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GENERALIZED WEAK CONTRACTION CONDITIONS FOR COMPATIBLE MAPPINGS OF TYPES INVOLVING CUBIC TERMS OF THE METRIC FUNCTION

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Abstract: In this paper, we prove some common fixed point theorems for pairs of compatible mappings of types satisfying the generalized ϕ-weak contraction condition involving cubic terms.

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Key Words: common fixed points, ϕ-weak contractions, compatible mappings of type (A), of type (B), of type (C) and of type (P)

1. Introduction

For the last four decades there has been a considerable interest to study common fixed point for a pair (or family) of mappings satisfying contractive conditions in metric spaces. Several interesting and elegant results were obtained in this

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direction by various authors. It was the turning point in the fixed point theory literature when the notion of commutativity mappings was used by Jungck [5] to obtain a generalization of Banachs fixed point theorem for a pair of mappings. This result was further generalized, extended and unified using various types of contractions and minimal commutative mappings.

Fixed point theorems statements basically involve sufficient conditions for the existence of fixed points. Therefore, one of the central concerns in fixed point theory is to find a minimal set of sufficient conditions which ensures the guarantee of fixed points or common fixed points. Common fixed point theorems for contraction type mappings necessarily require a commutativity condition, a condition on the containment of ranges of the mappings, continuity of one or more mappings besides a contraction condition. Mostly fixed point or common fixed point theorems attempt to weaken the above described condition. The study of common fixed points of pair of self-mappings satisfying contractive type conditions becomes more interesting when we extend such studies to the class of noncommuting contractive type mapping pair.

In 1969, Boyd and Wong [2] replaced the constant k in Banach contractive condition by an upper semi-continuous function as follows:

Let (X, d) be a complete metric space and $\psi : [0, \infty) \to [0, \infty)$ be upper semi-continuous from the right such that $0 \leq \psi(t) < t$ for all $t > 0$. If $T : X \to X$ satisfies $d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$, then it has a unique fixed point $x \in X$ and $\{T_n x\}$ converges to x for all $x \in X$.

The first ever attempt to relax the commutativity of mappings to a smaller subset of the domain of mappings was initiated by Sessa [14] in 1982 who gave the notion of weak commutativity. One can notice that the notion of weak commutativity is a point property, while the notion of compatibility is an iterate of sequence. Two self mappings f and g of a metric space (X, d) are said to be *weakly commuting* if $d(fgx, gfx) \leq d(gx, fx)$ for all $x \in X$.

Further, in 1986, Jungck [6] introduced more generalized commutativity, so called compatibility. Clearly commuting, weakly commuting mappings are compatible but converse need not be true (see [7]). Further, in 1993, Jungck et al. [8] defined the concept of compatible mappings of type (A), which is equivalent to the concept of compatible mappings under some conditions. In 1995, Pathak and Khan [10] introduced the notion of compatible mappings of type (B) and compared these mappings with compatible mappings and compatible mappings of type (A). In 1995, Pathak et. al. [11] introduced the notion of compatible mappings of type (P). Further in 1998, Pathak et. al. [12] introduced the notion of compatible mappings of type(C) and compared these mappings along with compatible mappings, compatible mappings of type

(A) and compatible mappings of type (B).

In 1997, Alber and Gueree-Delabriere [1] introduced the concept of weak contraction and in 2001, Rhoades [13] had shown that the results of Alber and Gueree-Delabriere [1] is also valid in complete metric spaces.

A mapping $T : X \to X$ is said to be *ϕ-weak contraction* if for each $x, y \in X$, there exists a function $\phi : [0, \infty) \to [0, \infty)$, $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)).$$

In 2013, Murthy and Vara Prasad [9] introduced a new type of inequality having cubic terms that extended and generalized the results of Alber and Gueree-Delabriere [1] and others cited in the literature of fixed point theory. Further Jain et al. [3] extended and generalized the result of Murthy and Vara Prasad [9] for pairs of mappings.

In this paper, we prove some common fixed point theorems for pairs of mappings of types satisfying the generalized ϕ-weak contractive condition involving various combination of the metric function.

2. Preliminaries

In this section, we give some basic definitions and results that are useful for proving our main results.

In 1986, Jungck [6] introduced the notion of compatible mappings as follows:

Definition 2.1. Two self mappings f and g on a metric space (X, d) are called *compatible* if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In 1993, Jungck et al. [8] introduced the notion of compatible mappings of type (A) as follows:

Definition 2.2. Two self mappings f and g on a metric space (X,d) are called *compatible of type* (A) if

$$\lim_{n\to\infty} d(ffx_n, gfx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(ggx_n, fgx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In 1995, Pathak and Khan [10] introduced the notion of compatible mappings of type (B) as follows:

Definition 2.3. Two self mappings f and g on a metric space (X,d) are called *compatible of type* (B) if

$$\lim_{n\to\infty} d(fgx_n, ggx_n) \leq \frac{1}{2}\Big[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, ffx_n)\Big]$$

and

$$\lim_{n\to\infty} d(gfx_n, ggx_n) \leq \frac{1}{2}\Big[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, ggx_n)\Big],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In 1998, Pathak et al. [12] introduced the notion of compatible mappings of type (C) as follows:

Definition 2.4. Two self mappings f and g on a metric space (X,d) are called compatible of type (C) if

$$\lim_{n\to\infty} d(fgx_n, ggx_n) \leq \frac{1}{3}\Big[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, ffx_n) + \lim_{n\to\infty} d(ft, ggx_n)\Big]$$

and

$$\lim_{n\to\infty} d(gfx_n, ffx_n) \leq \frac{1}{3}\Big[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, ggx_n) + \lim_{n\to\infty} d(gt, ffx_n)\Big],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In 1995, Pathak et al. [11] introduced the notion of compatible mappings of type (P) as follows:

Definition 2.5. Two self mappings f and g on a metric space (X,d) are called *compatible of type* (P) if

$$\lim_{n\to\infty} d(ffx_n, ggx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Now we give the relationships and properties of compatible mappings of types.

Proposition 2.6. [8] *Let S and T be compatible mappings of type (A) of a metric space (X,d) into itself. If one of S and T is continuous, then S and T are compatible.*

Proposition 2.7. [10] *Let S and T be compatible mappings of type (B) of a metric space (X,d) into itself. If $St = Tt$ for some $t \in X$, then $STt = SSt = TTt = TSt$.*

Proposition 2.8. [10] *Let S and T be compatible mappings of type (B) of a metric space (X,d) into itself. Suppose that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Then*

(i) $\lim_{n\to\infty} TTx_n = St$ *if S is continuous at t;*

(ii) $\lim_{n\to\infty} SSx_n = Tt$ *if T is continuous at t;*

(iii) $STt = TSt$ *and* $St = Tt$ *if S and T are continuous at t.*

Remark 2.9. In Proposition 2.7, if one assumes that S and T be compatible mappings of type (C) or of type (P) instead of type (B), the conclusion of the Proposition 2.7 still holds.

Remark 2.10. In Proposition 2.8, if one assumes that S and T be compatible mappings of type (C) or of type (P) instead of type (B), the conclusion of the proposition 2.8 still holds.

Remark 2.11. Notions of compatible mappings and its variants are independent to each other.

3. Fixed Points for Compatible Mappings of Types

Now we give the following theorem for compatible mappings of type (A).

Theorem 3.1. *Let S, T, A and B be four mappings of a complete metric space (X,d) into itself satisfying the following conditions:*

$$S(X) \subset B(X) \quad \text{and} \quad T(X) \subset A(X); \tag{C1}$$

$$\begin{aligned} &[1 + pd(Ax,By)]d^2(Sx,Ty) \\ &\leq p\max\{1/2[d^2(Ax,Sx)d(By,Ty) + d(Ax,Sx)d^2(By,Ty)], \\ &\qquad d(Ax,Sx)d(Ax,Ty)d(By,Sx), \\ &\qquad d(Ax,Ty)d(By,Sx)d(By,Ty))\} \\ &\quad + m(Ax,By) - \phi(m(Ax,By)) \end{aligned} \tag{C2}$$

for all $x, y \in X$*, where*

$$\begin{aligned} &m(Ax, By) \\ &= \max\{d^2(Ax, By), d(Ax, Sx)d(By, Ty), d(Ax, Ty)d(By, Sx), \\ &\quad 1/2[d(Ax, Sx)d(Ax, Ty) + d(By, Sx)d(By, Ty)]\}, \end{aligned}$$

$p \geq 0$ *is a real number and* $\phi : [0, \infty) \to [0, \infty)$ *is a continuous function with* $\phi(t) = 0$ *iff* $t = 0$ *and* $\phi(t) > 0$ *for each* $t > 0$*.*

$(C3)$ *one of* S, T, A *and* B *is continuous.*

Assume that the pairs A, S *and* B, T *are compatible of type* (A)*. Then* S, T, A *and* B *have a unique common fixed point in* X*.*

Proof. Suppose that A is continuous on X. Since A and S are compatible of type (A). From Proposition 2.6, A and S are compatible and so the result easily follows from [4, Theorem 3.2].

Similarly, if B is continuous and the pair B, T is compatible of type (A), then B and T are compatible and so the result easily follows from [4, Theorem 3.2].

Also, we can get the same results when S or T is continuous. This completes the proof. □

Next, we prove the following result for a generalized contractive mapping satisfying a compatibility of type (B).

Theorem 3.2. *Let* S, T, A *and* B *be four mappings of a complete metric space* (X, d) *into itself satisfying the conditions* $(C1)$*-*$(C3)$*. Assume that the pairs* A, S *and* B, T *are compatible of type* (B)*. Then* S, T, A *and* B *have a unique common fixed point in* X*.*

Proof. Let $x_0 \in X$ be an arbitrary point. From $(C1)$ we can find x_1 such that $Sx_0 = Bx_1 = y_0$ for this x_1 one can find $x_2 \in X$ such that $Tx_1 = Ax_2 = y_1$. Continuing in this way one can construct a sequence $\{y_n\}$ such that

$$y_{2n} = Sx_{2n} = Bx_{2n+1}, \quad y_{2n+1} = Tx_{2n+1} = Ax_{2n+2}, \quad n \geq 0.$$

From the proof of [4, Theorem 3.2], $\{y_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space, $\{y_n\}$ converges to a point z as $n \to \infty$. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converge to the same point z.

Now suppose that S is continuous. Then $\{SSx_{2n}\}$ and $\{SAx_{2n}\}$ converges to Sz as $n \to \infty$. Since the mappings A and S are compatible of type (B), it follows from the Proposition 2.8 that $\{AAx_{2n}\}$ converges to Sz as $n \to \infty$.

Now we claim that $z = Sz$.

For this put $x = Ax_{2n}$ and $y = x_{2n+1}$ in $(C2)$, we get

$$\begin{aligned}
&[1 + pd(AAx_{2n}, Bx_{2n+1})]d^2(SAx_{2n}, Tx_{2n+1}) \\
&\le p \max\{1/2[d^2(AAx_{2n}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \\
&\qquad + d(AAx_{2n}, SAx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1})], \\
&\qquad d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n}), \\
&\qquad d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1})\} \\
&\quad + m(AAx_{2n}, Bx_{2n+1}) - \phi(m(AAx_{2n}, Bx_{2n+1})),
\end{aligned}$$

where

$$\begin{aligned}
&m(AAx_{2n}, Bx_{2n+1}) \\
&= \max\{d^2(AAx_{2n}, Bx_{2n+1}), d(AAx_{2n}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1}), \\
&\quad d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n}), \\
&\quad 1/2[d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tx_{2n+1}) \\
&\quad + d(Bx_{2n+1}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1})]\}.
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}
m(Sz, z) &= \max\{d^2(Sz, z), d(Sz, Sz)d(z, z), d(Sz, z)d(z, Sz), \\
&\qquad 1/2[d(Sz, Sz)d(Sz, z) + d(z, Sz)d(z, z)]\} \\
&= d^2(Sz, z).
\end{aligned}$$

Hence we become

$$\begin{aligned}
&[1 + pd(Sz, z)]d^2(Sz, z) \\
&\le p \max\{1/2[0 + 0], 0, 0\} + d^2(Sz, z) - \phi(d^2(Sz, z)).
\end{aligned}$$

Thus we get $d^2(Sz, z) = 0$ implies that $Sz = z$. Since $S(X) \subset B(X)$ and hence there exists a point $u \in X$ such that $z = Sz = Bu$. We claim that $z = Tu$.

For this we put $x = Ax_{2n}$ and $y = u$ in $(C2)$ we get

$$\begin{aligned}
&[1 + pd(AAx_{2n}, Bu)]d^2(SAx_{2n}, Tu) \\
&\le p \max\{1/2[d^2(AAx_{2n}, SAx_{2n})d(Bu, Tu) \\
&\qquad + d(AAx_{2n}, SAx_{2n})d^2(Bu, Tu)], \\
&\qquad d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tu)d(Bu, SAx_{2n}), \\
&\qquad d(AAx_{2n}, Tu)d(Bu, SAx_{2n})d(Bu, Tu)\} \\
&\quad + m(AAx_{2n}, Bu) - \phi(m(AAx_{2n}, Bu)),
\end{aligned}$$

where

$$\begin{aligned}
&m(AAx_{2n}, Bu) \\
&= \max\{d^2(AAx_{2n}, Bu), d(AAx_{2n}, SAx_{2n})d(Bu, Tu), \\
&\quad d(AAx_{2n}, Tu)d(Bu, SAx_{2n}), 1/2[d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tu) \\
&\quad + d(Bu, SAx_{2n})d(Bu, Tu)]\}.
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}
m(Sz, Bu) &= \max\{d^2(z, z), d(z, z)d(z, Tu), d(z, Tu)d(z, z), \\
&\quad 1/2[d(z, z)d(z, Tu) + d(z, z)d(z, Tu)]\} \\
&= 0.
\end{aligned}$$

Hence we become

$$\begin{aligned}
&[1 + pd(z, z)]d^2(z, Tu) \\
&\leq p \max\{1/2[d^2(z, z)d(z, Tu) + d(z, z)d^2(z, Tu)], \\
&\quad d(z, z)d(z, Tu)d(z, z), d(z, Tu)d(z, z)d(z, Tu)\} + 0 - \phi(0).
\end{aligned}$$

This implies that $z = Tu = Sz$. Since the pair B, T is compatible of type $(B)$and $Bu = Tu = z$, by Proposition 2.7, we have $BTu = TBu$ and hence $Bz = BTu = TBu = Tz$.

Also we put $x = x_{2n}$ and $y = z$ in $(C2)$

$$\begin{aligned}
&[1 + pd(Ax_{2n}, Bz)]d^2(Sx_{2n}, Tz) \\
&\leq p \max\{1/2[d^2(Ax_{2n}, Sx_{2n})d(Bz, Tz) + d(Ax_{2n}, Sx_{2n})d^2(Bz, Tz)], \\
&\quad d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tz)d(Bz, Sx_{2n}), \\
&\quad d(Ax_{2n}, Tz)d(Bz, Sx_{2n})d(Bz, Tz)\} \\
&\quad + m(Ax_{2n}, Bz) - \phi(m(Ax_{2n}, Bz)),
\end{aligned}$$

where

$$\begin{aligned}
m(Ax_{2n}, Bz) = \max\{&d^2(Ax_{2n}, Bz), d(Ax_{2n}, Sx_{2n})d(Bz, Tz), \\
&d(Ax_{2n}, Tz)d(Bz, Sx_{2n}), 1/2[d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tz) \\
&+ d(Bz, Sx_{2n})d(Bz, Tz)]\}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&[1 + pd(z, Tz)]d^2(z, Tz) \\
&\leq p \max\{1/2[0 + 0], 0, 0\} + d^2(z, Tz) - \phi(d^2(z, Tz)).
\end{aligned}$$

This implies that $Tz = z$. Since $T(X) \subset A(X)$ and hence there exists a point $v \in X$ such that $z = Tz = Av$.

We claim that $z = Sv$.

For this we put $x = v$ and $y = z$ in $(C2)$ we get

$$\begin{aligned}
&[1 + pd(Av, Bz)]d^2(Sv, Tz) \\
&\leq p \max\{1/2[d^2(Av, Sv)d(Bz, Tz) + d(Av, Sv)d^2(Bz, Tz)], \\
&\qquad d(Av, Sv)d(Av, Tz)d(Bz, Sv), d(Av, Tz)d(Bz, Sv)d(Bz, Tz)\} \\
&\quad + m(Av, Bz) - \phi(m(Av, Bz)),
\end{aligned}$$

where

$$\begin{aligned}
m(Av, Bz) &= \max\{d^2(Av, Bz), d(Av, Sv)d(Bz, Tz), d(Av, Tz)d(Bz, Sv), \\
&\qquad 1/2[d(Av, Sv)d(Av, Tz) + d(Bz, Sv)d(Bz, Tz)]\} \\
&= \max\{d^2(z, z), d(z, Sv)d(Tz, Tz), d(z, z)d(z, Sv), \\
&\qquad 1/2[d(z, Sv)d(z, z) + d(z, Sv)d(Tz, Tz)]\} \\
&= 0.
\end{aligned}$$

Hence we become

$$\begin{aligned}
&[1 + pd(z, z)]d^2(Sv, z) \\
&\leq p \max\{1/2[d^2(z, Sv)d(z, z) + d(z, Sv)d^2(z, z)], \\
&\qquad d(z, Sv)d(z, z)d(z, Sv), d(z, z)d(z, Sv)d(z, z)\} + 0 - \phi(0).
\end{aligned}$$

This implies that $Sv = z$. Since the pair S, A is compatible of type (B) and $Sv = Av = z$, by Proposition 2.7 that $Sz = ASv = SAv = Az$. Hence $z = Az = Sz = Bz = Tz$. Therefore z is a common fixed point of S, T, A and B.

Now suppose that A is continuous. Then $\{AAx_{2n}\}$ and $\{ASx_{2n}\}$ converges to Az as $n \to \infty$. Since the mappings A and S are compatible of type (B), it follows from the Proposition 2.8 that $\{SSx_{2n}\}$ converges to Az as $n \to \infty$.

Now we prove that $z = Az$.

For this we put $x = Sx_{2n}$ and $y = x_{2n+1}$ in $(C2)$ we get

$$\begin{aligned}
&[1 + pd(ASx_{2n}, Bx_{2n+1})]d^2(SSx_{2n}, Tx_{2n+1}) \\
&\leq p \max\{1/2[d^2(ASx_{2n}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \\
&\qquad + d(ASx_{2n}, SSx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1})], \\
&\qquad d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n}), \\
&\qquad d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1})\} \\
&\quad + m(ASx_{2n}, Bx_{2n+1}) - \phi(m(ASx_{2n}, Bx_{2n+1})),
\end{aligned}$$

where

$$
\begin{aligned}
&m(ASx_{2n}, Bx_{2n+1}) \\
&= \max\{d^2(ASx_{2n}, Bx_{2n+1}), d(ASx_{2n}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1}), \\
&\qquad d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n}), \\
&\qquad 1/2[d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tx_{2n+1}) \\
&\qquad + d(Bx_{2n+1}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1})]\}.
\end{aligned}
$$

Letting $n \to \infty$, we get

$$
\begin{aligned}
m(Az, z) &= \max\{d^2(Az, z), d(Az, Az)d(z, z), d(Az, z)d(z, Az), \\
&\qquad 1/2[d(Az, Az)d(Az, z) + d(z, Az)d(z, z)]\} \\
&= d^2(Az, z).
\end{aligned}
$$

Hence we become

$$
\begin{aligned}
&[1 + pd(Az, z)]d^2(Az, z) \\
&\leq p\max\{1/2[0 + 0], 0, 0\} + d^2(Az, z) - \phi(d^2(Az, z)).
\end{aligned}
$$

Thus we get $d^2(Az, z) = 0$ implies that $Az = z$.

Next we claim that $Sz = z$.

For this put $x = z$ and $y = x_{2n+1}$ in $(C2)$.

$$
\begin{aligned}
&[1 + pd(Az, Bx_{2n+1})]d^2(Sz, Tx_{2n+1}) \\
&\leq p\max\{1/2[d^2(Az, Sz)d(z, z) + d(Az, Sz)d^2(z, z))], \\
&\qquad d(Az, Sz)d(Az, z)d(z, Sz), d(Az, z)d(z, Sz)d(z, z)\} \\
&\quad + m(Az, z) - \phi(m(Az, z)),
\end{aligned}
$$

where

$$
\begin{aligned}
m(Az, z) &= \max\{d^2(Az, z), d(Az, Sz)d(z, z), d(Az, z)d(z, Sz), \\
&\qquad 1/2[d(Az, Sz)d(Az, z) + d(z, Sz)d(z, z)]\} \\
&= 0.
\end{aligned}
$$

Hence we get

$$
[1 + pd(z, z)]d^2(Sz, z) \leq p\max\{1/2[0 + 0], 0, 0\} + 0 - \phi(0).
$$

Thus $d^2(Sz, z) = 0$. This implies that $Sz = z$. Since $S(X) \subset B(X)$ and hence there exists a point $w \in X$ such that $z = Sz = Bw$.

We claim that $z = Tw$.

For this we put $x = z$ and $y = w$ in $(C2)$ we get

$$\begin{aligned}
&[1 + pd(Az, Bw)]d^2(Sz, Tw) \\
&\le p \max\{1/2[d^2(Az, Sz)d(Bw, Tw) + d(Az, Sz)d^2(Bw, Tw)], \\
&\quad d(Az, Sz)d(Az, Tw)d(Bw, Sz), \\
&\quad d(Az, Tw)d(Bw, Sz)d(Bw, Tw)\} \\
&\quad + m(Az, Bw) - \phi(m(Az, Bw)),
\end{aligned}$$

where

$$\begin{aligned}
&m(Az, Bw) \\
&= \max\{d^2(Az, Bw), d(Az, Sz)d(Bw, Tw), d(Az, Tw)d(Bw, Sz), \\
&\quad 1/2[d(Az, Sz)d(Az, Tw) + d(Bw, Sz)d(Bw, Tw)]\} \\
&= \max\{d^2(z, z), d(z, z)d(z, Tw), d(z, Tw)d(z, z), \\
&\quad 1/2[d(z, z)d(Az, Tw) + d(z, z)d(z, Tw)]\} \\
&= 0.
\end{aligned}$$

Hence we become

$$\begin{aligned}
&[1 + pd(z, z)]d^2(z, Tw) \\
&\le p \max\{1/2[d^2(z, z)d(z, Tw) + d(z, z)d^2(z, Tw)], \\
&\quad d(z, z)d(z, Tw)d(z, z), d(z, Tw)d(z, z)d(z, Tw)\} + 0 - \phi(0).
\end{aligned}$$

This implies that $z = Tw$. Since the pair B, T is compatible of type (B) and $Bw = Tw = z$, by Proposition 2.7, we have $BTw = TBw$ and hence $Bz = BTw = TBw = Tz$. Also we have

$$\begin{aligned}
&[1 + pd(Az, Bz)]d^2(Sz, Tz) \\
&\le p \max\{1/2[d^2(Az, Sz)d(Bz, Tz) + d(Az, Sz)d^2(Bz, Tz)], \\
&\quad d(Az, Sz)d(Az, Tz)d(Bz, Sz), \\
&\quad d(Az, Tz)d(Bz, Sz)d(Bz, Tz)\} + m(Az, Bz) - \phi(m(Az, Bz)),
\end{aligned}$$

where

$$\begin{aligned}
m(Az, Bz) &= \max\{d^2(Az, Bz), d(Az, Sz)d(Bz, Tz), d(Az, Tz)d(Bz, Sz), \\
&\quad 1/2[d(Az, Sz)d(Az, Tz) + d(Bz, Sz)d(Bz, Tz)]\} \\
&= d^2(z, Tz).
\end{aligned}$$

Hence we become

$$[1+pd(z,Tz)]d^2(z,Tz) \le p\max\{1/2[0+0],0,0\} + d^2(z,Tz) - \phi(d^2(z,Tz)).$$

This implies that $z = Tz$. Hence, $z = Bz = Tz = Az = Sz$. Therefore, z is a common fixed point of S, T, A and B.

Similarly we can also complete the proof when B or T is continuous.

Finally, in order to uniqueness, suppose that z and w ($z \neq w$) are two common fixed points of S, T, A and B.

Put $x = z$ and $y = w$ in $(C2)$.

$$\begin{aligned}
[1+pd(z,w)]d^2(z,w) &= [1+pd(Az,Bw)]d^2(Sz,Tw)\\
&\le p\max\{0,0,0\} + m(Az,Bw) - \phi(m(Az,Bw))\\
&= d^2(z,w) - \phi(d^2(z,w)),
\end{aligned}$$

Thus we have $d^2(z,w) = 0$ and hence $z = w$. Therefore S, T, A and B have a unique common fixed point in X. This completes the proof. □

Next we give the following theorem for compatible mappings of type (C).

Theorem 3.3. *Let S, T, A and B be four mappings of a complete metric space (X,d) into itself satisfying the conditions $(C1)$-$(C3)$. Assume that the pairs A, S and B, T are compatible of type (C). Then S, T, A and B have a unique common fixed point in X.*

Proof. From the proof of Theorem 3.2, $\{y_n\}$ is a Cauchy sequence in X. Since (X,d) is a complete metric space, $\{y_n\}$ converges to a point z as $n \to \infty$. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converge to the same point z.

Now suppose that S is continuous. Then $\{SSx_{2n}\}$ and $\{SAx_{2n}\}$ converges to Sz as $n \to \infty$. Since the mappings A and S are compatible of type (C), it follows from Remark 2.10 that $\{AAx_{2n}\}$ converges to Sz as $n \to \infty$.

Now we claim that $z = Sz$.

For this put $x = Ax_{2n}$ and $y = x_{2n+1}$ in $(C2)$ we get

$$\begin{aligned}
&[1+pd(AAx_{2n},Bx_{2n+1})]d^2(SAx_{2n},Tx_{2n+1})\\
&\le p\max\{1/2[d^2(AAx_{2n},SAx_{2n})d(Bx_{2n+1},Tx_{2n+1})\\
&\qquad + d(AAx_{2n},SAx_{2n})d^2(Bx_{2n+1},Tx_{2n+1})],\\
&\qquad d(AAx_{2n},SAx_{2n})d(AAx_{2n},Tx_{2n+1})d(Bx_{2n+1},SAx_{2n}),\\
&\qquad d(AAx_{2n},Tx_{2n+1})d(Bx_{2n+1},SAx_{2n})d(Bx_{2n+1},Tx_{2n+1})\}\\
&\quad + m(AAx_{2n},Bx_{2n+1}) - \phi(m(AAx_{2n},Bx_{2n+1})),
\end{aligned}$$

where

$$\begin{aligned}&m(AAx_{2n}, Bx_{2n+1})\\&= \max\{d^2(AAx_{2n}, Bx_{2n+1}), d(AAx_{2n}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1}),\\&\quad d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n}),\\&\quad 1/2[d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tx_{2n+1})\\&\quad + d(Bx_{2n+1}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1})]\}.\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}m(Sz, z) &= \max\{d^2(Sz, z), d(Sz, Sz)d(z, z), d(Sz, z)d(z, Sz),\\&\qquad 1/2[d(Sz, Sz)d(Sz, z) + d(z, Sz)d(z, z)]\}\\&= d^2(Sz, z).\end{aligned}$$

Hence we become

$$\begin{aligned}&[1 + pd(Sz, z)]d^2(Sz, z)\\&\le p \max\{1/2[0 + 0], 0, 0\} + d^2(Sz, z) - \phi(d^2(Sz, z)).\end{aligned}$$

Thus we get $d^2(Sz, z) = 0$ implies that $Sz = z$. Since $S(X) \subset B(X)$ and hence there exists a point $u \in X$ such that $z = Sz = Bu$. We claim that $z = Tu$.

For this we put $x = Ax_{2n}$ and $y = u$ in $(C2)$ we get

$$\begin{aligned}&[1 + pd(AAx_{2n}, Bu)]d^2(SAx_{2n}, Tu)\\&\le p \max\{1/2[d^2(AAx_{2n}, SAx_{2n})d(Bu, Tu)\\&\qquad + d(AAx_{2n}, SAx_{2n})d^2(Bu, Tu)],\\&\qquad d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tu)d(Bu, SAx_{2n}),\\&\qquad d(AAx_{2n}, Tu)d(Bu, SAx_{2n})d(Bu, Tu)\}\\&\quad + m(AAx_{2n}, Bu) - \phi(m(AAx_{2n}, Bu)),\end{aligned}$$

where

$$\begin{aligned}&m(AAx_{2n}, Bu)\\&= \max\{d^2(AAx_{2n}, Bu), d(AAx_{2n}, SAx_{2n})d(Bu, Tu),\\&\quad d(AAx_{2n}, Tu)d(Bu, SAx_{2n}), 1/2[d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tu)\\&\quad + d(Bu, SAx_{2n})d(Bu, Tu)]\}.\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}m(z, Bu) &= \max\{d^2(z, z), d(z, z)d(z, Tu), d(z, Tu)d(z, z),\\&\qquad 1/2[d(z, z)d(z, Tu) + d(z, z)d(z, Tu)]\}\\&= 0.\end{aligned}$$

Hence we become

$$\begin{aligned}&[1+pd(z,z)]d^2(z,Tu)\\&\le p\max\{1/2[d^2(z,z)d(z,Tu)+d(z,z)d^2(z,Tu)],\\&\qquad d(z,z)d(z,Tu)d(z,z),d(z,Tu)d(z,z)d(z,Tu)\}+0-\phi(0).\end{aligned}$$

This implies that $z=Tu=Sz$. Since the pair B,T is compatible of type $(C)$and $Bu=Tu=z$, by Remark 2.9, we have $BTu=TBu$ and hence $Bz=BTu=TBu=Tz$.

Also we have put $x=x_{2n}$ and $y=z$ in $(C2)$.

$$\begin{aligned}&[1+pd(Ax_{2n},Bz)]d^2(Sx_{2n},Tz)\\&\le p\max\{1/2[d^2(Ax_{2n},Sx_{2n})d(Bz,Tz)+d(Ax_{2n},Sx_{2n})d^2(Bz,Tz)],\\&\qquad d(Ax_{2n},Sx_{2n})d(Ax_{2n},Tz)d(Bz,Sx_{2n}),\\&\qquad d(Ax_{2n},Tz)d(Bz,Sx_{2n})d(Bz,Tz)\}\\&\quad+m(Ax_{2n},Bz)-\phi(m(Ax_{2n},Bz)),\end{aligned}$$

where

$$\begin{aligned}&m(Ax_{2n},Bz)\\&=\max\{d^2(Ax_{2n},Bz),d(Ax_{2n},Sx_{2n})d(Bz,Tz),d(Ax_{2n},Tz)d(Bz,Sx_{2n}),\\&\qquad 1/2[d(Ax_{2n},Sx_{2n})d(Ax_{2n},Tz)+d(Bz,Sx_{2n})d(Bz,Tz)]\}.\end{aligned}$$

Letting $n\to\infty$, we get

$$m(z,Bz)=d^2(z,Tz).$$

Hence we become

$$\begin{aligned}&[1+pd(z,Tz)]d^2(z,Tz)\\&\le p\max\{1/2[0+0],0,0\}+d^2(z,Tz)-\phi(d^2(z,Tz)).\end{aligned}$$

This implies that $Tz=z$. Since $T(X)\subset A(X)$ and hence there exists a point $v\in X$ such that $z=Tz=Av$.

We claim that $z=Sv$.

For this we put $x=v$ and $y=z$ in $(C2)$ we get

$$\begin{aligned}&[1+pd(Av,Bz)]d^2(Sv,Tz)\\&\le p\max\{1/2[d^2(Av,Sv)d(Bz,Tz)+d(Av,Sv)d^2(Bz,Tz)],\\&\qquad d(Av,Sv)d(Av,Tz)d(Bz,Sv),d(Av,Tz)d(Bz,Sv)d(Bz,Tz)\}\\&\quad+m(Av,Bz)-\phi(m(Av,Bz)),\end{aligned}$$

where

$$\begin{aligned} m(Av, Bz) &= \max\{d^2(Av, Bz), d(Av, Sv)d(Bz, Tz), d(Av, Tz)d(Bz, Sv), \\ &\quad 1/2[d(Av, Sv)d(Av, Tz) + d(Bz, Sv)d(Bz, Tz)]\} \\ &= \max\{d^2(z, z), d(z, Sv)d(Tz, Tz), d(z, z)d(z, Sv), \\ &\quad 1/2[d(z, Sv)d(z, z) + d(z, Sv)d(Tz, Tz)]\} \\ &= 0. \end{aligned}$$

Hence we become

$$\begin{aligned} &[1 + pd(z, z)]d^2(Sv, z) \\ &\leq p\max\{1/2[d^2(z, Sv)d(z, z) + d(z, Sv)d^2(z, z)], \\ &\qquad d(z, Sv)d(z, z)d(z, Sv), d(z, z)d(z, Sv)d(z, z)\} + 0 - \phi(0). \end{aligned}$$

This implies that $Sv = z$. Since rhe pair S, A is compatible of type (C) and $Sv = Av = z$, by Remark 2.9 that $Sz = ASv = SAv = Az$. Hence $z = Az = Sz = Bz = Tz$. Therefore, z is a common fixed point of S, T, A and B.

Now suppose that A is continuous. Then $\{AAx_{2n}\}$ and $\{ASx_{2n}\}$ converges to Az as $n \to \infty$. Since the mappings A and S are compatible of type (C), it follows from the Remark 2.10 that $\{SSx_{2n}\}$ converges to

Now we claim that $z = Az$.

For this put $x = Sx_{2n}$ and $y = x_{2n+1}$ in $(C2)$ we get

$$\begin{aligned} &[1 + pd(ASx_{2n}, Bx_{2n+1})]d^2(SSx_{2n}, Tx_{2n+1}) \\ &\leq p\max\{1/2[d^2(ASx_{2n}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \\ &\qquad + d(ASx_{2n}, SSx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1}))], \\ &\qquad d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n}), \\ &\qquad d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1})\} \\ &\quad + m(ASx_{2n}, Bx_{2n+1}) - \phi(m(ASx_{2n}, Bx_{2n+1})), \end{aligned}$$

where

$$\begin{aligned} &m(ASx_{2n}, Bx_{2n+1}) \\ &= \max\{d^2(ASx_{2n}, Bx_{2n+1}), d(ASx_{2n}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1}), \\ &\qquad d(ASx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n}), \\ &\qquad 1/2[d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tx_{2n+1}) \\ &\qquad + d(Bx_{2n+1}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1})]\}, \end{aligned}$$

which implies that

$$\begin{aligned} m(Az,z) &= \max\{d^2(Az,z), d(Az,Az)d(z,z), d(Az,z)d(z,Az), \\ &\qquad 1/2[d(Az,Az)d(Az,z)+d(z,Az)d(z,z)]\} \\ &= d^2(Az,z). \end{aligned}$$

Hence we become

$$\begin{aligned} &[1+pd(Az,z)]d^2(Az,z) \\ &\le p\max\{1/2[0+0],0,0\}+d^2(Az,z)-\phi(d^2(Az,z)). \end{aligned}$$

Thus we get $d^2(Az,z)=0$ implies that $Az=z$.

Next we claim that $Sz=z$.

For this we put $x=z$ and $y=x_{2n+1}$ in $(C2)$.

$$\begin{aligned} &[1+pd(Az,Bx_{2n+1})]d^2(Sz,Tx_{2n+1}) \\ &\le p\max\{1/2[d^2(Az,Sz)d(Bx_{2n+1},Tx_{2n+1}) \\ &\qquad + d(Az,Sz)d^2(Bx_{2n+1},Tx_{2n+1}))], \\ &\qquad d(Az,Sz)d(Az,Tx_{2n+1})d(Bx_{2n+1},Sz), \\ &\qquad d(Az,Tx_{2n+1})d(Bx_{2n+1},Sz)d(Bx_{2n+1},Tx_{2n+1})\} \\ &\quad + m(Az,Bx_{2n+1})-\phi(m(Az,Bx_{2n+1})), \end{aligned}$$

where where

$$\begin{aligned} &m(Az,Bx_{2n+1}) \\ &= \max\{d^2(Az,Bx_{2n+1}), d(Az,Sz)d(Bx_{2n+1},Tx_{2n+1}), \\ &\qquad d(Az,Tx_{2n+1})d(Bx_{2n+1},Sz), 1/2[d(Az,Sz)d(Az,Tx_{2n+1}) \\ &\qquad + d(Bx_{2n+1},Sz)d(Bx_{2n+1},Tx_{2n+1})]\}, \end{aligned}$$

which implies that

$$\begin{aligned} m(Az,z) &= \max\{d^2(Az,z), d(Az,Sz)d(z,z), d(Az,z)d(z,Sz), \\ &\qquad 1/2[d(Az,Sz)d(Az,z)+d(z,Sz)d(z,z)]\} \\ &= 0. \end{aligned}$$

Hence we become

$$[1+pd(z,z)]d^2(Sz,z) \le p\max\{1/2[0+0],0,0\}+0-\phi(0).$$

Thus we get $d^2(Sz,z)=0$. This implies that $Sz=z$. Since $S(X)\subset B(X)$ and hence there exists a point $w\in X$ such that $z=Sz=Bw$.

We claim that $z = Tw$.

For this we put $x = z$ and $y = w$ in $(C2)$ we get

$$\begin{aligned}
&[1 + pd(Az, Bw)]d^2(Sz, Tw)\\
&\le p\max\{1/2[d^2(Az, Sz)d(Bw, Tw) + d(Az, Sz)d^2(Bw, Tw)],\\
&\qquad d(Az, Sz)d(Az, Tw)d(Bw, Sz),\\
&\qquad d(Az, Tw)d(Bw, Sz)d(Bw, Tw)\}\\
&\quad + m(Az, Bw) - \phi(m(Az, Bw)),
\end{aligned}$$

where

$$\begin{aligned}
&m(Az, Bw)\\
&= \max\{d^2(Az, Bw), d(Az, Sz)d(Bw, Tw), d(Az, Tw)d(Bw, Sz),\\
&\qquad 1/2[d(Az, Sz)d(Az, Tw) + d(Bw, Sz)d(Bw, Tw)]\}\\
&= \max\{d^2(z, z), d(z, z)d(z, Tw), d(z, Tw)d(z, z),\\
&\qquad 1/2[d(z, z)d(z, Tw) + d(z, z)d(z, Tw)]\}\\
&= 0.
\end{aligned}$$

Hence we become

$$[1 + pd(z, z)]d^2(z, Tw) \le p\max\{1/2[0 + 0], 0, 0\} + 0 - \phi(0).$$

This implies that $z = Tw$. Since the pair B, T is compatible of type (C) and $Bw = Tw = z$, by Remark 2.9, we have $BTw = TBw$ and hence $Bz = BTw = TBw = Tz$. Also we have

$$\begin{aligned}
&[1 + pd(Az, Bz)]d^2(Sz, Tz)\\
&\le p\max\{1/2[d^2(Az, Sz)d(Bz, Tz) + d(Az, Sz)d^2(Bz, Tz)],\\
&\qquad d(Az, Sz)d(Az, Tz)d(Bz, Sz), d(Az, Tz)d(Bz, Sz)d(Bz, Tz)\}\\
&\quad + m(Az, Bz) - \phi(m(Az, Bz)),
\end{aligned}$$

where

$$\begin{aligned}
m(Az, Bz) &= \max\{d^2(Az, Bz), d(Az, Sz)d(Bz, Tz), d(Az, Tz)d(Bz, Sz),\\
&\qquad 1/2[d(Az, Sz)d(Az, Tz) + d(Bz, Sz)d(Bz, Tz)]\}\\
&= d^2(z, Tz).
\end{aligned}$$

Hence we become

$$\begin{aligned}
&[1 + pd(z, Tz)]d^2(z, Tz)\\
&\le p\max\{1/2[0 + 0], 0, 0\} + d^2(z, Tz) - \phi(d^2(z, Tz)).
\end{aligned}$$

This implies that $z = Tz$. Hence $z = Bz = Tz = Az = Sz$. Therefore, z is a common fixed point of S, T, A and B.

Similarly we can also complete the proof when B or T is continuous.

Uniqueness follows easily. This completes the proof. □

Finally we give the following theorem for compatible mappings of type (P).

Theorem 3.4. *Let S, T, A and B be four mappings of a complete metric space (X, d) into itself satisfying the conditions $(C1)$-$(C3)$. Assume that the pairs A, S and B, T are compatible of type (P). Then S, T, A and B have a unique common fixed point in X.*

Proof. From the proof of Theorem 3.2, $\{y_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space, $\{y_n\}$ converges to a point z as $n \to \infty$. Consequently, the subsequences $\{Sx_{2n}\}$, $\{Ax_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converge to the same point z.

Now suppose that S is continuous. Then $\{SSx_{2n}\}$ and $\{SAx_{2n}\}$ converges to Sz as $n \to \infty$. Since the mappings A and S are compatible of type (P), it follows from Remark 2.10 that $\{AAx_{2n}\}$ converges to Sz as $n \to \infty$.

Now we claim that $z = Sz$.

For this put $x = Ax_{2n}$ and $y = x_{2n+1}$ in $(C2)$ we get

$$
\begin{aligned}
&[1 + pd(AAx_{2n}, Bx_{2n+1})]d^2(SAx_{2n}, Tx_{2n+1})\\
&\le p\max\{1/2[d^2(AAx_{2n}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1})\\
&\qquad + d(AAx_{2n}, SAx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1})],\\
&\qquad d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n}),\\
&\qquad d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1})\}\\
&\quad + m(AAx_{2n}, Bx_{2n+1}) - \phi(m(AAx_{2n}, Bx_{2n+1})),
\end{aligned}
$$

where

$$
\begin{aligned}
&m(AAx_{2n}, Bx_{2n+1})\\
&= \max\{d^2(AAx_{2n}, Bx_{2n+1}), d(AAx_{2n}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1}),\\
&\qquad d(AAx_{2n}, Tx_{2n+1})d(Bx_{2n+1}, SAx_{2n}),\\
&\qquad 1/2[d(AAx_{2n}, SAx_{2n})d(AAx_{2n}, Tx_{2n+1})\\
&\qquad + d(Bx_{2n+1}, SAx_{2n})d(Bx_{2n+1}, Tx_{2n+1})]\},
\end{aligned}
$$

which implies that

$$\begin{aligned} m(Sz, z) &= \max\{d^2(Sz, z), d(Sz, Sz)d(z, z), d(Sz, z)d(z, Sz), \\ &\quad 1/2[d(Sz, Sz)d(Sz, z) + d(z, Sz)d(z, z)]\} \\ &= d^2(Sz, z). \end{aligned}$$

Hence we become

$$[1 + pd(Sz, z)]d^2(Sz, z) \leq p \max\{1/2[0 + 0], 0, 0\} + d^2(Sz, z) - \phi(d^2(Sz, z)).$$

Thus we get $d^2(Sz, z) = 0$ implies that $Sz = z$. Since $S(X) \subset B(X)$ and hence there exists a point $u \in X$ such that $z = Sz = Bu$. Now we claim that $z = Tu$.

For this we put $x = x_{2n}$ and $y = u$ in $(C2)$ we get

$$\begin{aligned} &[1 + pd(Ax_{2n}, Bu)]d^2(Sx_{2n}, Tu) \\ &\leq p \max\{1/2[d^2(Ax_{2n}, Sx_{2n})d(Bu, Tu) + d(Ax_{2n}, Sx_{2n})d^2(Bu, Tu)], \\ &\qquad d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tu)d(Bu, Sx_{2n}), \\ &\qquad d(Ax_{2n}, Tu)d(Bu, Sx_{2n})d(Bu, Tu)\} \\ &\quad + m(Ax_{2n}, Bu) - \phi(m(Ax_{2n}, Bu)), \end{aligned}$$

where

$$\begin{aligned} &m(Ax_{2n}, Bu) \\ &= \max\{d^2(Ax_{2n}, Bu), d(Ax_{2n}, Sx_{2n})d(Bu, Tu), d(Ax_{2n}, Tu)d(Bu, Sx_{2n}), \\ &\qquad 1/2[d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tu) + d(Bu, Sx_{2n})d(Bu, Tu)]\}, \end{aligned}$$

which implies that

$$\begin{aligned} m(z, z) &= \max\{d^2(z, z), d(z, z)d(z, Tu), d(z, Tu)d(z, z), \\ &\quad 1/2[d(z, z)d(z, Tu) + d(z, z)d(z, Tu)]\} \\ &= 0. \end{aligned}$$

Hence we become

$$\begin{aligned} &[1 + pd(z, z)]d^2(z, Tu) \\ &\leq p \max\{1/2[d^2(z, z)d(z, Tu) + d(z, z)d^2(z, Tu)], \\ &\qquad d(z, z)d(z, Tu)d(z, z), d(z, Tu)d(z, z)d(z, Tu)\} + 0 - \phi(0). \end{aligned}$$

This implies that $z = Tu$. Since the pair B, T is compatible of type (P) and $Bu = Tu = z$, by Remark 2.9, we have $TTu = BBu$, which implies that $d(Bz, Tz) = 0$ and hence $Bz = Tz$.

Now we claim that $Tz = z$.

Put $x = x_{2n}$ and $y = z$ in $(C2)$.

$$\begin{aligned}
&[1 + pd(Ax_{2n}, Bz)]d^2(Sx_{2n}, Tz)\\
&\le p\max\{1/2[d^2(Ax_{2n}, Sx_{2n})d(Bz, Tz) + d(Ax_{2n}, Sx_{2n})d^2(Bz, Tz)],\\
&\qquad d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tz)d(Bz, Sx_{2n}),\\
&\qquad d(Ax_{2n}, Tz)d(Bz, Sx_{2n})d(Bz, Tz)\}\\
&\quad + m(Ax_{2n}, Bz) - \phi(m(Ax_{2n}, Bz)),
\end{aligned}$$

where

$$\begin{aligned}
m(Ax_{2n}, Bz) = \max\{&d^2(Ax_{2n}, Bz), d(Ax_{2n}, Sx_{2n})d(Bz, Tz),\\
&d(Ax_{2n}, Tz)d(Bz, Sx_{2n}), 1/2[d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tz)\\
&+ d(Bz, Sx_{2n})d(Bz, Tz)]\},
\end{aligned}$$

which implies that

$$m(z, Tz) = d^2(z, Tz).$$

Hence we become

$$\begin{aligned}
&[1 + pd(z, Tz)]d^2(z, Tz)\\
&\le p\max\{1/2[0 + 0], 0, 0\} + d^2(z, Tz) - \phi(d^2(z, Tz)).
\end{aligned}$$

This implies that $Tz = z$. Hence $Bz = Tz = z$. Since $T(X) \subset A(X)$ and hence there exists a point $v \in X$ such that $z = Tz = Av$.

We claim that $z = Sv$.

For this we put $x = v$ and $y = z$ in $(C2)$ we get

$$\begin{aligned}
&[1 + pd(Av, Bz)]d^2(Sv, Tz)\\
&\le p\max\{1/2[d^2(Av, Sv)d(Bz, Tz) + d(Av, Sv)d^2(Bz, Tz)],\\
&\qquad d(Av, Sv)d(Av, Tz)d(Bz, Sv), d(Av, Tz)d(Bz, Sv)d(Bz, Tz)\}\\
&\quad + m(Av, Bz) - \phi(m(Av, Bz)),
\end{aligned}$$

where

$$\begin{aligned}
m(Av, Bz) &= \max\{d^2(Av, Bz), d(Av, Sv)d(Bz, Tz), d(Av, Tz)d(Bz, Sv),\\
&\qquad 1/2[d(Av, Sv)d(Av, Tz) + d(Bz, Sv)d(Bz, Tz)]\}\\
&= \max\{d^2(z, z), d(z, Sv)d(Tz, Tz), d(z, z)d(z, Sv),\\
&\qquad 1/2[d(z, Sv)d(z, z) + d(z, Sv)d(Tz, Tz)]\}\\
&= 0.
\end{aligned}$$

Hence we become

$$\begin{aligned}&[1+pd(z,z)]d^2(Sv,z)\\&\le p\max\{1/2[d^2(z,Sv)d(z,z)+d(z,Sv)d^2(z,z)],\\&\quad d(z,Sv)d(z,z)d(z,Sv),d(z,z)d(z,Sv)d(z,z)\}+0-\phi(0).\end{aligned}$$

This implies that $Sv = z$. Since the pair S, A is compatible of type (P) and $Sv = Av = z$, by Remark 2.9, we have $SSv = AAv$ which implies that $d(Sz, Az) = 0$. Thus $Sz = Az$. Hence $z = Az = Sz = Bz = Tz$. Therefore, z is a common fixed point of S, T, A and B.

Similarly we can complete the proof when A or B or T is continuous.

The uniqueness follows easily. This completes the proofs. $\square$

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