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CHARACTERIZATIONS FOR PRINCIPAL IDEAL GRAPHS OF RECTANGULAR GROUPS

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Abstract: A principal right ideal graph of a semigroup S is the graph whose vertex set is S and any two vertices x and y $(x \neq y)$ are adjacent if and only if $xS \cap yS \neq \emptyset$. We denote the principal right ideal graph of a semigroup S by Γ_S. A principal left ideal graph of a semigroup S is defined dually and is denoted by ${}_S\Gamma$. We define a principal ideal graph of a semigroup S as the graph ${}_S\Gamma_S$ with S is the vertex set and any two vertices x and y $(x \neq y)$ are adjacent in ${}_S\Gamma_S$ if and only if $Sx \cap Sy \neq \emptyset$ and $xS \cap yS \neq \emptyset$. A rectangular band is defined as a direct product of a left zero semigroup and a right zero semigroup. A rectangular group is defined as a direct product of a group and a rectangular band. The principal ideal graph of a rectangular group is studied in this paper. First, we characterize the principal right ideal graphs and the principal left ideal graphs of a rectangular groups. Finally we characterize the principal ideal graphs of a rectangular groups.

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1. Introduction

Algebraic graph theory is a branch of mathematics in which algebraic methods are applied to problems about graphs. The graph theory is very useful in the theory of group and semigroup. The graph can be used to visualize the structure and problems in the semigroup theory. Throughout the past few years,

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there have been many defined a new type of graphs on semigroups. In 1975, Rosenfeld considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs. The concepts of fuzzy graph has been studied in [9]. In [3] Csakany and Pollak defined intersection graphs of nontrivial proper subgroups of groups. The commuting graphs of rings have been studied in [1] and [2]. The Cayley digraph was defined for the first time by Arthur Cayley in 1878 to study finite groups. Cayley digraphs of semigroups have been studied extensively and many results have been interesting, see for example [5], [6], [7] and [8]. In 2012, the principal ideal graphs of a semigroup was defined by Indu and John in [4]. They studied the principal ideal graphs of a rectangular bands. Inspired by these study, we aim to characterize the principal ideal graphs of a rectangular groups.

2. Basic Definitions

Let S be a semigroups. The *principal right ideal graph of* S is the graph Γ_S with S is the vertex set and any two vertices x and y $(x \neq y)$ are adjacent in Γ_S if and only if $xS \cap yS \neq \emptyset$. The *principal left ideal graph of* S is the graph ${}_S\Gamma$ with S is the vertex set and any two vertices x and y $(x \neq y)$ are adjacent in ${}_S\Gamma$ if and only if $Sx \cap Sy \neq \emptyset$. We now define the *principal ideal graph of* S as the graph ${}_S\Gamma_S$ with S is the vertex set and any two vertices x and y $(x \neq y)$ are adjacent in ${}_S\Gamma_S$ if and only if $xS \cap yS \neq \emptyset$ and $Sx \cap Sy \neq \emptyset$ (i.e. ${}_S\Gamma_S = {}_S\Gamma \cap \Gamma_S$).

All sets in this paper are assume to be finite. An element z of a semigroup S is a *left(right) zero* of S if $zs = z(sz = z)$ for all $s \in S$, z is a *zero* of S if it is both a left and right zero of S. A semigroup all of whose elements are left(right) zeros is a *left(right) zero semigroup.* A direct product of a left zero and a right zero semigroup is called a *rectangular band.* A *rectangular groups* is a direct product of a group and a rectangular band. The *cardinality* of a set X, denoted by $|X|$, is the number of elements in X. For any family of nonempty set $\{X_i | i \in I\}$, let $\dot{\cup}_{i \in I} X_i$ denote the disjoint union of $X_i, i \in I$.

Let G_1 and G_2 be graphs. A mapping $\varphi : V(G_1) \to V(G_2)$ is called a *graph homomorphism* if any two vertices u and v of G_1 are adjacent in G_1 implies $\varphi(u)$ and $\varphi(v)$ are adjacent in G_2, i.e. φ preserves edge. We write $\varphi : G_1 \to G_2$. If $\varphi : G_1 \to G_2$ is a bijective graph homomorphism and φ^{-1} is also a graph homomorphism, then φ is called a *graph isomorphism.* If a graph isomorphism $\varphi : G_1 \to G_2$ exists, then the graphs are called isomorphic and we write $G_1 \cong G_2$.

Let $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ be graphs such that $V_i \cap V_j = \emptyset$ for all $i \neq j$. The disjoint union of $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ is defined as $\dot\cup_{i=1}^n (V_i, E_i) := \left(\dot\cup_{i=1}^n V_i, \dot\cup_{i=1}^n E_i\right)$. A subgraph F of a graph G is called a *strong subgraph* of G if and only if whenever u and v are vertices of F if u and v are adjacent in G, then u and v are adjacent in F as well.

3. Principal Right/Left Ideal Graphs of Rectangular Groups

We characterize a principal right and left ideal graphs of a rectangular groups in this section. First, we describe the necessary and sufficient conditions for any two elements in a rectangular group S are adjacent in Γ_S.

Lemma 1. *Let $S = G \times L \times R$ be a rectangular group and (g_1, l_1, r_1), $(g_2, l_2, r_2) \in S$. Then there exists an edge between (g_1, l_1, r_1) and (g_2, l_2, r_2) in Γ_S if and only if $l_1 = l_2$.*

Proof Let $S = G \times L \times R$ be a rectangular group and (g_1, l_1, r_1), $(g_2, l_2, r_2) \in S$.

($\Rightarrow$) Assume that there exists an edge between (g_1, l_1, r_1) and (g_2, l_2, r_2) in Γ_S. Then by the definition of Γ_S, we thus get $(g_1, l_1, r_1)S \cap (g_2, l_2, r_2)S \neq \emptyset$. Hence there exists (a, i, α) and $(b, j, \beta) \in S$ such that

$$\begin{aligned} (g_1, l_1, r_1)(a, i, \alpha) &= (g_2, l_2, r_2)(b, j, \beta) \\ \Rightarrow (g_1 a, l_1, \alpha) &= (g_2 b, l_2, \beta) \\ \Rightarrow \qquad l_1 &= l_2. \end{aligned}$$

($\Leftarrow$) Let $l_1 = l_2$. Now we have $(g_1, l_1, r_1)(g_1^{-1}, i, r_1) = (g_1 g_1^{-1}, l_1, r_1) = (e, l_1, r_1)$, where e is the identity element in G. Also we have,

$$(g_2, l_2, r_2)(g_2^{-1}, i, r_1) = (g_2 g_2^{-1}, l_2, r_1) = (e, l_2, r_1) = (e, l_1, r_1),$$

as $l_1 = l_2$. This means that, $(g_1, l_1, r_1)S \cap (g_2, l_2, r_2)S \neq \emptyset$. Therefore there exists an edge between (g_1, l_1, r_1) and (g_2, l_2, r_2) in Γ_S. □

Similarly to Lemma 1, we have the following lemma.

Lemma 2. *Let $S = G \times L \times R$ be a rectangular group and (g_1, l_1, r_1), $(g_2, l_2, r_2) \in S$. Then there exists an edge between (g_1, l_1, r_1) and (g_2, l_2, r_2) in ${}_S\Gamma$ if and only if $r_1 = r_2$.*

The next theorem, we characterize a principal right ideal graphs of a rectangular groups. From now on, p_i denotes the projection into the i^{th} component.

Theorem 3. *A graph (V, E) is a principal right ideal graph of a rectangular group if and only if the following conditions holds:*

(1) (V, E) is the disjoint union of m subgraphs $(V_1, E_1), (V_2, E_2), \ldots, (V_m, E_m)$ for some $m \in \mathbb{N}$;

(2) for all $i = 1, 2, \ldots m$, (V_i, E_i) is a complete graph and contains n subgraphs (V_{i1}, E_{i1}), $(V_{i2}, E_{i2}), \ldots, (V_{in}, E_{in})$ such that $V_i = \dot\cup_{j=1}^{n} V_{ij}$ for some $n \in \mathbb{N}$;

(3) for all $j = 1, 2, \ldots n$, (V_{ij}, E_{ij}) is a complete graph with t vertices for some $t \in \mathbb{N}$.

Proof ($\Rightarrow$) Let (V, E) be a principal right ideal graph of a rectangular group. Then there exists a rectangular group $S = G \times L \times R$ where $G = \{g_1, g_2, \ldots, g_t\}$ is a group, $L = \{l_1, l_2, \ldots, l_m\}$ is a left zero semigroup and $R = \{r_1, r_2, \ldots, r_n\}$ is a right zero semigroup, such that $(V, E) \cong \Gamma_S$. Hence we will prove that (1), (2) and (3) are true for Γ_S.

(1) For each $i \in \{1, 2, \ldots, m\}$, set $V_i = G \times \{l_i\} \times R$, and $E_i = E(\Gamma_S) \cap E(K_{|V_i|})$ where $K_{|V_i|}$ is a complete graph with V_i is the vertex set. Hence $S = \dot\cup_{i=1}^{m} V_i$ and (V_i, E_i) is a strong subgraph of Γ_S. We show that (V_1, E_1), $(V_2, E_2), \ldots, (V_m, E_m)$ are disjoint. By Lemma 1, there is no edge between (h, l, r) and (h', l', r') in Γ_S for all $l \neq l'$. Then by the definition of V_i, we thus get (V_1, E_1), (V_2, E_2), $\ldots$, (V_m, E_m) are disjoint. Therefore $\Gamma_S = \dot\cup_{i=1}^{m}(V_i, E_i)$.

(2) For all $i = 1, 2, \ldots, m$, by the definition of V_i and Lemma 1, we get that there is an edge between any two vertices in (V_i, E_i). This means that (V_i, E_i) is a complete.

For each $j \in \{1, 2, \ldots, n\}$, set $V_{ij} = G \times \{l_i\} \times \{r_j\}$, and $E_{ij} = E_i \cap E(K_{|V_{ij}|})$ where $K_{|V_{ij}|}$ is a complete graph with V_{ij} is the vertex set. Therefore (V_{ij}, E_{ij}) is a strong subgraph of (V_i, E_i) and $V_i = \dot\cup_{j=1}^{n} V_{ij}$.

(3) Since (V_i, E_i) is a complete, this implies that (V_{ij}, E_{ij}) is a complete because it is a strong subgraph of (V_i, E_i). Also we have

$$\begin{aligned} |V_{ij}| &= |G \times \{l_i\} \times \{r_j\}| \\ &= |G| \times |\{l_i\}| \times |\{r_j\}| \\ &= |G|. \\ &= t. \end{aligned}$$

Hence (V_{ij}, E_{ij}) is a complete graph with t vertices.

($\Leftarrow$) From (1) and (2), we get that $V = \dot{\cup}_{i=1}^{m}\dot{\cup}_{j=1}^{n}V_{ij}$. From (3), we let $V_{ij} = \{v_{ij}^1, v_{ij}^2, \ldots, v_{ij}^t\} = \dot{\cup}_{k=1}^{t}\{v_{ij}^k\}$. Let $S = G \times L \times R$ be a rectangular group, where $G = \{g_1, g_2, \ldots, g_t\}$ is a group, $L = \{l_1, l_2, \ldots, l_m\}$ is a left zero semigroup and $R = \{r_1, r_2, \ldots, r_n\}$ is a right zero semigroup. We will show that $(V, E) \cong \Gamma_S$. Define $f : V \to S$ by $f(v_{ij}^k) = (g_k, l_i, r_j)$. Since $|V| = t \times m \times n = |G| \times |L| \times |R| = |S|$, it easily seen that f is a bijection from V to S. We will show that f and f^{-1} are homomorphism from (V, E) to Γ_S.

Let $a, b \in V$, assume that there exists an edge between a and b in (V, E). By (1), we have $a, b \in V_i$ for some $i \in \{1, 2, \ldots, m\}$. We let $a = v_{ij}^k$ and $b = v_{ij'}^{k'}$ for some $j, j' \in \{1, 2, \ldots, n\}$ and $k, k' \in \{1, 2, \ldots, t\}$. Therefore $f(a) = f(v_{ij}^k) = (g_k, l_i, r_j)$ and $f(b) = f(v_{ij'}^{k'}) = (g_{k'}, l_i, r_{j'})$. By Lemma 1, there exists an edge between (g_k, l_i, r_j) and $(g_{k'}, l_i, r_{j'})$ in Γ_S. This means that there is an edge between $f(a)$ and $f(b)$ in Γ_S. Hence f is a homomorphism.

Conversely suppose that there exists an edge between $f(a)$ and $f(b)$ in Γ_S. By Lemma 1, we get that $p_2(f(a)) = p_2(f(b))$. We let $l_i = p_2(f(a)) = p_2(f(b))$ for some $i \in \{1, 2, \ldots, m\}$. It follows that $a, b \in V_i$. From (2), there exists an edge between a and b in (V_i, E_i). This implies that there exists an edge between a and b in (V, E). Therefore f^{-1} is a homomorphism. Hence $(V, E) \cong \Gamma_S$. □

From Theorem 3, we have the following corollary which describe the number of components in the principal right ideal graph of a rectangular group.

Corollary 4. *Let $S = G \times L \times R$ be a rectangular group with $|G| = t, |L| = m$ and $|R| = n$. Then Γ_S is a disconnected graph with m components in which each component is complete with tn vertices.*

Proof Let $S = G \times L \times R$ be a rectangular group with $|G| = t, |L| = m$ and $|R| = n$. By (1) and (2) of Theorem 3, we get that $\Gamma_S = \dot{\cup}_{i=1}^{m}(V_i, E_i)$ and (V_i, E_i) is a complete respectively. By the proof of (1) again, we have $V_i = G \times \{l_i\} \times R$ for all $i \in \{1, 2, \ldots, m\}$. Hence $|V_i| = |G \times \{l_i\} \times R| = |G| \times |\{l_i\}| \times |R| = tn$. □

Similarly to Theorem 3 and by using Lemma 2, the next theorem we characterize a principal left ideal graphs of a rectangular groups.

Theorem 5. *A graph (V, E) is a principal left ideal graph of a rectangular group if and only if the following conditions holds:*

(1) (V, E) is the disjoint union of n subgraphs $(V_1, E_1), (V_2, E_2), \ldots, (V_n, E_n)$ for some $n \in \mathbb{N}$;

(2) for all $i = 1, 2, \ldots, n$, (V_i, E_i) is a complete graph and contains m subgraphs (V_{i1}, E_{i1}), $(V_{i2}, E_{i2}), \ldots, (V_{im}, E_{im})$ such that $V_i = \dot{\cup}_{j=1}^{m}V_{ij}$ for

some $m \in \mathbb{N}$*;*

(3) for all $j = 1, 2, \ldots, m, (V_{ij}, E_{ij})$ *is a complete graph with* t *vertices for some* $t \in \mathbb{N}$*.*

From Theorem 5, we have the following corollary which describe the number of components in the principal left ideal graph of a rectangular group.

Corollary 6. *Let* $S = G \times L \times R$ *be a rectangular group with* $|G| = t, |L| = m$ *and* $|R| = n$*. Then* ${}_S\Gamma$ *is a disconnected graph with* n *components in which each component is complete with* tm *vertices.*

The following corollary describe the number of edges in the principal right ideal graph of a rectangular group.

Corollary 7. *Let* $S = G \times L \times R$ *be a rectangular group with* $|G| = t, |L| = m$ *and* $|R| = n$*. Then* Γ_S *has* $\frac{tn(tn-1)m}{2}$ *edges.*

Proof Let $S = G \times L \times R$ be a rectangular group with $|G| = t, |L| = m$ and $|R| = n$. By Corollary 4, we thus get Γ_S has m components in which each component is complete with tn vertices. Therefore each component in Γ_S has $\frac{tn(tn-1)}{2}$ edges. Because there are m components, then the total number of edges in ${}_S\Gamma$ is $\frac{tn(tn-1)m}{2}$. □

Similarly to Corollary 7 and by using Corollary 6, we have the following corollary.

Corollary 8. *Let* $S = G \times L \times R$ *be a rectangular group with* $|G| = t, |L| = m$ *and* $|R| = n$*. Then* ${}_S\Gamma$ *has* $\frac{tm(tm-1)n}{2}$ *edges.*

Example 1. Let $S = \mathbb{Z}_3 \times L_2 \times R_2$ be a rectangular group, where $\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$, $L_2 = \{l_1, l_2\}$ and $R_2 = \{r_1, r_2\}$.

In Figure 1, we see that Γ_S is the disjoint union of two subgraphs (V_1, E_1) and (V_2, E_2), where

$$V_1 = \{\bar{0}l_1r_1, \bar{1}l_1r_1, \bar{2}l_1r_1, \bar{0}l_1r_2, \bar{1}l_1r_2, \bar{2}l_1r_2\}$$

and

$$V_2 = \{\bar{0}l_2r_1, \bar{1}l_2r_1, \bar{2}l_2r_1, \bar{0}l_2r_2, \bar{1}l_2r_2, \bar{2}l_2r_2\}.$$

It is easily seen that (V_1, E_1) and (V_2, E_2) are complete. Moreover, we see that (V_1, E_1) contains two strong subgraphs (V_{11}, E_{11}) and (V_{12}, E_{12}) such that for each strong subgraph is a complete, where $V_{11} = \{\bar{0}l_1r_1, \bar{1}l_1r_1, \bar{2}l_1r_1\}$ and $V_{12} = \{\bar{0}l_1r_2, \bar{1}l_1r_2, \bar{2}l_1r_2\}$.

Similarly, we see that (V_2, E_2) contains two strong subgraphs (V_{21}, E_{21}) and (V_{22}, E_{22}) such that for each strong subgraph is a complete, where
$V_{21} = \{\bar{0}l_2r_1, \bar{1}l_2r_1, \bar{2}l_2r_1\}$ and $V_{22} = \{\bar{0}l_2r_2, \bar{1}l_2r_2, \bar{2}l_2r_2\}$.

And we have $|V_{11}| = |V_{12}| = |V_{21}| = |V_{22}| = 3$.

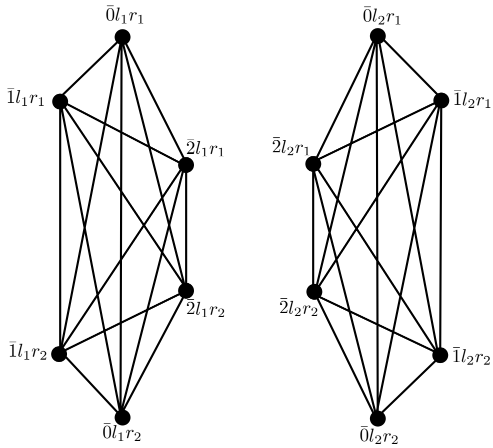

Figure 1: Principal right ideal graph Γ_S.

4. Principal Ideal Graphs of Rectangular Groups

In this section, we characterize a principal ideal graphs of a rectangular groups. By Lemma 1 and 2, we have the following lemma which describe the necessary and sufficient conditions for any two elements in a rectangular group S are adjacent in ${}_S\Gamma_S$.

Lemma 9. *Let $S = G \times L \times R$ be a rectangular group and (g_1, l_1, r_1), $(g_2, l_2, r_2) \in S$. Then there exists an edge between (g_1, l_1, r_1) and (g_2, l_2, r_2) in ${}_S\Gamma_S$ if and only if $l_1 = l_2$ and $r_1 = r_2$.*

The next theorem, we characterize a principal ideal graphs of a rectangular groups.

Theorem 10. *A graph (V, E) is a principal ideal graph of a rectangular group if and only if the following conditions holds:*

(1) $(V, E) = \dot{\cup}_{i=1}^{m} \dot{\cup}_{j=1}^{n} (V_{ij}, E_{ij})$ for some $m, n \in \mathbb{N}$;

(2) for all i and j, (V_{ij}, E_{ij}) is a complete graph with t vertices for some $t \in \mathbb{N}$.

Proof ($\Rightarrow$) Let (V,E) be a principal ideal graph of a rectangular group. Then there exists a rectangular group $S = G\times L\times R$ where $G=\{g_1,g_2,\ldots,g_t\}$ is a group, $L=\{l_1,l_2,\ldots,l_m\}$ is a left zero semigroup and $R=\{r_1,r_2,\ldots,r_n\}$ is a right zero semigroup, such that $(V,E)\cong {}_S\Gamma_S$. Hence we will prove that (1) and (2) are true for ${}_S\Gamma_S$.

(1) For each $i\in\{1,2,\ldots,m\}$ and $j\in\{1,2,\ldots,n\}$, set $V_{ij}=G\times\{l_i\}\times\{r_j\}$, and $E_i = E({}_S\Gamma_S)\cap E(K_{|V_{ij}|})$ where $K_{|V_{ij}|}$ is a complete graph with V_{ij} is the vertex set. Hence $S=\dot{\cup}_{i=1}^{m}\dot{\cup}_{j=1}^{n}V_{ij}$ and (V_{ij},E_{ij}) is a strong subgraph of ${}_S\Gamma_S$. Let $(h,i,j)\in V_{ij}$ and $(h',i',j')\in V_{i'j'}$ By Lemma 9, there is no edge between (h,i,j) and (h',i',j') in ${}_S\Gamma_S$ for all $i\neq i'$ or $j\neq j'$. Then we thus get (V_{ij},E_{ij}) and $(V_{i'j'},E_{i'j'})$ are disjoint for all $i\neq i'$ or $j\neq j'$. Therefore ${}_S\Gamma_S=\dot{\cup}_{i=1}^{m}\dot{\cup}_{j=1}^{n}(V_{ij},E_{ij})$.

(2) For all i and j, by the definition of V_{ij} and Lemma 9, we get that there is an edge between any two vertices in (V_{ij},E_{ij}). This means that (V_{ij},E_{ij}) is a complete.

We have

$$\begin{aligned}|V_{ij}| &= |G\times\{l_j\}\times\{r_i\}|\\ &= |G|\times|\{l_j\}|\times|\{r_i\}|\\ &= |G|.\\ &= t.\end{aligned}$$

Hence (V_{ij},E_{ij}) is a complete graph with t vertices.

($\Leftarrow$) From (2), we let $V_{ij}=\{v_{ij}^1,v_{ij}^2,\ldots,v_{ij}^t\}=\dot{\cup}_{k=1}^{t}\{v_{ij}^k\}$. From (1), we get that $V=\dot{\cup}_{i=1}^{m}\dot{\cup}_{j=1}^{n}V_{ij}=\dot{\cup}_{i=1}^{m}\dot{\cup}_{j=1}^{n}\dot{\cup}_{k=1}^{t}\{v_{ij}^k\}$. Let $S=G\times L\times R$ be a rectangular group, where $G=\{g_1,g_2,\ldots,g_t\}$ is a group, $L=\{l_1,l_2,\ldots,l_m\}$ is a left zero semigroup and $R=\{r_1,r_2,\ldots,r_n\}$ is a right zero semigroup. We will show that $(V,E)\cong {}_S\Gamma_S$. Define $f:V\to S$ by $f(v_{ij}^k)=(g_k,l_i,r_j)$. Since $|V|=t\times m\times n=|G|\times|L|\times|R|=|S|$, it easily seen that f is a bijection from V to S. We will show that f and f^{-1} are homomorphism from (V,E) to ${}_S\Gamma_S$.

Let $a,b\in V$, assume that there exists an edge between a and b in (V,E). By (1), we have $a,b\in V_{ij}$ for some $i\in\{1,2,\ldots,m\}$ and $j\in\{1,2,\ldots,n\}$. Let $a=v_{ij}^k$ and $b=v_{ij}^{k'}$ for some $k,k'\in\{1,2,\ldots,t\}$. Therefore $f(a)=f(v_{ij}^k)=(g_k,l_i,r_j)$ and $f(b)=f(v_{ij}^{k'})=(g_{k'},l_i,r_j)$. By Lemma 9, there exists an edge between (g_k,l_i,r_j) and $(g_{k'},l_i,r_j)$ in ${}_S\Gamma_S$. This means that there is an edge between $f(a)$ and $f(b)$ in ${}_S\Gamma_S$. Hence f is a homomorphism.

Conversely suppose that there exists an edge between $f(a)$ and $f(b)$ in ${}_S\Gamma_S$. By Lemma 9 again, we get that $p_2(f(a))=p_2(f(b))$ and $p_3(f(a))=p_3(f(b))$. We let $l_i=p_2(f(a))=p_2(f(b))$ and $r_j=p_3(f(a))=p_3(f(b))$ for some

$i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$. This means that $a, b \in V_{ij}$. From (2), there exists an edge between a and b in (V_{ij}, E_{ij}). It follows that there exists an edge between a and b in (V, E). Therefore f^{-1} is a homomorphism. Hence $(V, E) \cong {}_S\Gamma_S$. □

By Theorem 10, we have the following theorem.

Theorem 11. *Let $H = \dot{\cup}_{i=1}^{w} H_i$ be a finite disjoint union of complete graphs H_i such that for some $t \in \mathbb{N}$, $|V(H_i)| = t$ for all $i = 1, 2, \ldots, w$. Then there is a rectangular group S such that ${}_S\Gamma_S \cong H$.*

Proof Let $H = \dot{\cup}_{i=1}^{w} H_i$ be a finite disjoint union of complete graphs H_i and $|V(H_i)| = t$ for all $i = 1, 2, \ldots, w$. And let $S = G \times L \times R$ be a rectangular group with $|G| = t, |L| = m$ and $|R| = n$ such that $mn = w$. By (1) and (2) of Theorem 10, we get that ${}_S\Gamma_S = \dot{\cup}_{i=1}^{m} \dot{\cup}_{j=1}^{n} (V_{ij}, E_{ij})$ and for each (V_{ij}, E_{ij}) is a complete respectively. It follows that ${}_S\Gamma_S$ is a disjoint union of $mn = w$ complete graphs. By the proof of (1) again, we have $V_{ij} = G \times \{L_i\} \times \{r_j\}$. Hence $|V_{ij}| = |G \times \{l_i\} \times \{r_j\}| = |G| \times |\{l_i\}| \times |\{r_j\}| = t$. Therefore ${}_S\Gamma_S$ is a disjoint union of w components in which each component is complete with t vertices. Hence ${}_S\Gamma_S \cong \dot{\cup}_{i=1}^{w} H_i = H$. □

From Theorem 11 we have the following corollary which describe the number of components in the principal ideal graph of a rectangular group.

Corollary 12. *Let $S = G \times L \times R$ be a rectangular group with $|G| = t, |L| = m$ and $|R| = n$. Then the principal ideal graph ${}_S\Gamma_S$ is a disconnected graph with mn components in which each component is complete with t vertices.*

By Corollary 12, the following corollary is immediate.

Corollary 13. *Let $S = G \times L \times R$ be a rectangular group. Then the principal ideal graph ${}_S\Gamma_S$ is a connected graph if and only if $|L| = |R| = 1$.*

The following corollary describe the number of edges in the principal ideal graph of a rectangular group.

Corollary 14. *Let $S = G \times L \times R$ be a rectangular group with $|G| = t, |L| = m$ and $|R| = n$. Then the principal ideal graph ${}_S\Gamma_S$ has $\frac{t(t-1)mn}{2}$ edges.*

Proof Let $S = G \times L \times R$ be a rectangular group with $|G| = t, |L| = m$ and $|R| = n$. By Corollary 12, the principal ideal graph ${}_S\Gamma_S$ has mn components in which each component is complete with t vertices. Therefore each component in ${}_S\Gamma_S$ has $\frac{t(t-1)}{2}$ edges. Because there are mn components, then the total number of edges in ${}_S\Gamma_S$ is $\frac{t(t-1)mn}{2}$. □

Example 2. Let $S = \mathbb{Z}_3 \times L_2 \times R_2$ be a rectangular group, where $\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$, $L_2 = \{l_1, l_2\}$ and $R_2 = \{r_1, r_2\}$.

In Figure 2, it is easily seen that the principal ideal graph ${}_S\Gamma_S$ has 4 components ($|L||R| = 4$). For each component is a complete with 3 vertices ($|G| = 3$).

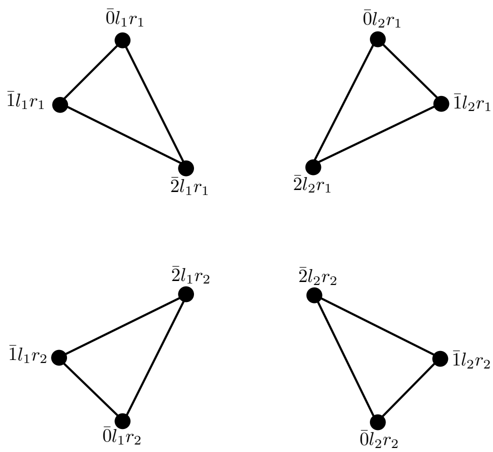

Figure 2: Principal ideal graph ${}_S\Gamma_S$.

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