^2s^3x^2+6h^5r^2s^3xx_n-3h^5r^2s^3x_n^2+20h^5r^2s^2x^2-40h^5r^2s^2xx_n+20h^5r^2s^2x_n^2+39h^5r^2sx^2-78h^5r^2sxx_n+39h^5r^2sx_n^2+6h^5rs^4x^2-12h^5rs^4x$

$x_n+6h^5rs^4x_n^2-28h^5rs^3x^2+56h^5rs^3xx_n-28h^5rs^3x_n^2+39h^5rs^2x^2-78h^5rs^2xx_n+39h^5rs^2x_n^2-2h^5s^5x^2+4h^5s^5xx_n-2h^5s^5x_n^2+8h^5s^4x^2-16h^5s^4xx_n+8h^5s^4x_n^2-9h^5s^3x^2+18h^5s^3xx_n-9h^5s^3x_n^2-2h^4r^4x^3+6h^4r^4x^2x_n-6h^4r^4xx_n^2+2h^4r^4x_n^3+6h^4r^3sx^3-18h^4r^3sx^2x_n+18h^4r^3sxx_n^2-6h^4r^3sx_n^3+8h^4r^3x^3-24h^4r^3x^2x_n+24h^4r^3xx_n^2-8h^4r^3x_n^3-3h^4r^2s^2x^3+9h^4r^2s^2x^2x_n-9h^4r^2s^2xx_n^2+3h^4r^2s^2x_n^3-28h^4r^2sx^3+84h^4r^2sx^2x_n-84h^4r^2sxx_n^2+28h^4r^2sx_n^3-9h^4r^2x^3+27h^4r^2x^2x_n-27h^4r^2xx_n^2+9h^4r^2x_n^3+6h^4rs^3x^3-18h^4rs^3x^2x_n+18h^4rs^3xx_n^2-6h^4rs^3x_n^3-28h^4rs^2x^3+84h^4rs^2x^2$

$x_n-84h^4rs^2xx_n^2+28h^4rs^2x_n^3-87h^4rsx^3+261h^4rsx^2x_n-261h^4rsxx_n^2+87h^4rsx_n^3-2h^4s^4x^3+6h^4s^4x^2x_n-6h^4s^4xx_n^2+2h^4s^4x_n^3+8h^4s^3x^3-24h^4s^3x^2x_n+24h^4s^3xx_n^2-8h^4s^3x_n^3-9h^4s^2x^3+27h^4s^2x^2x_n-27h^4s^2xx_n^2+9h^4s^2x_n^3-2h^3r^3x^4+8h^3r^3x^3x_n-12h^3r^3x^2x_n^2+8h^3r^3xx_n^3-2h^3r^3x_n^4+6h^3r^2sx^4-24h^3r^2sx^3x_n+36h^3r^2sx^2x_n^2-24h^3r^2sxx_n^3+6h^3r^2sx_n^4+8h^3r^2$

$x^4-32h^3r^2x^3x_n+48h^3r^2x^2x_n^2-32h^3r^2xx_n^3+8h^3r^2x_n^4+6h^3rs^2x^4-24h^3rs^2x^3x_n+36h^3rs^2x^2x_n^2-24h^3rs^2xx_n^3+6h^3rs^2x_n^4+92h^3rsx^4-368h^3r$

$sx^3x_n+552h^3rsx^2x_n^2-368h^3rsxx_n^3+92h^3rsx_n^4+33h^3rx^4-132h^3rx^3x_n+198h^3rx^2x_n^2-132h^3rxx_n^3+33h^3rx_n^4-2h^3s^3x^4+8h^3s^3x^3x_n-12h^3s^3x^2x_n^2+8h^3s^3xx_n^3-2h^3s^3x_n^4+8h^3s^2x^4-32h^3s^2x^3x_n+48h^3s^2x^2x_n^2-32h^3s^2xx_n^3+8h^3s^2x_n^4+33h^3sx^4-132h^3sx^3x_n+198h^3sx^2x_n^2-132h^3sxx_n^3+33h^3sx_n^4-2h^2r^2x^5+10h^2r^2x^4x_n-20h^2r^2x^3x_n^2+20h^2r^2x^2x_n^3-10h^2r^2xx_n^4+2h^2r^2x_n^5-27h^2rsx^5+135h^2rsx^4x_n-270h^2rsx^3x_n^2+270h^2rsx^2x_n^3-135h^2rsxx_n^4+27h^2rsx_n^5-40h^2rx^5+200h^2rx^4x_n-400h^2rx^3x_n^2+400h^2rx^2x_n^3-200h^2rx$

$x_n^4+40h^2rx_n^5-2h^2s^2x^5+10h^2s^2x^4x_n-20h^2s^2x^3x_n^2+20h^2s^2x^2x_n^3-10h^2s^2xx_n^4+2h^2s^2x_n^5-40h^2sx^5+200h^2sx^4x_n-400h^2sx^3x_n^2+400h^2sx^2x_n^3-200h^2sxx_n^4+40h^2sx_n^5-15h^2x^5+75h^2x^4x_n-150h^2x^3x_n^2+150h^2x^2x_n^3-75h^2xx_n^4+15h^2x_n^5+13hrx^6-78hrx^5x_n+$

$195hrx^4x_n^2 - 260hrx^3x_n^3 + 195hrx^2x_n^4 - 78hrxx_n^5 + 13hrx_n^6 + 13hsx^6 - 78hsx^5x_n + 195hsx^4x_n^2 - 260hsx^3x_n^3 + 195hsx^2x_n^4 - 78hsxx_n^5 + 13hsx_n^6 + 20hx^6 - 120hx^5x_n + 300hx^4x_n^2 - 400hx^3x_n^3 + 300hx^2x_n^4 - 120hxx_n^5 + 20hx_n^6 - 7x^7 + 49x^6x_n - 147x^5x_n^2 + 245x^4x_n^3 - 245x^3x_n^4 + 147x^2x_n^5 - 49xx_n^6 + 7x_n^7)$,

$\gamma_r = \frac{(x-x_n)^{10}}{(720h^6r^2(r-s)^2(r-1)^2} - \frac{(x-x_n)^7}{210h^3r^2(r-s)^3(r-1)^2}(r^2s^2 + 4r^2s + r^2 - rs^3 - 2rs^2 + rs - 2s^3 - 2s^2) - \frac{s^2(x-x_n)^5}{60hr(r-s)^2(r-1)^2} - \frac{(x-x_n)^9}{504h^5r^2(r-s)^3(r-1)^2}(r^2+rs+2r-2s^2-2s) + \frac{h^2(x-x_n)^2}{5040r^2(r-s)^3(r-1)^2}(3r^9 - 10r^8s - 10r^8 + 9r^7s^2 + 36r^7s + 9r^7 - 36r^6s^2 - 36r^6s + 42r^5s^2 - 4rs^8 + 18rs^7 - 24rs^6 + 2s^9 - 8s^8 + 9s^7) + \frac{(x-x_n)^8}{336h^4r^2(r-s)^3(r-1)^2}(2r^2s + 2r^2 - rs^2 + 2rs + r - s^3 - 4s^2 - s) - \frac{s(x-x_n)^6}{120h^2r^2(r-s)^3(r-1)^2}(-2r^2s - 2r^2 + 2rs^2 + rs + s^2) - \frac{h^3s(x-x_n)}{5040r(r-s)^2(r-1)^2}(3r^7 - 7r^6s - 10r^6 + 2r^5s^2 + 26r^5s + 9r^5 + 2r^4s^3 - 10r^4s^2 - 27r^4s + 2r^3s^4 - 10r^3s^3 + 15r^3s^2 + 2r^2s^5 - 10r^2s^4 + 15r^2s^3 + 2rs^6 - 10rs^5 + 15rs^4 - 2s^7 + 8s^6 - 9s^5)$,

$\gamma_s = \frac{(x-x_n)^{10}}{720h^6s^2(r-s)^2(s-1)^2} + \frac{(x-x_n)^7}{210h^3s^2(r-s)^3(s-1)^2}(-r^3s - 2r^3 + r^2s^2 - 2r^2s - 2r^2 + 4rs^2 + rs + s^2) - \frac{r^2(x-x_n)^5}{60hs(r-s)^2(s-1)^2} + \frac{(x-x_n)^9}{504h^5s^2(r-s)^3(s-1)^2}(-2r^2 + rs - 2r + s^2 + 2s) + \frac{(x-x_n)^8}{336h^4s^2(r-s)^3(s-1)^2}(r^3 + r^2s + 4r^2 - 2rs^2 - 2rs + r - 2s^2 - s) - \frac{h^2(x-x_n)^2}{5040s^2(r-s)^3(s-1)^2}(2r^9 - 4r^8s - 8r^8 + 18r^7s + 9r^7 - 24r^6s + 9r^2s^7 - 36r^2s^6 + 42r^2s^5 - 10rs^8 + 36rs^7 - 36rs^6 + 3s^9 - 10s^8 + 9s^7) + \frac{r(x-x_n)^6}{120h^2s^2(r-s)^3(s-1)^2}(2r^2s + r^2 - 2rs^2 + rs - 2s^2) - \frac{h^3r(x-x_n)}{5040s(r-s)^2(s-1)^2}(-2r^7 + 2r^6s + 8r^6 + 2r^5s^2 - 10r^5s - 9r^5 + 2r^4s^3 - 10r^4s^2 + 15r^4s + 2r^3s^4 - 10r^3s^3 + 15r^3s^2 + 2r^2s^5 - 10r^2s^4 + 15r^2s^3 - 7rs^6 + 26rs^5 - 27rs^4 + 3s^7 - 10s^6 + 9s^5)$,

$\gamma_1 = -\frac{(x-x_n)(x_n-x+hr)(x_n-x+hs)}{5040h^6(r-1)^2(s-1)^2}(6h^7r^6s - 2h^7r^7 + 4h^7r^6 - 3h^7r^5s^2 - 14h^7r^5s - 3h^7r^4s^3 + 10h^7r^4s^2 - 3h^7r^3s^4 + 10h^7r^3s^3 - 3h^7r^2s^5 + 10h^7r^2s^4 + 6h^7rs^6 - 14h^7rs^5 - 2h^7s^7 + 4h^7s^6 - 2h^6r^6x + 2h^6r^6x_n + 6h^6r^5sx - 6h^6r^5sx_n + 4h^6r^5x - 4h^6r^5x_n - 3h^6r^4s^2x + 3h^6r^4s^2x_n - 14h^6r^4sx + 14h^6r^4sx_n - 3h^6r^3s^3x + 3h^6r^3s^3x_n + 10h^6r^3s^2x - 10h^6r^3s^2x_n - 3h^6r^2s^4x + 3h^6r^2s^4x_n + 10h^6r^2s^3x - 10h^6r^2s^3x_n + 6h^6rs^5x - 6h^6rs^5x_n - 14h^6rs^4x + 14h^6rs^4x_n - 2h^6s^6x + 2h^6s^6x_n + 4h^6s^5x - 4h^6s^5x_n - 2h^5r^5x^2 + 4h^5r^5xx_n - 2h^5r^5x_n^2 + 6h^5r^4sx^2 - 12h^5r^4sxx_n + 6h^5r^4sx_n^2 + 4h^5r^4x^2 - 8h^5r^4xx_n + 4h^5r^4x_n^2 - 3h^5r^3s^2x^2 + 6h^5r^3s^2xx_n - 3h^5r^3s^2x_n^2 - 14h^5r^3sx^2 + 28h^5r^3sxx_n - 14h^5r^3sx_n^2 - 3h^5r^2s^3x^2 + 6h^5r^2s^3xx_n - 3h^5r^2s^3x_n^2 + 10h^5r^2s^2x^2 - 20h^5r^2s^2x$

$x_n + 10h^5r^2s^2x_n^2 + 6h^5rs^4x^2 - 12h^5rs^4xx_n + 6h^5rs^4x_n^2 - 14h^5rs^3x^2 + 28h^5rs^3xx_n - 14h^5rs^3x_n^2 - 2h^5s^5x^2 + 4h^5s^5xx_n - 2h^5s^5x_n^2 + 4h^5s^4x^2 - 8h^5s^4xx_n + 4h^5s^4x_n^2 - 2h^4r^4x^3 + 6h^4r^4x^2x_n - 6h^4r^4xx_n^2 + 2h^4r^4x_n^3 + 6h^4r^3sx^3 - 18h^4r^3sx^2x_n + 18h^4r^3sxx_n^2 - 6h^4r^3sx_n^3 + 4h^4r^3x^3 - 12h^4r^3x^2x_n + 12h^4r^3xx_n^2 - 4h^4r^3x_n^3 - 3h^4r^2s^2x^3 + 9h^4r^2s^2x^2x_n - 9h^4r^2s^2xx_n^2 + 3h^4r^2s^2x_n^3$

$- 14h^4r^2sx^3 + 42h^4r^2sx^2x_n - 42h^4r^2sxx_n^2 + 14h^4r^2sx_n^3 + 6h^4rs^3x^3 - 18h^4rs^3x^2x_n + 18h^4rs^3xx_n^2 - 6h^4rs^3x_n^3 - 14h^4rs^2x^3 + 42h^4rs^2x^2x_n - 42h^4rs^2xx_n^2 + 14h^4rs^2x_n^3 - 2h^4s^4x^3 + 6h^4s^4x^2x_n - 6h^4s^4xx_n^2 + 2h^4s^4x_n^3 + 4h^4s^3x^3 - 12h^4s^3x^2x_n + 12h^4s^3xx_n^2 - 4h^4s^3x_n^3 - 2h^3r^3x^4 + 8h^3r^3x^3x_n - 12h^3r^3x^2x_n^2 + 8h^3r^3xx_n^3 - 2h^3r^3x_n^4 + 6h^3r^2sx^4 - 24h^3r^2sx^3x_n + 36h^3r^2sx^2$

$x_n^2 - 24h^3r^2sxx_n^3 + 6h^3r^2sx_n^4 + 4h^3r^2x^4 - 16h^3r^2x^3x_n + 24h^3r^2x^2x_n^2 - 16h^3r^2xx_n^3 +$

$4h^3r^2x_n^4 + 6h^3rs^2x^4 - 24h^3rs^2x^3x_n + 36h^3rs^2x^2x_n^2 - 24h^3rs^2x$
$x_n^3 + 6h^3rs^2x_n^4 + 46h^3rsx^4 - 184h^3rsx^3x_n + 276h^3rsx^2x_n^2 - 184h^3rsxx_n^3 + 46h^3rsx_n^4 -$
$2h^3s^3x^4 + 8h^3s^3x^3x_n - 12h^3s^3x^2x_n^2 + 8h^3s^3xx_n^3 - 2h^3s^3x_n^4 + 4h^3s^2x^4 - 16h^3s^2x^3x_n +$
$24h^3s^2x^2x_n^2 - 16h^3s^2xx_n^3 + 4h^3s^2x_n^4 - 2h^2r^2x^5 + 10h^2r^2x^4x_n - 20h^2r^2x^3x_n^2 + 20h^2r^2x^2x_n^3 -$
$10h^2r^2xx_n^4 + 2h^2r^2x_n^5 - 27h^2rsx^5 + 135h^2rsx^4x_n - 270h^2rsx^3x_n^2 + 270h^2rsx^2x_n^3 -$
$135h^2rsxx_n^4 + 27h^2rsx_n^5 - 20h^2rx^5 + 100h^2rx^4x_n - 200h^2rx^3x_n^2 + 200h^2rx^2x_n^3 - 100h^2rxx_n^4 +$
$20h^2rx_n^5 - 2h^2s^2x^5 + 10h^2s^2x^4x_n - 20h^2s^2x^3x_n^2 + 20h^2s^2x^2x_n^3 - 10h^2s^2xx_n^4 + 2h^2s^2x_n^5 -$
$20h^2sx^5 + 100h^2sx^4x_n - 200h^2sx^3x_n^2 + 200h^2sx^2x_n^3 - 100h^2sxx_n^4 + 20h^2sx_n^5 + 13hrx^6 -$
$78hrx^5x_n + 195hrx^4x_n^2 - 260hrx^3x_n^3 + 195hrx^2x_n^4 - 78hrxx_n^5 + 13hrx_n^6 + 13hsx^6 -$
$78hsx^5x_n + 195hsx^4x_n^2 - 260hsx^3x_n^3 + 195hsx^2x_n^4 - 78hsxx_n^5 + 13hsx_n^6 + 10hx^6 -$
$60hx^5x_n + 150hx^4x_n^2 - 200hx^3x_n^3 + 150hx^2x_n^4 - 60hxx_n^5 + 10hx_n^6 - 7x^7 + 49x^6x_n -$
$147x^5x_n^2 + 245x^4x_n^3 - 245x^3x_n^4 + 147x^2x_n^5 - 49xx_n^6 + 7x_n^7)$.

**Appendix II**

$B_{19}^{[0]} = \frac{-h^3r^3}{2520s^3}(-2r^5s - 2r^5 + 8r^4s^2 + 12r^4s + 8r^4 - 9r^3s^3 - 28r^3s^2 - 28r^3s - 9r^3 +$
$27r^2s^3 + 12r^2s^2 + 27r^2s + 48rs^3 + 48rs^2 - 336s^3)$,
$B_{29}^{[0]} = \frac{-h^3s^3}{2520r^3}(-9r^3s^3 + 27r^3s^2 + 48r^3s - 336r^3 + 8r^2s^4 - 28r^2s^3 + 12r^2s^2 + 48r^2s -$
$2rs^5 + 12rs^4 - 28rs^3 + 27rs^2 - 2s^5 + 8s^4 - 9s^3)$,
$B_{39}^{[0]} = \frac{-h^3}{2520r^3s^3}(-336r^3s^3 + 48r^3s^2 + 27r^3s - 9r^3 + 48r^2s^3 + 12r^2s^2 - 28r^2s + 8r^2 +$
$27rs^3 - 28rs^2 + 12rs - 2r - 9s^3 + 8s^2 - 2s)$,
$B_{49}^{[0]} = \frac{-h^2r^2}{2520s^3}(-10r^5s - 10r^5 + 36r^4s^2 + 53r^4s + 36r^4 - 36r^3s^3 - 108r^3s^2 - 108r^3s -$
$36r^3 + 90r^2s^3 + 24r^2s^2 + 90r^2s + 168rs^3 + 168rs^2 - 882s^3)$,
$B_{59}^{[0]} = \frac{-h^2s^2}{2520r^3}(-36r^3s^3 + 90r^3s^2 + 168r^3s - 882r^3 + 36r^2s^4 - 108r^2s^3 + 24r^2s^2 + 168r^2s -$
$10rs^5 + 53rs^4 - 108rs^3 + 90rs^2 - 10s^5 + 36s^4 - 36s^3)$,
$B_{69}^{[0]} = \frac{-h^2}{2520r^3s^3}(-882r^3s^3 + 168r^3s^2 + 90r^3s - 36r^3 + 168r^2s^3 + 24r^2s^2 - 108r^2s + 36r^2 +$
$90rs^3 - 108rs^2 + 53rs - 10r - 36s^3 + 36s^2 - 10s)$,
$B_{79}^{[0]} = \frac{-hr}{420s^3}(-5r^5s - 5r^5 + 16r^4s^2 + 23r^4s + 16r^4 - 14r^3s^3 - 40r^3s^2 - 40r^3s - 14r^3 +$
$28r^2s^3 + 28r^2s + 56rs^3 + 56rs^2 - 210s^3)$,
$B_{89}^{[0]} = \frac{hs}{420r^3}(14r^3s^3 - 28r^3s^2 - 56r^3s + 210r^3 - 16r^2s^4 + 40r^2s^3 - 56r^2s + 5rs^5 - 23rs^4 +$
$40rs^3 - 28rs^2 + 5s^5 - 16s^4 + 14s^3)$,
$B_{99}^{[0]} = \frac{h}{420r^3s^3}(210r^3s^3 - 56r^3s^2 - 28r^3s + 14r^3 - 56r^2s^3 + 40r^2s - 16r^2 - 28rs^3 + 40rs^2 -$
$23rs + 5r + 14s^3 - 16s^2 + 5s)$.
$B_{11}^{[1]} = \frac{h^3r^3}{2520(r-s)^3(r-1)^3}(14r^6 - 56r^5s - 56r^5 + 74r^4s^2 + 234r^4s + 74r^4 - 30r^3s^3 - 325r^3s^2 -$
$325r^3s - 30r^3 + 135r^2s^3 + 480r^2s^2 + 135r^2s - 204rs^3 - 204rs^2 + 84s^3)$,
$B_{21}^{[1]} = \frac{-h^3s^7}{2520r^3(r-s)^3(r-1)^3}(-14r^3s^2 + 63r^3s - 84r^3 + 16r^2s^3 - 59r^2s^2 + 42r^2s + 60r^2 -$
$4rs^4 + 6rs^3 + 25rs^2 - 54rs + 2s^4 - 8s^3 + 9s^2)$,
$B_{31}^{[1]} = \frac{-h^3}{2520r^3(r-s)^3(r-1)^3}(-84r^3s^2 + 63r^3s - 14r^3 + 60r^2s^3 + 42r^2s^2 - 59r^2s + 16r^2 -$
$54rs^3 + 25rs^2 + 6rs - 4r + 9s^3 - 8s^2 + 2s)$,
$B_{41}^{[1]} = \frac{h^2r^2}{2520(r-s)^3(r-1)^3}(105r^6 - 385r^5s - 385r^5 + 468r^4s^2 + 1457r^4s + 468r^4 - 180r^3s^3 -$

$1836r^3s^2 - 1836r^3s - 180r^3 + 720r^2s^3 + 2418r^2s^2 + 720r^2s - 966rs^3 - 966rs^2 + 378s^3)$,
$B_{51}^{[1]} = \frac{-h^2s^6}{2520r^3(r-s)^3(r-1)^3}(-63r^3s^2 + 252r^3s - 294r^3 + 75r^2s^3 - 243r^2s^2 + 138r^2s + 210r^2 - 20rs^4 + 25rs^3 + 108rs^2 - 198rs + 10s^4 - 36s^3 + 36s^2)$,
$B_{61}^{[1]} = \frac{-h^2}{2520r^3(r-s)^3(r-1)^3}(-294r^3s^2 + 252r^3s - 63r^3 + 210r^2s^3 + 138r^2s^2 - 243r^2s + 75r^2 - 198rs^3 + 108rs^2 + 25rs - 20r + 36s^3 - 36s^2 + 10s)$,
$B_{71}^{[1]} = \frac{hr}{420(r-s)^3(r-1)^3}(105r^6 - 350r^5s - 350r^5 + 388r^4s^2 + 1187r^4s + 388r^4 - 140r^3s^3 - 1342r^3s^2 - 1342r^3s - 140r^3 + 490r^2s^3 + 1554r^2s^2 + 490r^2s - 574rs^3 - 574rs^2 + 210s^3)$,
$B_{81}^{[1]} = \frac{-hs^5}{420r^3(r-s)^3(r-1)^3}(-28r^3s^2 + 98r^3s - 98r^3 + 35r^2s^3 - 98r^2s^2 + 42r^2s + 70r^2 - 10rs^4 + 10rs^3 + 46rs^2 - 70rs + 5s^4 - 16s^3 + 14s^2)$,
$B_{91}^{[1]} = \frac{-h}{420r^3(r-s)^3(r-1)^3}(-98r^3s^2 + 98r^3s - 28r^3 + 70r^2s^3 + 42r^2s^2 - 98r^2s + 35r^2 - 70rs^3 + 46rs^2 + 10rs - 10r + 14s^3 - 16s^2 + 5s)$,
$B_{12}^{[1]} = \frac{h^3r^7}{2520s^3(r-s)^3(s-1)^3}(-4r^4s + 2r^4 + 16r^3s^2 + 6r^3s - 8r^3 - 14r^2s^3 - 59r^2s^2 + 25r^2s + 9r^2 + 63rs^3 + 42rs^2 - 54rs - 84s^3 + 60s^2)$,
$B_{22}^{[1]} = \frac{-h^3s^3}{2520(r-s)^3(s-1)^3}(-30r^3s^3 + 135r^3s^2 - 204r^3s + 84r^3 + 74r^2s^4 - 325r^2s^3 + 480r^2s^2 - 204r^2s - 56rs^5 + 234rs^4 - 325rs^3 + 135rs^2 + 14s^6 - 56s^5 + 74s^4 - 30s^3)$,
$B_{32}^{[1]} = \frac{h^3}{2520s^3(r-s)^3(s-1)^3}(60r^3s^2 - 54r^3s + 9r^3 - 84r^2s^3 + 42r^2s^2 + 25r^2s - 8r^2 + 63rs^3 - 59rs^2 + 6rs + 2r - 14s^3 + 16s^2 - 4s)$,
$B_{42}^{[1]} = \frac{h^2r^6}{2520s^3(r-s)^3(s-1)^3}(-20r^4s + 10r^4 + 75r^3s^2 + 25r^3s - 36r^3 - 63r^2s^3 - 243r^2s^2 + 108r^2s + 36r^2 + 252rs^3 + 138rs^2 - 198rs - 294s^3 + 210s^2)$,
$B_{52}^{[1]} = \frac{-h^2s^2}{2520(r-s)^3(s-1)^3}(-180r^3s^3 + 720r^3s^2 - 966r^3s + 378r^3 + 468r^2s^4 - 1836r^2s^3 + 2418r^2s^2 -$

$966r^2s - 385rs^5 + 1457rs^4 - 1836rs^3 + 720rs^2 + 105s^6 - 385s^5 + 468s^4 - 180s^3)$,
$B_{62}^{[1]} = \frac{h^2}{2520s^3(r-s)^3(s-1)^3}(210r^3s^2 - 198r^3s + 36r^3 - 294r^2s^3 + 138r^2s^2 + 108r^2s - 36r^2 + 252rs^3 - 243rs^2 + 25rs + 10r - 63s^3 + 75s^2 - 20s)$,
$B_{72}^{[1]} = \frac{hr^5}{420s^3(r-s)^3(s-1)^3}(-10r^4s + 5r^4 + 35r^3s^2 + 10r^3s - 16r^3 - 28r^2s^3 - 98r^2s^2 + 46r^2s + 14r^2 + 98rs^3 + 42rs^2 - 70rs - 98s^3 + 70s^2)$,
$B_{82}^{[1]} = \frac{-hs}{420(r-s)^3(s-1)^3}(-140r^3s^3 + 490r^3s^2 - 574r^3s + 210r^3 + 388r^2s^4 - 1342r^2s^3 + 1554r^2s^2 - 574r^2s - 350rs^5 + 1187rs^4 - 1342rs^3 + 490rs^2 + 105s^6 - 350s^5 + 388s^4 - 140s^3)$,
$B_{92}^{[1]} = \frac{h}{420s^3(r-s)^3(s-1)^3}(70r^3s^2 - 70r^3s + 14r^3 - 98r^2s^3 + 42r^2s^2 + 46r^2s - 16r^2 + 98rs^3 - 98rs^2 + 10rs + 5r - 28s^3 + 35s^2 - 10s)$,
$B_{13}^{[1]} = \frac{-h^3r^7}{2520(r-1)^3(s-1)^3}(2r^4s - 4r^4 - 8r^3s^2 + 6r^3s + 16r^3 + 9r^2s^3 + 25r^2s^2 - 59r^2s - 14r^2 - 54rs^3 + 42rs^2 + 63rs + 60s^3 - 84s^2)$,
$B_{23}^{[1]} = \frac{-h^3s^7}{2520(r-1)^3(s-1)^3}(9r^3s^2 - 54r^3s + 60r^3 - 8r^2s^3 + 25r^2s^2 + 42r^2s - 84r^2 + 2rs^4 + 6rs^3 - 59rs^2 + 63rs - 4s^4 + 16s^3 - 14s^2)$,
$B_{33}^{[1]} = \frac{-h^3}{2520(r-1)^3(s-1)^3}(-84r^3s^3 + 204r^3s^2 - 135r^3s + 30r^3 + 204r^2s^3 - 480r^2s^2 + 325r^2s - 74r^2 - 135rs^3 + 325rs^2 - 234rs + 56r + 30s^3 - 74s^2 + 56s - 14)$,
$B_{43}^{[1]} = \frac{-h^2r^6}{2520(r-1)^3(s-1)^3}(10r^4s - 20r^4 - 36r^3s^2 + 25r^3s + 75r^3 + 36r^2s^3 + 108r^2s^2 - 243r^2s - 63r^2 - 198rs^3 + 138rs^2 + 252rs + 210s^3 - 294s^2)$,
$B_{53}^{[1]} = \frac{-h^2s^6}{2520(r-1)^3(s-1)^3}(36r^3s^2 - 198r^3s + 210r^3 - 36r^2s^3 + 108r^2s^2 + 138r^2s - 294r^2 +$

$10rs^4+25rs^3-243rs^2+252rs-20s^4+75s^3-63s^2)$,
$B_{63}^{[1]}=\frac{-h^2}{2520(r-1)^3(s-1)^3}(-378r^3s^3+966r^3s^2-720r^3s+180r^3+966r^2s^3-2418r^2s^2+1836r^2s-468$
$r^2-720rs^3+1836rs^2-1457rs+385r+180s^3-468s^2+385s-105)$,
$B_{73}^{[1]}=\frac{-hr^5}{420(r-1)^3(s-1)^3}(5r^4s-10r^4-16r^3s^2+10r^3s+35r^3+14r^2s^3+46r^2s^2-98r^2s-28r^2-70rs^3+42rs^2+98rs+70s^3-98s^2)$,
$B_{83}^{[1]}=\frac{-hs^5}{420(r-1)^3(s-1)^3}(14r^3s^2-70r^3s+70r^3-16r^2s^3+46r^2s^2+42r^2s-98r^2+5rs^4+10rs^3-98rs^2+98rs-10s^4+35s^3-28s^2)$,
$B_{93}^{[1]}=\frac{-h}{420(r-1)^3(s-1)^3}(-210r^3s^3+574r^3s^2-490r^3s+140r^3+574r^2s^3-1554r^2s^2+1342r^2s-388r^2-490rs^3+1342rs^2-1187rs+350r+140s^3-388s^2+350s-105)$.
$D_{19}^{[0]}=\frac{h^4r^4}{5040s^2}(2r^4-8r^3s-8r^3+9r^2s^2+36r^2s+9r^2-48rs^2-48rs+84s^2)$,
$D_{29}^{[0]}=\frac{h^4s^4}{5040r^2}(9r^2s^2-48r^2s+84r^2-8rs^3+36rs^2-48rs+2s^4-8s^3+9s^2)$,
$D_{39}^{[0]}=\frac{h^4}{5040r^2s^2}(84r^2s^2-48r^2s+9r^2-48rs^2+36rs-8r+9s^2-8s+2)$,
$D_{49}^{[0]}=\frac{h^3r^3}{2520s^2}(5r^4-18r^3s-18r^3+18r^2s^2+72r^2s+18r^2-84rs^2-84rs+126s^2)$,
$D_{59}^{[0]}=\frac{h^3s^3}{2520r^2}(18r^2s^2-84r^2s+126r^2-18rs^3+72rs^2-84rs+5s^4-18s^3+18s^2)$,
$D_{69}^{[0]}=\frac{h^3}{2520r^2s^2}(126r^2s^2-84r^2s+18r^2-84rs^2+72rs-18r+18s^2-18s+5)$,
$D_{79}^{[0]}=\frac{h^2r^2}{840s^2}(5r^4-16r^3s-16r^3+14r^2s^2+56r^2s+14r^2-56rs^2-56rs+70s^2)$,
$D_{89}^{[0]}=\frac{h^2s^2}{840r^2}(14r^2s^2-56r^2s+70r^2-16rs^3+56rs^2-56rs+5s^4-16s^3+14s^2)$,
$D_{99}^{[0]}=\frac{h^2}{840r^2s^2}(70r^2s^2-56r^2s+14r^2-56rs^2+56rs-16r+14s^2-16s+5)$.
$D_{11}^{[1]}=\frac{-h^4r^4}{5040(r-s)^2(r-1)^2}(3r^4-10r^3s-10r^3+9r^2s^2+36r^2s+9r^2-36rs^2-36rs+42s^2)$,
$D_{21}^{[1]}=\frac{-h^4s^7}{5040r^2(r-s)^2(r-1)^2}(24r-9s-18rs+4rs^2+8s^2-2s^3)$,
$D_{31}^{[1]}=\frac{-h^4}{5040r^2(r-s)^2(r-1)^2}(4r+8s-18rs+24rs^2-9s^2-2)$,
$D_{41}^{[1]}=\frac{-h^3r^3}{1260(r-s)^2(r-1)^2}(5r^4-15r^3s-15r^3+12r^2s^2+48r^2s+12r^2-42rs^2-42rs+42s^2)$,
$D_{51}^{[1]}=\frac{-h^3s^6}{2520r^2(r-s)^2(r-1)^2}(42r-18s-36rs+9rs^2+18s^2-5s^3)$,
$D_{61}^{[1]}=\frac{-h^3}{2520r^2(r-s)^2(r-1)^2}(9r+18s-36rs+42rs^2-18s^2-5)$,
$D_{71}^{[1]}=\frac{-h^2r^2}{840(r-s)^2(r-1)^2}(15r^4-40r^3s-40r^3+28r^2s^2+112r^2s+28r^2-84rs^2-84rs+70s^2)$,
$D_{81}^{[1]}=\frac{-h^2s^5}{840r^2(r-s)^2(r-1)^2}(28r-14s-28rs+8rs^2+16s^2-5s^3)$,
$D_{91}^{[1]}=\frac{-h^2}{840r^2(r-s)^2(r-1)^2}(8r+16s-28rs+28rs^2-14s^2-5)$,
$D_{12}^{[1]}=\frac{h^4r^7}{5040s^2(r-s)^2(s-1)^2}(9r-24s+18rs-4r^2s-8r^2+2r^3)$,
$D_{22}^{[1]}=\frac{-h^4s^4}{5040(r-s)^2(s-1)^2}(9r^2s^2-36r^2s+42r^2-10rs^3+36rs^2-36rs+3s^4-10s^3+9s^2)$,
$D_{32}^{[1]}=\frac{-h^4}{5040s^2(r-s)^2(s-1)^2}(8r+4s-18rs+24r^2s-9r^2-2)$,
$D_{42}^{[1]}=\frac{h^3r^6}{2520s^2(r-s)^2(s-1)^2}(18r-42s+36rs-9r^2s-18r^2+5r^3)$,
$D_{52}^{[1]}=\frac{-h^3s^3}{1260(r-s)^2(s-1)^2}(12r^2s^2-42r^2s+42r^2-15rs^3+48rs^2-42rs+5s^4-15s^3+12s^2)$,
$D_{62}^{[1]}=\frac{-h^3}{2520s^2(r-s)^2(s-1)^2}(18r+9s-36rs+42r^2s-18r^2-5)$,
$D_{72}^{[1]}=\frac{h^2r^5}{840s^2(r-s)^2(s-1)^2}(14r-28s+28rs-8r^2s-16r^2+5r^3)$,

$D_{82}^{[1]} = \frac{-h^2s^2}{840(r-s)^2(s-1)^2}(28r^2s^2-84r^2s+70r^2-40rs^3+112rs^2-84rs+15s^4-40s^3+28s^2)$,
$D_{92}^{[1]} = \frac{-h^2}{840s^2(r-s)^2(s-1)^2}(16r+8s-28rs+28r^2s-14r^2-5)$,
$D_{13}^{[1]} = \frac{h^4r^7}{5040(r-1)^2(s-1)^2}(2r^3-8r^2s-4r^2+9rs^2+18rs-24s^2)$,
$D_{23}^{[1]} = \frac{h^4s^7}{5040(r-1)^2(s-1)^2}(9r^2s-24r^2-8rs^2+18rs+2s^3-4s^2)$,
$D_{33}^{[1]} = \frac{-h^4}{5040(r-1)^2(s-1)^2}(42r^2s^2-36r^2s+9r^2-36rs^2+36rs-10r+9s^2-10s+3)$,
$D_{43}^{[1]} = \frac{h^3r^6}{2520(r-1)^2(s-1)^2}(5r^3-18r^2s-9r^2+18rs^2+36rs-42s^2)$,
$D_{53}^{[1]} = \frac{h^3s^6}{2520(r-1)^2(s-1)^2}(18r^2s-42r^2-18rs^2+36rs+5s^3-9s^2)$,
$D_{63}^{[1]} = \frac{-h^3}{1260(r-1)^2(s-1)^2}(42r^2s^2-42r^2s+12r^2-42rs^2+48rs-15r+12s^2-15s+5)$,
$D_{73}^{[1]} = \frac{h^2r^5}{840(r-1)^2(s-1)^2}(5r^3-16r^2s-8r^2+14rs^2+28rs-28s^2)$,
$D_{83}^{[1]} = \frac{h^2s^5}{840(r-1)^2(s-1)^2}(14r^2s-28r^2-16rs^2+28rs+5s^3-8s^2)$,
$D_{93}^{[1]} = \frac{-h^2}{840(r-1)^2(s-1)^2}(70r^2s^2-84r^2s+28r^2-84rs^2+112rs-40r+28s^2-40s+15)$.