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# INTEGRAL REPRESENTATION OF THE SOLUTION OF THE CAUCHY PROBLEM FOR AUTONOMOUS LINEAR NEUTRAL FRACTIONAL SYSTEM

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**Abstract:** The aim of the paper is to obtain a variation of constant formula for an autonomous linear fractional neutral system with Caputo type derivatives and multiple delays. The obtained results improve and extend the corresponding results even in the particular case of fractional system with one constant delay and coincide with the corresponding ones in the case of first order neutral linear differential system.

**AMS Subject Classification:** 34A08, 34A30
**Key Words:** fractional derivatives, neutral fractional system, general solution

## 1. Introduction

The fractional calculus has more than three centuries history, but it has attracted considerable attention and interest in the last few decades as an important tool for modelling of various real problems as viscoelastic systems, diffusion processes, signal and control processing, etc. Detailed information on fractional calculus theory and applications can be found in the monographs of Podlubny [11], Kiryakova [5] and Kilbas et al [4].

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It is well known that the study of linear fractional equations (integral representation, several types of stability and etc.) is an evergreen theme for research. Concerning fundamental and qualitative investigations for linear fractional ordinary differential equations and systems we refer to [1], [3], [4], [10], [11] and the references therein. Some interesting results in this direction are obtained also using the Laplace transform method in [7], [8], [11]. About works concerning fractional differential systems with delay we point out [16], [18], [19], [20] and [2], [9], [12], [13], [14], [15] devoted to the case of distributed delay.

The problem of existence of a fundamental (Cauchy) matrix for linear homogenous fractional system (delayed or neutral) leads to prove that the initial problem (IP) for this system with discontinuous initial function has a unique solution. As far as we know there are only two results concerning an initial problem for this system with discontinuous initial function - [6] for the delayed case with Riemann-Liouville fractional derivatives and [17] for the neutral case with Caputo fractional derivatives.

The aim of our work is using the results obtained in [17] to prove an integral representation for the general solution of the Cauchy problem for autonomous linear neutral fractional system with Caputo type derivatives and multiple delays in the homogeneous and nonhomogeneous cases. Note that our results extend and improve the results in [19] even in the case of one constant delay considered there.

The paper is organized as follows. In Section 2 are given the necessary definitions of Riemann-Liouville and Caputo fractional derivatives, as well as some of their properties. Section 3 is devoted to the problem statement. In Section 4 as main result are proved integral representations of the general solution of an autonomous linear neutral fractional system with Caputo type derivatives and multiple delays in the homogeneous and nonhomogeneous cases.

## 2. Preliminaries

Below are described the definitions of Riemann–Liouville and Caputo fractional derivatives and some their properties necessary for our exposition, mainly to avoid possible misunderstandings. For details and other properties we refer to [4] and [11].

Let denote by $L_1^{loc}(\mathbb{R},\mathbb{R})$ the linear space of all locally Lebesgue integrable functions $f:\mathbb{R}\to\mathbb{R}$. Then for each $a\in\mathbb{R}$ and $f\in L_1^{loc}(\mathbb{R},\mathbb{R})$ the left-sided fractional integral operators of order $\alpha\in(0,1)$ and Riemann–Louville fractional derivative are defined for $t > a$ with

$$(D_{a+}^{-\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds,$$
$$_{RL}D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dt}\right) \int_a^t (t-s)^{-\alpha} f(s)ds$$

respectively. The Caputo fractional left side derivative (see Kilbas et al. [4], p. 91) is defined with

$$_{C}D_{a+}^{\alpha} f(t) = {}_{RL}D_{a+}^{\alpha}[f(s) - f(a)](t)$$

and hence

$$_{C}D_{a+}^{\alpha} f(t) = {}_{RL}D_{a+}^{\alpha} f(t) - \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha}.$$

Concerning the Laplace transform $(\mathfrak{L}f)(p) = \int_0^\infty e^{-pt} f(t)dt$, $p \in \mathbb{C}$ we need the properties:

(j) $(\mathfrak{L}D_{0+}^{-\alpha} f)(p) = p^{-\alpha}(\mathfrak{L}f)(p)$;
(jj)$(\mathfrak{L}_{RL}D_{0+}^{\alpha} f)(p) = p^{\alpha}(\mathfrak{L}f)(p) - [_{RL}D^{\alpha-1}\ f(t)]_{t=0}$;
(jjj)$(\mathfrak{L}_{C}D_{0+}^{\alpha} f)(p) = p^{\alpha}(\mathfrak{L}f)(p) - p^{\alpha-1}f(0)$.

Below are used the following notations: $\mathbb{C}_+ = \{p \in \mathbb{C} | \operatorname{Re} p > 0\}$, $\bar{\mathbb{C}}_+ = \{p \in \mathbb{C} | \operatorname{Re} p \geq 0\}$, $\mathbb{C}_- = \mathbb{C} \setminus \bar{\mathbb{C}}_+$ and $I_\beta(p) = diag(p^{\beta_1}, ..., p^{\beta_n})$ for $p \in \mathbb{C}$, $I_\beta(0) = 0$, where $\beta_k \in [-1, 1], k \in \langle n \rangle = \{1, 2, ..., n\}, n \in \mathbb{N}$.

## 3. Problem Statement

In the present work we consider an nonhomogeneous autonomous linear neutral fractional system with multiple delays in the form

$$D_{0+}^{\alpha}[X(t) - CX(t-\tau)] = AX(t) + \sum_{i=1}^{m} B^i X(t-\sigma_i) + F(t), \tag{1}$$

and the corresponding homogeneous one

$$D_{0+}^{\alpha}[X(t) - CX(t-\tau)] = AX(t) + \sum_{i=1}^{m} B^i X(t-\sigma_i) \tag{2}$$

where $A, C, B^i \in \mathbb{R}^{n \times n}$, $i \in \langle m \rangle$, $m \in \mathbb{N}$, $X, F : \bar{\mathbb{R}}_+ \to \mathbb{R}^n$, $\bar{\mathbb{R}}_+ = [0, \infty)$, $\mathbb{R}_+ = (0, \infty)$, $J^* = [-h, \infty)$, $\tau, \sigma_1, ..., \sigma_m > 0$, $h = \max(\sigma_1, ..., \sigma_m, \tau)$, $\alpha = (\alpha_1, ..., \alpha_n)$,

$\alpha_k \in (0,1)$, $k \in \langle n\rangle$ and $D^{\alpha_k}_{0+}$ denotes the left side Caputo fractional derivative ${}_C D^{\alpha_k}_{0+}$ with lower terminal zero. Above are used also the following notations $D^{\alpha}_{0+} = diag(D^{\alpha_1}_{0+}, ..., D^{\alpha_n}_{0+})$, $D^{\alpha}_{0+}X(t) = (D^{\alpha_1}_{0+}x_1(t), ..., D^{\alpha_n}_{0+}x_n(t))^T$, $X(t) = (x_1(t), ..., x_n(t))^T$, $F(t) = (f_1(t), ..., f_n(t))^T$, $A = \{a_{kj}\}^n_{k,j=1}$, $B^i = \{b^i_{kj}\}^n_{k,j=1}$.

Let denote the space of initial functions by $\mathbf{C}$, $\mathbf{C} = C([-h,0], \mathbb{R}^n)$ and consider the following initial conditions for the system (1) or (2) with derivatives in Caputo sense:

$$X(t) = \Phi(t),\ \Phi \in \mathbf{C},\ (x_k(t) = \phi_k(t),\ k \in \langle n\rangle),\ t \in [-h, 0] \tag{3}$$

**Definition 1.** The vector function $X(t) = (x_1(t), ..., x_n(t))^T$ is a continuous solution of the IP (1), (3) (or IP (2), (3) ) in the interval $\bar{\mathbb{R}}_+$ if $X|_{\bar{\mathbb{R}}_+} \in C(\bar{\mathbb{R}}_+, \mathbb{R}^n)$ satisfies the system (1) (or (2)) for all $t \in \mathbb{R}_+$ and the initial condition (3) for $t \in [-h, 0]$.

Let $s \in \bar{\mathbb{R}}_+$ be an arbitrary fixed number, $J_s = [s, \infty)$ and consider the following matrix IP

$$D^{\alpha}_{0+}[Q(t,s) - CQ(t-\tau, s)] = AQ(t,s) + \sum_{i=1}^{m} B^i Q(t - \sigma_i, s) \tag{4}$$

with initial condition

$$Q(t,s) = \begin{cases} I, & t = s \\ 0, & -\infty < t < s \end{cases} \tag{5}$$

**Definition 2.** For each $s \in \bar{\mathbb{R}}_+$ the matrix valued function $t \to Q(t,s) = \{\gamma_{kj}(t,s)\}^n_{k,j=1}$, $Q(\cdot, s) : J_s \to \mathbb{R}^{n\times n}$ is called a solution of the IP (4), (5) for $t \in J_s$, if $Q(\cdot, s)$ is continuous in $t$ on $J_s$ and satisfies the matrix equation (4) for $t \in (s, \infty)$ as well as the initial condition (5) too.

In the case when $s = 0$ the matrix $Q(t) = Q(t, 0)$ will be called fundamental (or Cauchy) matrix for the system (2) and in the case when $s > 0$ the matrix $Q(t,s)$ will be called expanded fundamental matrix for the same system.

**Remark 3.** Note that from Corollary 8 in [17] it follows that the IP (4), (5) has a unique solution. Moreover from Theorem 6 in [17] it follows that the IP (1), (3) has a unique continuous solution for each $\Phi \in \mathbf{C} = C([-h, 0], \mathbb{R}^n)$ and locally bounded $F \in L^{loc}_1(\bar{\mathbb{R}}_+, \mathbb{R}^n)$.

## 4. Main Results

**Theorem 4.** *The fundamental matrix $Q(t)$ of (2) has the following representation*

$$Q(t) = (\mathfrak{L}^{-1}[I_{\alpha-1}(p)G^{-1}(p)])(t) \tag{6}$$

*where*

$$G(p) = I_\alpha(p) - e^{-p\tau}CI_\alpha(p) - A - \sum_{i=1}^{m} B^i e^{-p\sigma_i}$$

*is the characteristic matrix of* (2) *(see [14]).*

*Proof.* In virtue of Theorem 3 [14] we can correct apply the Laplace transform to both sides of (2) and then we obtain

$$\begin{aligned} &I_\alpha(p)(\mathfrak{L}Q(t))(p) - CI_\alpha(p)(\mathfrak{L}Q(t-\tau))(p) \\ &= I_{\alpha-1}(p) + A(\mathfrak{L}Q(t))(p) + \sum_{i=1}^{m} B^i(\mathfrak{L}Q(t-\sigma_i))(p). \end{aligned} \tag{7}$$

Since

$$\begin{aligned} (\mathfrak{L}Q(t-\tau))(p) &= \int_0^\infty e^{-pt}Q(t-\tau)dt = e^{-p\tau}\int_{-\tau}^\infty e^{-ps}Q(s)ds \\ &= e^{-p\tau}(\mathfrak{L}Q(t))(p) + e^{-p\tau}\int_{-\tau}^0 e^{-pt}Q(t)dt \\ &= e^{-p\tau}(\mathfrak{L}Q(t))(p) \end{aligned}$$

and analogically $(\mathfrak{L}Q(t-\sigma_i))(p) = e^{-p\sigma_i}(\mathfrak{L}Q(t))(p)$ then from (7) it follows that

$$\begin{aligned} I_\alpha(p)(\mathfrak{L}Q)(p) - e^{-p\tau}CI_\alpha(p)(\mathfrak{L}Q)(p) = I_{\alpha-1}(p) + A(\mathfrak{L}Q)(p) \\ + \sum_{i=1}^{m} B^i e^{-p\sigma_i}(\mathfrak{L}Q)(p) \end{aligned}$$

and hence

$$[I_\alpha(p) - e^{-p\tau}CI_\alpha(p) - A - \sum_{i=1}^{m} B^i e^{-p\sigma_i}](\mathfrak{L}Q)(p) = I_{\alpha-1}(p)$$

from where it follows the theorem statement. □

Introduce for $i \in \langle m \rangle$ the following functions:

$$\Phi_\tau^*(t) = \begin{cases} \Phi(t), & t \in (-\tau, 0) \\ 0 & , t \in \mathbb{R} \setminus (-\tau, 0) \end{cases}$$

$$\Phi_\tau(t) = \begin{cases} \Phi(t), & t \in [-\tau, 0] \\ 0 & , t \in \mathbb{R} \setminus [-\tau, 0] \end{cases}$$

$$\Phi_{\sigma_i}(t) = \begin{cases} \Phi(t), & t \in [-\sigma_i, 0] \\ 0 & , t \in \mathbb{R} \setminus [-\sigma_i, 0] \end{cases}$$

**Theorem 5.** *Let $\Phi \in \mathbf{C}$ be an arbitrary initial function and $X_\Phi(t)$ be the corresponding unique solution of the IP* (2), (3).

*Then $X_\Phi(t)$ can be represented in the form:*

$$\begin{aligned} X_\Phi(t) &= Q(t)(\Phi(0) + \Phi(-\tau)) \\ &+ \frac{C}{\Gamma(2^{-1})} \int_0^t D_{0+}^{\frac{1}{2}} Q(t-s) d\Big(\int_{-\tau}^{0} (s - \eta - \tau)^{-\frac{1}{2}} \Phi(\eta) d\eta\Big) \\ &- C\Phi_\tau(t-\tau) + \sum_{i=1}^{m} \Big(B^i \int_{-\sigma_i}^{0} D_{0+}^{1-\alpha} Q(t - \eta - \sigma) \Phi(\eta) d\eta\Big) \\ &- \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \Big(B^i \int_{-\sigma_i}^{0} (t - \eta - \sigma)^{1-\alpha} \Phi(\eta) d\eta\Big) \end{aligned} \tag{8}$$

*where $Q(t)$ is the fundamental matrix of* (2)

*Proof.* Substitute $X_\Phi(t)$ in (2) and applying the Laplace transform to both sides of (2) we obtain

$$I_\alpha(p)(\mathfrak{L}X_\Phi)(p) - I_\alpha(p)Ce^{-p\tau}(\mathfrak{L}X_\Phi)(p) - I_\alpha(p)Ce^{-p\tau} \int_{-\tau}^{0} e^{-pt}\Phi(t)dt$$

$$= I_{\alpha-1}(p)(\Phi(0) + \Phi(-\tau)) + A(\mathfrak{L}X_\Phi)(p) + \sum_{i=1}^{m} B^i e^{-p\sigma_i} (\mathfrak{L}X_\Phi)(p)$$

$$+ \sum_{i=1}^{m} B^i e^{-p\sigma_i} \int_{-\sigma_i}^{0} e^{-pt}\Phi(t)dt$$

and hence

$$G(p)(\mathfrak{L}X_\Phi)(p) = I_{\alpha-1}(p)(\Phi(0)+\Phi(-\tau))$$
$$+ I_\alpha(p)Ce^{-p\tau}\int_{-\tau}^{0} e^{-pt}\Phi(t)dt + \sum_{i=1}^{m} B^i e^{-p\sigma_i}\int\limits_{-\sigma_i}^{0} e^{-pt}\Phi(t)dt \tag{9}$$

From (9) it follows

$$\begin{aligned}
(\mathfrak{L}X_\Phi)(p) &= I_{\alpha-1}(p)G^{-1}(p)(\Phi(0)+\Phi(-\tau)) \\
&+ I_{\alpha-1}(p)G^{-1}(p)I_1(p)Ce^{-p\tau}\int\limits_{-\tau}^{0} e^{-pt}\Phi(t)dt \\
&+ I_{\alpha-1}(p)G^{-1}(p)I_{1-\alpha}(p)\sum_{i=1}^{m} B^i e^{-p\sigma_i}\int\limits_{-\sigma_i}^{0} e^{-pt}\Phi(t)dt \\
&= (\mathfrak{L}Q(t))(p)(\Phi(0)+\Phi(-\tau)) \\
&+ (\mathfrak{L}Q(t))(p)I_1(p)Ce^{-p\tau}\int\limits_{-\tau}^{0} e^{-pt}\Phi(t)dt \\
&+ (\mathfrak{L}Q(t))(p)I_{1-\alpha}(p)\sum_{i=1}^{m} B^i e^{-p\sigma_i}\int\limits_{-\sigma_i}^{0} e^{-pt}\Phi(t)dt
\end{aligned} \tag{10}$$

Then applying (jjj) we obtain

$$
\begin{aligned}
&I_{\alpha-1}(p)G^{-1}(p)I_1(p)Ce^{-p\tau}\int\limits_{-\tau}^{0} e^{-pt}\Phi(t)dt\\
&=(\mathfrak{L}Q(t))(p)I_1(p)Ce^{-p\tau}\int\limits_{-\tau}^{0} e^{-pt}\Phi(t)dt\\
&=I_{\frac{1}{2}}(p)(\mathfrak{L}Q(t))(p)I_{\frac{1}{2}}(p)\int_{-\tau}^{0} e^{-p(t+\tau)}C\Phi(t)dt\\
&=I_{\frac{1}{2}}(p)(\mathfrak{L}Q(t))(p)I_{\frac{1}{2}}(p)\int\limits_{-\tau}^{\infty} e^{-p(t+\tau)}C\Phi_\tau(t)dt\\
&=I_{\frac{1}{2}}(p)(\mathfrak{L}Q(t))(p)I_{\frac{1}{2}}(p)\int\limits_{0}^{\infty} e^{-ps}C\Phi_\tau(s-\tau)ds\\
&=I_{\frac{1}{2}}(p)(\mathfrak{L}Q(t))(p)I_{\frac{1}{2}}(p)(\mathfrak{L}C\Phi_\tau(t-\tau))(p)\\
&=[(\mathfrak{L}D_{0+}^{\frac{1}{2}}Q(t))(p)-I_{-\frac{1}{2}}(p)]I_{\frac{1}{2}}(p)(\mathfrak{L}C\Phi_\tau^*(t-\tau))(p)\\
&=\mathfrak{L}D_{0+}^{\frac{1}{2}}Q(t))(p)I_{\frac{1}{2}}(p)(\mathfrak{L}C\Phi_\tau^*(t-\tau))(p)-(\mathfrak{L}C\Phi_\tau^*(t-\tau))(p)\\
&=(\mathfrak{L}D_{0+}^{\frac{1}{2}}Q(t))(p)(\mathfrak{L}D_{0+}^{\frac{1}{2}}C\Phi_\tau^*(t-\tau))(p)-(\mathfrak{L}C\Phi_\tau(t-\tau))(p)
\end{aligned}
\tag{11}
$$

Analogically for each $i\in\langle m\rangle$ we obtain

$$
\begin{aligned}
&(\mathfrak{L}Q(t))(p)I_{1-\alpha}(p)B^ie^{-p\sigma_i}\int\limits_{-\sigma_i}^{0} e^{-pt}\Phi(t)dt\\
&=(\mathfrak{L}Q(t))(p)I_{1-\alpha}(p)B^i\int\limits_{-\sigma_i}^{0} e^{-p(t+\sigma_i)}\Phi(t)dt\\
&=I_{1-\alpha}(p)(\mathfrak{L}Q(t))(p)B^i\int\limits_{0}^{\infty} e^{-ps}\Phi_{\sigma_i}(s-\sigma_i)ds\\
&=I_{1-\alpha}(p)(\mathfrak{L}Q(t))(p)(\mathfrak{L}B^i\Phi_{\sigma_i}(t-\sigma_i))(p)\\
&=[(\mathfrak{L}D_{0+}^{1-\alpha}Q(t))(p)-I_{-\alpha}(p)](\mathfrak{L}B^i\Phi_{\sigma_i}(t-\sigma_i))(p)\\
&=(\mathfrak{L}D_{0+}^{1-\alpha}Q(t))(p)(\mathfrak{L}B^i\Phi_{\sigma_i}(t-\sigma_i))(p)-(\mathfrak{L}B^iD_{0+}^{-\alpha}\Phi_{\sigma_i}(t-\sigma_i))(p)
\end{aligned}
\tag{12}
$$

From (10), (11) and (12) it follows

$$
\begin{aligned}
&(\mathfrak{L}X_\Phi)(p) = (\mathfrak{L}Q(t))(p)(\Phi(0) + \Phi(-\tau)) \\
&+ (\mathfrak{L}D_{0+}^{\frac{1}{2}}Q(t))(p)(\mathfrak{L}D_{0+}^{\frac{1}{2}}C\Phi_\tau^*(t-\tau))(p) - (\mathfrak{L}C\Phi_\tau(t-\tau))(p) \\
&+ (\mathfrak{L}D_{0+}^{1-\alpha}Q(t))(p)\sum_{i=1}^{m}(\mathfrak{L}B^i\Phi_{\sigma_i}(t-\sigma_i))(p) \\
&- \sum_{i=1}^{m}(\mathfrak{L}B^iD_{0+}^{-\alpha}\Phi_{\sigma_i}(t-\sigma_i))(p)
\end{aligned}
\tag{13}
$$

Then applying to (13) the inverse Laplace transform we obtain

$$
\begin{aligned}
&X_\Phi(t) = Q(t)(\Phi(0) + \Phi(-\tau)) + D_{0+}^{\frac{1}{2}}Q(t) * D_{0+}^{\frac{1}{2}}C\Phi_\tau^*(t-\tau) \\
&- C\Phi_\tau(t-\tau) + D_{0+}^{1-\alpha}Q(t) * \sum_{i=1}^{m}(B^i\Phi_{\sigma_i}(t-\sigma_i) \\
&- \sum_{i=1}^{m}B^iD_{0+}^{-\alpha}\Phi_{\sigma_i}(t-\sigma_i)
\end{aligned}
\tag{14}
$$

It is not very difficult to see that

$$
\begin{aligned}
&{}_CD_{0+}^{\frac{1}{2}}\Phi_\tau^*(t-\tau) ={}_{RL} D_{0+}^{\frac{1}{2}}\Phi_\tau^*(t-\tau) - \frac{\Phi_\tau^*(-\tau)}{\Gamma(1-\alpha)}t^{-\frac{1}{2}} \\
&={}_{RL} D_{0+}^{\frac{1}{2}}\Phi_\tau^*(t-\tau) = \Gamma^{-1}(\frac{1}{2})\left(\frac{d}{dt}\right)\int_0^t (t-s)^{-\frac{1}{2}}\Phi_\tau^*(s-\tau)ds \\
&= \Gamma^{-1}(\frac{1}{2})\left(\frac{d}{dt}\right)\int_0^t (t-s)^{-\frac{1}{2}}\Phi_\tau(s-\tau)ds \\
&= \Gamma^{-1}(\frac{1}{2})\left(\frac{d}{dt}\right)\int_0^\tau (t-s)^{-\frac{1}{2}}\Phi_\tau(s-\tau)ds \\
&= \Gamma^{-1}(\frac{1}{2})\left(\frac{d}{dt}\right)\int_{-\tau}^0 (t-\eta-\tau)^{-\frac{1}{2}}\Phi(\eta)d\eta
\end{aligned}
\tag{15}
$$

and then from (15) it follows

$$\int_0^t D_{0+}^{\frac{1}{2}} Q(t-s) D_{0+}^{\frac{1}{2}} C\Phi_\tau^*(s-\tau)ds$$
$$= \Gamma^{-1}(\frac{1}{2}) \int_0^t D_{0+}^{\frac{1}{2}} Q(t-s) C \left(\frac{d}{ds}\right) \int\limits_{-\tau}^{0} (s-\eta-\tau)^{-\frac{1}{2}} \Phi(\eta)d\eta \;)ds \tag{16}$$
$$= \Gamma^{-1}(\frac{1}{2}) \int_0^t D_{0+}^{\frac{1}{2}} Q(t-s) C d( \int\limits_{-\tau}^{0} (s-\eta-\tau)^{-\frac{1}{2}} \Phi(\eta)d\eta \;).$$

Then from (14) and (16) we obtain

$$\begin{aligned}
X_\Phi(t) &= Q(t)(\Phi(0)+\Phi(-\tau)) + \int_0^t D_{0+}^{\frac{1}{2}} Q(t-s) C D_{0+}^{\frac{1}{2}} \Phi_\tau^*(s-\tau)ds \\
&- C\Phi_\tau(t-\tau) + \sum_{i=1}^m B^i \int_0^t D_{0+}^{1-\alpha} Q(t-s)\Phi_{\sigma_i}(s-\sigma_i)ds \\
&- \frac{1}{\Gamma(\alpha)} \sum_{i=1}^m B^i \int_0^t (t-s)^{1-\alpha} \Phi_{\sigma_i}(s-\sigma_i)ds \\
&= Q(t)(t)(\Phi(0)+\Phi(-\tau)) + C\int_0^t D_{0+}^{\frac{1}{2}} Q(t-s) D_{0+}^{\frac{1}{2}} \Phi_\tau^*(s-\tau)ds \\
&- C\Phi_\tau(t-\tau) + \sum_{i=1}^m B^i \int_0^{\sigma_i} D_{0+}^{1-\alpha} Q(t-s)\Phi_{\sigma_i}(s-\sigma_i)ds \\
&- \frac{1}{\Gamma(\alpha)} \sum_{i=1}^m B^i \int_0^{\sigma_i} (t-s)^{1-\alpha} \Phi_{\sigma_i}(s-\sigma_i)ds \\
&= Q(t)(t)(\Phi(0)+\Phi(-\tau)) \\
&+ \frac{C}{\Gamma(2^{-1})} \int_0^t D_{0+}^{\frac{1}{2}} Q(t-s) d( \int\limits_{-\tau}^{0} (s-\eta-\tau)^{-\frac{1}{2}} \Phi(\eta)d\eta \;) \\
&- C\Phi_\tau(t-\tau) + \sum_{i=1}^m B^i \int_{-\sigma_i}^0 D_{0+}^{1-\alpha} Q(t-\eta-\sigma_i)\Phi(\eta)d\eta \\
&- \frac{1}{\Gamma(\alpha)} \sum_{i=1}^m B^i \int_{-\sigma_i}^0 (t-\eta-\sigma_i)^{1-\alpha} \Phi(\eta)d\eta
\end{aligned} \tag{17}$$

and from (17) it follows the theorem statement. □

**Theorem 6.** *Let the function $F \in L_1^{loc}(\bar{\mathbb{R}}_+, \mathbb{R}^n)$ be exponentially bounded. Then the solution $X^F(t)$ of the IP* (1), (3) *with initial function $\Phi(t) \equiv 0,\ t \in [-h, 0]$ has the following representation:*

$$X^F(t) = \int_0^t D_{0+}^{1-\alpha} Q(t-s)F(s)ds + D_{0+}^{-\alpha} F(t) \tag{18}$$

*where $Q(t)$ is the fundamental matrix of the system* (2).

*Proof.* As in the proof of Theorem 5 substitute $X_\Phi(t)$ in (1) and applying to both sides the Laplace transform we obtain

$$\begin{aligned} &I_\alpha(p)[(\mathfrak{L}X^F(t))(p) - C(\mathfrak{L}X^F(t-\tau))(p)] \\ &- I_{1-\alpha}(p)[X^F(0) - CX^F(-\tau)] \\ &= A(\mathfrak{L}X^F(t))(p) + \sum_{i=1}^{m} B^i(\mathfrak{L}X^F(t-\sigma_i))(p) + (\mathfrak{L}F(t))(p) \end{aligned} \tag{19}$$

Taking into account that $X^F(t) \equiv 0,\ t \in [-h, 0]$ from (19) it follows

$$\begin{aligned} &[I_\alpha(p) - e^{-p\tau}CI_\alpha(p) - A - \sum_{i=1}^{m} B^i e^{-p\sigma_i}](\mathfrak{L}X^F(t))(p) \\ &= G(p)(\mathfrak{L}X^F(t))(p) = (\mathfrak{L}F(t))(p) \end{aligned} \tag{20}$$

Since $G^{-1}(p) = I_{1-\alpha}(p)(\mathfrak{L}Q(t))(p)$ then (20) implies that

$$\begin{aligned} &(\mathfrak{L}X^F(t))(p) = G^{-1}(p)(\mathfrak{L}F(t))(p) \\ &= I_{1-\alpha}(p)(\mathfrak{L}Q(t))(p)(\mathfrak{L}F(t))(p) \\ &= [(\mathfrak{L}D_{0+}^{1-\alpha}Q(t))(p) + I_{-\alpha}(p)](\mathfrak{L}F(t))(p) \\ &= (\mathfrak{L}D_{0+}^{1-\alpha}Q(t))(p)(\mathfrak{L}F(t))(p) + (\mathfrak{L}D_{0+}^{-\alpha}F(t))(p) \end{aligned} \tag{21}$$

Applying to the both sides of (21) the inverse Laplace transform we obtain the representation (18). □

**Corollary 7.** *Let the function $F \in L_1^{loc}(\bar{\mathbb{R}}_+, \mathbb{R}^n)$ be exponentially bounded. Then for every initial function $\Phi \in \mathbf{C}$ the corresponding unique solution*

$X_\Phi^F(t)$ *of the IP* (1),(3) *has the following integral representation:*

$$\begin{aligned} X_\Phi^F(t) &= Q(t)(\Phi(0)+\Phi(-\tau)) \\ &+ \frac{C}{\Gamma(2^{-1})}\int_0^t D_{0+}^{\frac{1}{2}}Q(t-s)d(\int\limits_{-\tau}^{0}(s-\eta-\tau)^{-\frac{1}{2}}\Phi(\eta)d\eta\ ) \\ &- C\Phi_\tau(t-\tau)+\sum_{i=1}^{m}B^i\int_{-\sigma_i}^{0}D_{0+}^{1-\alpha}Q(t-\eta-\sigma)\Phi(\eta)d\eta \\ &- \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m}B^i\int_{-\sigma_i}^{0}(t-\eta-\sigma)^{1-\alpha}\Phi(\eta)d\eta \\ &+ \int\limits_{o}^{t}D_{0+}^{1-\alpha}Q(t-s)F(s)ds + D_{0+}^{-\alpha}F(t) \end{aligned} \tag{22}$$

*where* $Q(t)$ *is the fundamental matrix of system* (2).

*Proof.* Let $\Phi \in \mathbf{C}$ be an arbitrary initial function and let the functions $X_\Phi(t)$ and $X^F(t)$ are defined with equalities (8) and (18) respectively. Then according the superposition principle the function $X_\Phi^F(t) = X_\Phi(t) + X^F(t)$ is the unique solution of the IP (1),(3). Then the statement of Corollary 7 follows immediately from Theorems 5 and 6. □

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