International Journal of Pure and Applied Mathematics
Volume 119 No. 1 2018, 31-51
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v119i1.3
ijpam.eu

EXISTENCE OF FUNDAMENTAL MATRIX FOR NEUTRAL LINEAR FRACTIONAL SYSTEM WITH DISTRIBUTED DELAYS

Andrey Zahariev[1], Hristo Kiskinov[2 §]
[1,2]Faculty of Mathematics and Informatics
University of Plovdiv
4 Tzar Asen, 4000, Plovdiv, BULGARIA

Abstract: The aim of the present paper is to obtain sufficient conditions for existing and uniqueness of the solution of the Cauchy problem with discontinuous initial conditions for nonhomogeneous neutral linear fractional differential system with distributed delay and Caputo type derivatives. As an application is obtained that the homogeneous system has a continuous in $[a, \infty)$, $a \in \mathbb{R}$ fundamental matrix, which result extends the corresponding ones even in the particular case of fractional system with one constant delay and lower terminal at zero of the fractional derivatives.

AMS Subject Classification: 34A08, 34A12, 34A30
Key Words: fractional derivatives, neutral fractional system, fundamental matrix

1. Introduction

The fractional calculus and respectively the fractional differential equations have a lot of applications in various fields of the science. For a good introduction on fractional calculus theory and fractional differential equations see the monographs of Kilbas et al. [9], Kiryakova [10], Podlubny [17] and Fecan et al. [4]. The distributed order fractional differential equations is discussed in Jiao et al. [6] and for an application oriented exposition see Diethelm [3]. We refer the

Received: September 10, 2017
Revised: February 11, 2018
Published: June 4, 2018

§Correspondence author

monograph of Stamova, Stamov [18] where impulsive fractional differential and functional differential equations as well as several applications are considered.

As in the integer order the main advantage of the fractional differential equations with delay is the possibility to describe evolution of the processes which depends on the past history. Single order fractional systems of retarded and neutral type with distributed delays are studied in [19], [20], [21], [22] and in [2] is considered the case with Caputo type distributed order fractional derivatives.

It is well known that the fundamental matrix is a mainstay tool for establishing a formula for the general solution for linear fractional differential equations and/or systems (ordinary or with delay). That's why the problem of existence of a fundamental matrix is an important evergreen theme for research and this explains the largest amount of papers devoted to this problem. For linear fractional ordinary differential equations and systems detailed information concerning this theme is given in [1], [8], [9], [16], [17], [26] and the references therein. Several results in the autonomous case using the Laplace transform method are obtained in [13], [14], [17]. From the works concerning the problem of establishing a formula for the general solution for fractional differential equations and/or systems with delay we point out [24], [25] and [23] for the case of singular systems.

It is known that in the general case the existence of a fundamental matrix for a homogeneous delayed (or neutral) fractional differential system is a corollary form the solvability of an initial problem for this system with discontinuous initial function. As far as we know, there are no results concerning initial problem for delayed fractional differential equations with discontinuous initial function except [11]. Without to solve this problem, a consideration of the existence of fundamental matrix without proof (except in some partial autonomous cases) is generally speaking incorrect. Note that almost all results establishing a formula for the general solution of a system are obtained from many authors, using the definition of Caputo type derivative applicable only in the particular case when the functions are absolutely continuous. Our work is motivated also from the fact that in the mentioned works is investigated only the case of one constant delay and all used fractional derivatives have lower terminal at zero.

In the present work we consider a neutral linear delayed system of incommensurate type with distributed delay with derivatives in Caputo sense. For this system we study two important problems. First of them is to clear the problem with existence and the uniqueness of the solutions of an initial problem (IP) in the case of discontinuous initial conditions. The other one is to obtain as corollary that the non autonomous homogeneous system has a fundamental matrix $G(t,s) = \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ which is continuous for $t \in [a,\infty), a \in \mathbb{R}$. The obtained result extends the corresponding one even in the particular case of fractional system with one constant delay and $a = 0$. Moreover the proposed conditions coincide with the conditions which guaranty the same result in the case of integer order linear differential equations with distributed delay. In our work we use the more general definition of Caputo type derivative without the restrictive assumption that the functions must be absolutely continuous.

The paper is organized as follows: In Section 2, we recall some needed definitions of Riemann-Liouville and Caputo fractional derivatives, as well as the needed part of their properties. In the same section the problem statement can be found and some notations are introduced. Section 3 is devoted to the existence and the uniqueness of the solutions of the Cauchy problem for delayed linear incommensurate fractional differential system with distributed delays in the case of Caputo derivatives with piecewise continuous initial function. As an application of the results from Section 3, in Section 4 is obtained that the homogeneous system has a continuous in $[a,\infty),\ a \in \mathbb{R}$ fundamental matrix, which result extends the corresponding ones even in the particular case of fractional system with one constant delay and lower terminal at zero of the fractional derivatives.

2. Preliminaries and Problem Statement

For convenience and to avoid possible misunderstandings, below we recall only the definitions of Riemann–Liouville and Caputo fractional derivatives and some needed their properties. For details and other properties we refer to [9, 10, 17].

Let $\alpha \in (0,1)$ be an arbitrary number and denote by $L_1^{loc}(\mathbb{R},\mathbb{R})$ the linear space of all locally Lebesgue integrable functions $f : \mathbb{R} \to \mathbb{R}$. Then for each $a \in \mathbb{R}$ and $f \in L_1^{loc}(\mathbb{R},\mathbb{R})$ the left-sided fractional integral operator of order α is defined by

$$(D_{a+}^{-\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad (D_{a+}^{0} f)(t) = f(t), \quad t > a.$$

The corresponding left side Riemann–Liouville fractional derivative for $t > a$ is defined by

$$_{RL}D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_a^t (t-s)^{-\alpha} f(s)ds \right)$$

and the Caputo fractional left side derivative ${}_CD^{\alpha}_{a+}$ is defined by

$$ {}_CD^{\alpha}_{a+}f(t) = {}_{RL}D^{\alpha}_{a+}[f(s) - f(a)](t) = {}_{RL}D^{\alpha}_{a+}f(t) - \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha}. $$

We will use the following relations (see Kilbas et al. [9]):

$$ \text{(a) } D^{-\alpha}_{a+}\,{}_CD^{\alpha}_{a+}f(t) = f(t) - f(a); \quad \text{(b) } {}_CD^{\alpha}_{a+}D^{-\alpha}_{a+}f(t) = f(t). $$

Consider the nonhomogeneous and homogeneous linear delayed systems of incommensurate type with distributed delay in the following general form:

$$ \begin{aligned} &D^{\alpha}_{a+}\Big(X(t) - \sum_{l=1}^{r}\int_{-\tau}^{0}[d_\theta V^l(t,\theta)]X(t+\theta)\Big) \\ &= \sum_{i=0}^{m}\int_{-\sigma}^{0}[d_\theta U^i(t,\theta)]X(t+\theta) + F(t), \end{aligned} \tag{1} $$

$$ \begin{aligned} &D^{\alpha}_{a+}\Big(X(t) - \sum_{l=1}^{r}\int_{-\tau}^{0}[d_\theta V^l(t,\theta)]X(t+\theta)\Big) \\ &= \sum_{i=0}^{m}\int_{-\sigma}^{0}[d_\theta U^i(t,\theta)]X(t+\theta). \end{aligned} \tag{2} $$

where $r, m \in \mathbb{N}$, $k \in \langle n\rangle = \{1, 2, ..., n\}$, $\tau, \sigma > 0$, $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_k \in (0,1)$, $X, F : J_a \to \mathbb{R}^n$, $U^i, V^l : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n}$, $J_a = [a, \infty)$, $a \in \mathbb{R}$, $X(t) = (x_1(t), ..., x_n(t))^T$, $F(t) = (f_1(t), ..., f_n(t))^T$, $V^l(t,\theta) = \{v^l_{kj}(t,\theta)\}^n_{k,j=1}$, $l \in \langle r\rangle$, $U^i(t,\theta) = \{u^i_{kj}(t,\theta)\}^n_{k,j=1}$, $i \in \langle m\rangle_0 = \langle m\rangle \bigcup \{0\}$,

$$ D^{\alpha}_{a+}X(t) = (D^{\alpha_1}_{a+}x_1(t), ..., D^{\alpha_n}_{a+}x_n(t))^T, $$

and $D^{\alpha_k}_{a+}$ denotes the left side Caputo fractional derivative ${}_CD^{\alpha_k}_{a+}$.

The system (1) described with more details has the form

$$ \begin{aligned} &D^{\alpha_k}_{a+}\Big(x_k(t) - \sum_{l=1}^{r}\Big(\sum_{j=1}^{n}\int_{-\tau}^{0}x_j(t+\theta)d_\theta v^l_{kj}(t,\theta)\Big)\Big) \\ &= \sum_{i=0}^{m}\Big(\sum_{j=1}^{n}\int_{-\sigma}^{0}x_j(t+\theta)d_\theta u^i_{kj}(t,\theta)\Big) + f_k(t) \end{aligned} $$

In addition we will use also the following notations: $\mathbb{R}_+ = (0, \infty)$, $\bar{\mathbb{R}}_+ = [0, \infty)$, $h = \max(\tau, \sigma)$, $J_* = [a-h, \infty)$, $J_{a+M} = [a-h, a+M]$, $M \in \mathbb{R}_+$.

Let $Y : J_a \times \mathbb{R} \to \mathbb{R}^{n\times n},\ a \in \mathbb{R},\ Y(t,\theta) = \{y_{k,j}(t,\theta)\}_{k,j=1}^n$. Then $|Y(t,\theta)| = \sum\limits_{k,j=1}^{n} |y_{k,j}(t,\theta)|$ and $BV[-h,0]$ denotes the linear space of matrix valued functions $Y(t,\theta)$ with bounded variation in θ on $[-h,0]$ for every $t \in J_a$ and $Var_{[-h,0]}Y(t,\cdot) = \sum\limits_{k,j=1}^{n} Var_{[-h,0]}y_{k,j}(t,\cdot)$. Everywhere below for $Y(t) = (y_1(t), ..., y_n(t))^T : J_a \to \mathbb{R}^n$ $\beta = (\beta_1, ..., \beta_n)$, $\beta_k \in [-1,1]$, $k \in \langle n\rangle$ we will use the notation $I_\beta(Y(t)) = \mathrm{diag}(y_1^{\beta_1}(t), ..., y_n^{\beta_n}(t))$ and $I \in \mathbb{R}^{n\times n}$ be the identity matrix.

As usual a vector function $\Phi = (\phi_1, ..., \phi_n)^T : [a-h,a] \to \mathbb{R}^n$ is called piecewise continuous on $[a-h,a]$ (it will be noted $\Phi \in PC([a-h,a],\mathbb{R}^n)$) if the interval $[a-h,a]$ can be broken into a finite number of subintervals on which the function is continuous on each open subinterval and has a finite limit at the end points of each subinterval. We will denote the set of all jumps point of $\Phi \in PC([a-h,a],\mathbb{R}^n)$ with S^Φ.

With $\mathbf{C}_a^* \subset PC([a-h,a],\mathbb{R}^n) \cap BV([a-h,a],\mathbb{R}^n)$,$a \in \mathbb{R}$ we denote the Banach space of all right continuous vector functions with bounded variation $\Phi \in \mathbf{C}_a^*$ with norm $||\Phi|| = \sup\limits_{t\in[a-h,a]} |\Phi(t)| = \sup\limits_{t\in[a-h,a]} \sum\limits_{k=1}^{n} |\phi_k(t)| < \infty$ and by $\mathbf{C}_a \subset \mathbf{C}_a{}^*$ the subspace of all continuous functions in $\mathbf{C}_a{}^*$.

In view of the applications and for simplicity, in our exposition in Section 3 we will assume that every initial vector function $\Phi \in \mathbf{C}_a^*$ has at most one jump point $t_\Phi \in [a-h,a]$, i.e. $S^\Phi = \{t_\Phi\}$.

Consider the following initial conditions for the system (1):

$$X(t) = \Phi(t)(x_k(t) = \phi_k(t),\ k \in \langle n\rangle),\ t \in [a-h,a],\ \Phi \in \mathbf{C}_a^* \tag{3}$$

We say that for the kernels $U^i, V^l : \bar{\mathbb{R}}_+ \times \mathbb{R} \to \mathbb{R}^{n\times n}$, $i \in \langle m\rangle_0$, $l \in \langle r\rangle$ the conditions **(S)** are fulfilled if the following conditions hold (see [12],[5]):

(S1) The functions $(t,\theta) \to U^i(t,\theta)$,$(t,\theta) \to V^l(t,\theta)$ are measurable in $(t,\theta) \in J_a \times \mathbb{R}$ and normalized so that $U^i(t,\theta) = 0$,$V^l(t,\theta) = 0$ for $\theta \geq 0$ and $U^i(t,\theta) = U^i(t,-\sigma)$ for $\theta \leq -\sigma$, $V^l(t,\theta) = V^l(t,-\tau)$ for $\theta \leq -\tau$, $t \in J_a$.

(S2) For each $t \in J_a$ the kernels $U^i(t,\theta)$ and $V^l(t,\theta)$ are continuous from the left in θ on $(-\sigma,0)$ and $(-\tau,0)$ respectively and $U^i(t,\cdot), V^l(t,\cdot) \in BV[-h,0]$.

(S3) The Lebesgue decompositions of the kernels $U^i(t,\theta)$ and $V^l(t,\theta)$ for

$t \in J_a$ and $\theta \in [-h, 0]$ have the form:

$$U^i(t,\theta) = \aleph^i(t,\theta) + \int_{-\sigma}^{\theta} B(t,s)ds + \Upsilon(t,\theta),$$
$$V^l(t,\theta) = \bar{\aleph}^l(t,\theta) + \int_{-\tau}^{\theta} \bar{B}(t,s)ds + \bar{\Upsilon}(t,\theta)$$

where $\Upsilon(t,\theta) = \{g_{k,j}(t,\theta)\}_{k,j=1}^n$, $\bar{\Upsilon}(t,\theta) = \{\bar{g}_{k,j}(t,\theta)\}_{k,j=1}^n \in C(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$, $\aleph^i(t,\theta) = \{a^i_{kj}(t)H(\theta + \sigma_i(t))\}_{k,j=1}^n$, $\bar{\aleph}^l(t,\theta) = \{\bar{a}^l_{kj}(t)H(\theta + \tau_l(t))\}_{k,j=1}^n$, $H(t)$ is the Heaviside function, $A^i(t) = \{a^i_{kj}(t)\}_{k,j=1}^n, B(t,\theta) = \{b^j_k(t,\theta)\}_{k,j=1}^n \in L_1^{loc}(J_a \times \mathbb{R}, \mathbb{R}^{n\times n})$ are locally bounded, $\bar{A}^l(t) = \{\bar{a}^l_{kj}(t)\}_{k,j=1}^n \in C(J_a, \mathbb{R}^{n\times n})$, $\bar{B}(t,\theta) = \{\bar{b}^j_k(t,\theta)\}_{k,j=1}^n \in C(J_a, \mathbb{R}^{n\times n})$, $\sigma_i(t)$, $\tau_l(t) \in C(J_a, \bar{\mathbb{R}}_+)$, $\sigma_0(t) \equiv 0$, $\min\limits_{i\in\langle m\rangle_0} (\min\limits_{t\in[a,a+\sigma]} (t - \sigma_i(t))) = a - \sigma$, $\min\limits_{l\in\langle r\rangle}(\min\limits_{t\in[a,a+\tau]} (t - \tau_l(t))) = a - \tau$.

(S4) There exist locally bounded functions $z^i_u, z^l_v \in L_1^{loc}(J_a, \mathbb{R}_+)$ such that $\mathrm{Var}_{[-\sigma,0]}U^i(t,\cdot) \le z^i_u(t)$ and $\mathrm{Var}_{[-\tau,0]}V^l(t,\cdot) \le z^l_v(t)$ for each $t \in J_a$.

(S5) For each $t^* \in J_a$ the following relations hold:
$\int_{-\sigma}^0 |U^i(t,\theta) - U^i(t^*,\theta)|d\theta \to 0$, $\int_{-\tau}^0 |V^l(t,\theta) - V^l(t^*,\theta)|d\theta \to 0$ when $t \to t^*$.

(S6) The sets $S^i_\Phi = \{t \in J_a |\ t - \sigma_i(t) \in S^\Phi\}$, $S^l_\Phi = \{t \in J_a |\ t - \tau_l(t) \in S^\Phi\}$ for every $i \in \langle m\rangle$ and $l \in \langle r\rangle$ do not have limit points.

(S7) The kernel $V^l(t,\theta)$ is uniformly nonatomic at zero (see [12]), i.e. for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that
$Var_{[-\delta,0]}V^l(t,\cdot) < \varepsilon$.

Remark 1. Note that the Condition **(S6)** is ultimately fulfilled in the case when all delays are strictly monotonic functions (or in the particular case of constant delays).

Definition 2. The vector function $X(t)$ is a solution of the IP (1), (3) in $J_{a+M}(J_*)$ if $X|_{[a,a+M]} \in C([a, a + M],\ \mathbb{R}^n)$ $(X|_{J_a} \in C(J_a, \mathbb{R}^n))$ satisfies the system (1) for all $t \in (a, M]$ $(t \in (a, \infty))$ and the initial condition (3) for $t \in [a - h, a]$.

Consider the following auxiliary system

$$
\begin{aligned}
X(t) &= \Phi(a) - \sum_{l=1}^{r} \int_{-\tau}^{0} [d_\theta V^l(a,\theta)\,]\Phi(a+\theta) \\
&+ \sum_{l=1}^{r} \int_{-\tau}^{0} [d_\theta V^l(t,\theta)\,]X(t+\theta) \\
&+ I_{-1}(\Gamma(\alpha)) \int_{a}^{t} I_{\alpha-1}(t-\eta)F(\eta)d\eta \\
&+ I_{-1}(\Gamma(\alpha)) \int_{a}^{t} I_{\alpha-1}(t-\eta)[\sum_{i=0}^{m} \int_{-\sigma}^{0} [d_\theta U^i(\eta,\theta)\,]X(\eta+\theta)]d\eta
\end{aligned}
\tag{4}
$$

or in more detailed form for $k \in \langle n \rangle$

$$
\begin{aligned}
x_k(t) &= \phi_k(a) - \sum_{l=1}^{r} (\sum_{j=1}^{n} \int_{-\tau}^{0} \phi_j(a+\theta)d_\theta v^l_{kj}(a,\theta)) \\
&+ \sum_{l=1}^{r} (\sum_{j=1}^{n} \int_{-\tau}^{0} x_j(t+\theta)d_\theta v^l_{kj}(t,\theta)) \\
&+ \frac{1}{\Gamma(\alpha_k)} \int_{a}^{t} (t-\eta)^{\alpha_k-1} f_k(\eta)d\eta \\
&+ \frac{1}{\Gamma(\alpha_k)} \int_{a}^{t} (t-\eta)^{\alpha_k-1}[\sum_{i=0}^{m} (\sum_{j=1}^{n} \int_{-\sigma}^{0} x_j(\eta+\theta)d_\theta u^i_{kj}(\eta,\theta))]\,d\eta
\end{aligned}
\tag{5}
$$

with the initial condition (3).

Definition 3. The vector function $X(t)$ is a solution of the IP (4), (3) in $J_{a+M}(J_*)$ if $X|_{[a,a+M]} \in C([a,a+M],\ \mathbb{R}^n)$ $(X|_{J_a} \in C(J_a,\mathbb{R}^n))$ satisfies the system (4) for all $t \in (a,M]$ $(t \in (a,\infty))$ and the initial condition (3) for $t \in [a-h,a]$.

3. Main results

Lemma 4. *Let the following conditions be fulfilled:*

1. *The conditions (S) hold.*
2. *The function $F \in L_1^{loc}(J_a, \mathbb{R}^n)$ is locally bounded.*

Then every solution $X(t)$ of IP (1), (3) *is a solution of the IP* (4), (3) *and vice versa.*

Proof. The proof is standard and partially based of the proof of the Lemma 1 in [21] for the case of continuous initial functions. For every solution in J_a of the IP (1), (3) $X(t)$ we apply the operator $D_{a+}^{-\alpha}$ to the both sides of (1) and in virtue of (a) we obtain that (4) holds. Thus we prove that $X(t)$ is a solution of the IP (4), (3). Analogically let $X(t)$ be a solution of the IP (4), (3). Then applying the operator D_{a+}^{α} to both sides of (4) and in virtue of (b) it follows that $X(t)$ is a solution of the IP (1), (3). □

Let for every $\Phi \in \mathbf{C}_a^*$ consider the corresponding linear space

$$\begin{aligned} E^\Phi = \{G : J_* \to \mathbb{R}^n |\ \ G|_{J_a} \in C(J_a, \mathbb{R}^n) \cap BV([a, a+\tau], \mathbb{R}^n),\\ G(t) = \Phi(t),\ t \in [a-h, a]\} \end{aligned}$$

Then for each $M \in \mathbb{R}_+$ and $\Phi \in \mathbf{C}_a^*$ define the set

$$E_M^\Phi = \{G_M : J_{a+M} \to \mathbb{R}^n |\ \ G_M = G|_{[a,a+M]},\ G \in E^\Phi\}$$

with a metric function $d_M^\Phi : E_M^\Phi \times E_M^\Phi \to \bar{\mathbb{R}}_+$,

$$d_M^\Phi(G_M, G_M^*) = \sum_{k=1}^n \sup_{t \in J_{a+M}} |g_k(t) - g_k^*(t)|$$

for each $G_M, G_M^* \in E_M^\Phi$. Obviously the set E_M^Φ is a complete metric space concerning this metric.

Using (5) for each $M \in \mathbb{R}_+$, $t \in (a, a+M)$ and for every $G_M = (g_1, ..., g_n)^T \in$

E_M^Φ define for $k \in \langle n\rangle$ the operator $\Re_k g_k(t)$ by

$$
\begin{aligned}
\Re_k g_k(t) = \phi_k(a) - \sum_{l=1}^{r}(\sum_{j=1}^{n}\int_{-\tau}^{0}\phi_j(a+\theta)d_\theta v_{kj}^l(a,\theta))\\
+\sum_{l=1}^{r}(\sum_{j=1}^{n}\int_{-\tau}^{0} g_j(t+\theta)d_\theta v_{kj}^l(t,\theta))\\
+\frac{1}{\Gamma(\alpha_k)}\int_a^t (t-\eta)^{\alpha_k-1} f_k(\eta)d\eta\\
+\frac{1}{\Gamma(\alpha_k)}\int_a^t (t-\eta)^{\alpha_k-1}[\sum_{i=0}^{m}(\sum_{j=1}^{n}\int_{-\sigma}^{0} g_j(\eta+\theta)d_\theta u_{kj}^i(\eta,\theta))]\,d\eta
\end{aligned}
\tag{6}
$$

with additional conditions

$$
\begin{aligned}
\Re_k g_k(t) &= \phi_k(t),\ t\in[a-h,a]\\
\Re_k g_k(a+M) &= \lim_{t\to(a+M)-0}\Re_k g_k(t)
\end{aligned}
\tag{7}
$$

Theorem 5. *Let the following conditions be fulfilled:*

1. *The conditions of Lemma 4 hold.*
2. *The initial vector function $\Phi \in \mathbf{C}_a^*$ has at most one jump point $t_\Phi \in [a-h,a]$.*

Then there exists $M^0 \in \mathbb{R}_+$ such that the operator

$$(\Re G_{M^0})(t) = (\Re_1 g_1(t),...,\Re_n g_n(t))^T$$

has a unique fixed point in the complete metric space $E_{M^0}^\Phi$, i.e. the IP (4), (3) *has a unique local solution with interval of existence J_{a+M^0}*

Proof. Let $\Phi \in \mathbf{C}_a^*$ be arbitrary and $t_l = \min\limits_{t\in[a,a+\tau]}(t-\tau_l(t)), l\in\langle r\rangle$. Then without loss of generality we can assume that $a-\tau = t_r \le ... \le t_1 \le t_0 = a$ and since $\Phi \in \mathbf{C}_a^*$ is right continuous at t_Φ, then we can conclude that $t_\Phi \in (a-h,a]$. Then either $t_\Phi \in (a-h,a-\tau]$ or $t_\Phi \in (a-\tau,a]$.

Introduce the operator $\Re$ with (6) and (7) for each $M \in \mathbb{R}_+, t\in(a,a+M)$.

(i) Let consider first the case when $t_\Phi \in (a-h,a-\tau]$ and then obviously we have that $\tau < h$. In this case the third addend in the right side of (6) is a continuous function for $t \in [a, a+M]$, $M \leq \tau$ and in virtue of Lemma 1 in [15] the fourth and fifth addends are continuous functions for $t \in [a, a+M]$ too. From (7) it follows that $\Re_k g_k(a+M) = \lim_{t \to (a+M)-0} \Re_k g_k(t)$, $\Re_k g_k(a) = \phi_k(a)$ and hence $\Re_k g_k(t) = \phi_k(t)$ for $t \in [a-h, a]$. Taking into account the conditions (S) we can conclude that for each $G_M \in E_M^\Phi$ we have $\Re G_M \in E_M^\Phi$.

Let $G_M, G_M^* \in E_M^\Phi$ be arbitrary and let

$$C^U = \max_{i \in \langle m \rangle_0} \Big(\sup_{s \in J_{a+M}} Var_{\theta \in [-\sigma, 0]} U^i(s, \theta) \Big).$$

Then for every $M \leq \tau$ from (6) it follows that the estimation

$$\begin{aligned} |\Re_k g_k(t) - \Re_k g_k^*(t)| &\leq \sum_{l=1}^{r} \Big(\sum_{j=1}^{n} \Big| \int_{-\tau}^{0} (g_j(t+\theta) - g_j^*(t+\theta)) d_\theta v_{kj}^l(t, \theta) \Big| \Big) \\ &+ \frac{1}{\Gamma(\alpha_k)} \int_a^t (t-s)^{\alpha_k - 1} \sum_{i=0}^{m} \Big(\sum_{j=1}^{n} \Big| \int_{-\sigma}^{0} (g_j(s+\theta) - g_j^*(s+\theta)) du_{kl}^i(s, \theta) \Big| \, ds \Big) \\ &\leq \frac{(t-a)^{\alpha_k} C^U}{\Gamma(1+\alpha_k)} \sum_{l=1}^{n} \sup_{t \in J_{a+M}} |(g_l(t) - g_l^*(t)| \end{aligned} \tag{8}$$

holds for each $t \in [a, a+M]$ and $k \in \langle n \rangle$.

Let $M^0 = \min\big(\big(\frac{\Gamma(1+\alpha_k)}{2nC^U} \big)^{\frac{1}{\alpha_k}}, \tau \big)$. Then for each $t \in [a, a+M^0]$, from (8) and conditions (S) it follows that the inequality

$$\frac{(t-a)^{\alpha_k} C^U}{\Gamma(1+\alpha_k)} \leq \frac{1}{2n} \tag{9}$$

holds for every $k \in \langle n \rangle$. Hence from (8) and (9) it follows that

$$d_{M^0}(\Re G_{M^0}, \Re G_{M^0}^*) \leq \frac{1}{2} d_{M^0}(G_{M^0}, G_{M^0}^*),$$

i.e. the operator $\Re$ is contractive in $E_{M^0}^\Phi$.

(ii) Let consider the case when $t_\Phi \in (a-\tau, a]$. From conditions (S) it follows that

$$\begin{aligned} \sum_{l=1}^{r} \Big(\sum_{j=1}^{n} \int_{-\tau}^{0} g_j(t+\theta) d_\theta v_{kj}^l(t, \theta) \Big) &= \sum_{l=1}^{r} \Big(\sum_{j=1}^{n} \bar{a}_{kj}^l(t) g_j(t - \tau_l(t)) \Big) \\ &+ \sum_{l=1}^{r} \Big(\sum_{j=1}^{n} \int_{-\tau}^{0} g_j(t+\theta) d_\theta \tilde{v}_{kj}^l(t, \theta) \Big) \end{aligned} \tag{10}$$

where $\tilde{v}^l_{kj}(t,\theta)$ denotes the sum of the absolutely continuous and singular parts in the Lebesgue decomposition of the function $v^l_{kj}(t,\theta)$. Then there exists a maximal number $l^* \in \langle r-1\rangle_0$, such that $t_\Phi \in (t_{l^*+1}, t_{l^*}]$ and hence $t_{l^*+1} < t_\Phi$. Hence there exists a point $M^* \in (a, a+\tau]$ such that for $l^* \geq 1$ the inequalities $\max\limits_{l^*+1\leq l\leq r}(\sup\limits_{t\in[a,a+M^*]}(t-\tau_l(t))) < t_\Phi$ and $\min\limits_{1\leq l\leq l^*}(\min\limits_{t\in[a,a+M^*]}(t-\tau_l(t))) \geq t_{l^*} \geq t_\Phi$ hold and when $l^* = 0$ ($t_\Phi = a$ and $t_1 < a$) only the first one. Therefore we can conclude that the first addend in the right side of (10) is a continuous function for $t \in [a, a+M^*]$. For the second addend for each $j \in \langle n\rangle$ and $t \in [a, a+\tau]$ we have that $t+\theta \leq t$ and hence $Var_{\theta\in[-\tau,0]}g_j(t+\theta) < \infty$ and $\lim\limits_{t\to t^*}\int\limits_{-\tau}^{0}|g_j(t^*+\theta) - g_j(t+\theta)|d\theta = 0$ for $t^* \in [a, a+\tau]$. Then for each $l \in \langle r\rangle,\ j \in \langle n\rangle,\ k \in \langle n\rangle$ we have

$$\begin{aligned}\int\limits_{-\tau}^{0} g_j(t+\theta)d_\theta\tilde{v}^l_{kj}(t,\theta) = g_j(t+\theta)\tilde{v}^l_{kj}(t,\theta)|^{\theta=0}_{\theta=-\tau}\\ -\int\limits_{-\tau}^{0}\tilde{v}^l_{kj}(t,\theta)d_\theta g_j(t+\theta).\end{aligned} \tag{11}$$

It is well known that (11) holds if at least one of the integrals in the equation (11) exists. Since $\tilde{v}^l_{kj}(t,\theta)$ is a continuous function for $t \in J_a$ and $\theta \in [-\tau, 0]$, then the integral in the left side of (11) exists and hence the other one in the right side exists too. Taking into account that $Var_{\theta\in[-\tau,0]}g_j(t+\theta) < \infty$ in virtue of Lemma 1 in [15] we can conclude that the integral in the right side of (11) is a continuous function for $t \in [a, a+M^*]$. Since the right side of (10) is a continuous function for $t \in [a, a+M^*]$ then the left side of (10) is a continuous function for $t \in [a, a+M^*]$ too and hence for each $M \leq M^*$ we can conclude that for each $G_M \in E^\Phi_M$ we have that $\Re G_M \in E^\Phi_M$. Then as in case (i) we can obtain that there exists $M^0 \leq M^*$ such that the operator $\Re$ is contractive in$E^\Phi_{M^0}$. □

Theorem 6. *Let the conditions of Theorem 5 hold.*
Then the IP (4), (3) *has a unique solution in* J_*.

Proof. We will use mathematical induction to proof the statement of the theorem. According Theorem 5 there exists $M^0 > 0$ such that the IP (4), (3) has a unique solution in J_{a+M^0}. Denote by $X^{\max}(t) = (x^{\max}_1(t), ..., x^{\max}_n(t))$ the maximal solution of the IP (4), (3) with intervals of existence $J_{\max}$, i.e. $X^{\max}(t)$ is a continuation of every other solution of the IP (4), (3).

Let assume that $J_{\max} = [a-h, a+M^{\max})$ and consider the case when $M^{\max} \geq \tau$. Then obviously the right side of (4) can be prolonged as continuous function at $t = a + M^{\max}$ and hence (4) holds for $t = a + M^{\max}$ too. In the case when $M^{\max} < \tau$ as in the proof of Theorem 5 point (ii) we can prove that the right side of (4) can be prolonged as continuous function also at $t = a + M^{\max}$ and therefore (4) holds for $t = a + M^{\max}$ too. Therefore we obtain a solution which is a prolongation of $X^{\max}(t)$ since it has as interval of existence $[a-h, a+M^{\max}]$, which contradicts of our assumption that $X^{\max}(t)$ is a maximal solution of the IP (4), (3). Thus we proved that the interval of existence for $X^{\max}(t)$ is $J_{\max} = [a-h, a+M^{\max}] = J_{a+M^{\max}}$.

Let assume that the the interval of existence for $X^{\max}(t)$ is $J_{a+M^{\max}}$ and consider the case when $M^{\max} < \tau$. Denote by $M_\varepsilon = M^{\max} + \varepsilon$, where $\varepsilon \in (0, \tau - M^{\max})$ is arbitrary number. Then we can consider a new IP for the system (4) in the interval $[a+M^{\max}, a+M_\varepsilon]$, with initial interval $[a+M^{\max}-h, a+M^{\max}]$. As initial function $\Phi^{X^{\max}}(t) = (\phi_1^{X^{\max}}(t), ..., \phi_n^{X^{\max}}(t))$ in this interval we use the function $X^{\max}(t)$, i.e. $\Phi^{X^{\max}}(t) = X^{\max}(t)$ for $t \in [a+M^{\max}-h, a+M^{\max}]$.

As above introduce the complete metric space

$$E^{\Phi}_{M_\varepsilon} = \{G_{M_\varepsilon} : J_{a+M_\varepsilon} \to \mathbb{R}^n | \ \ G_{M_\varepsilon} = G|_{[a,a+M_\varepsilon]}, \ G \in E^{\Phi}\}$$

with metric function

$$d^{\Phi}_{M_\varepsilon}(G_{M_\varepsilon}, G^*_{M_\varepsilon}) = \sum_{k=1}^{n} \sup_{t \in J_{a+M_\varepsilon}} |g_k(t) - g^*_k(t)|.$$

Using (6) for every $G_{M_\varepsilon} = (g_1, ..., g_n)^T \in E^{\Phi}_{M_\varepsilon}$ define for each $\varepsilon \in (0, \tau - M^{\max})$, $t \in (a+M^{\max}, a+M_\varepsilon)$ and $k \in \langle n \rangle$ the operators $\Re_k g_k(t)$ with equality (6) and the additional conditions

$$\begin{aligned} \Re_k g_k(t) &= \phi_k^{X^{\max}}(t), \ t \in [a+M^{\max}-h, a+M^{\max}] \\ \Re_k g_k(a+M_\varepsilon) &= \lim_{t \to (a+M_\varepsilon)-0} \Re_k g_k(t) \end{aligned} \tag{12}$$

Let denote by $t_l = \min\limits_{t \in [a+M^{\max}, a+\tau]} (t - \tau_l(t)), l \in \langle r \rangle$. Then without loss of generality we can assume that $a + M^{\max} - \tau \leq t_r \leq ... \leq t_1 \leq t_0 = a + M^{\max}$ Then either $t_r \geq a$ or there exists a maximal number $l^* \in \langle r-1 \rangle_0$, such that $t_{l^*} \geq a$ and $t_{l^*+1} < a$. Then we have that either $t_r < a$ or $t_r \geq a$ (note that $t_r > a + M^{\max} - \tau$). Hence either $t_\Phi \leq t_r$ or $t_\Phi \in (t_r, a]$.

(i) Consider the case when $t_\Phi \leq t_r$ Then is simply to see that for each $\varepsilon \in (0, \tau - M^{\max})$ the right side of (6) is a continuous function in $[a+M^{\max}, a+M_\varepsilon]$

and hence the operator $\Re$ defined with (6) and satisfying (12) maps $E^{\Phi}_{M_\varepsilon}$ into $E^{\Phi}_{M_\varepsilon}$.

(ii) Let assume that $t_\Phi \in (t_r, a]$. Then obviously $a \in (t_{l^*+1}, t_{l^*}]$ and there exists a number $\tilde{l}$, $l^* \leq \tilde{l} \leq r-1$, such that $t_\Phi \in (t_{\tilde{l}+1}, t_{\tilde{l}}]$. Then according condition (S3) there exists $\varepsilon_* \in (0, \tau - M^{\max})$ such that the right side of (6) is a continuous function for $t \in [a + M^{\max}, a + M_{\varepsilon_*}]$.

Hence in both considered cases (i) and (ii) we proved that the right side of (6) is a continuous function for $t \in [a + M^{\max}, a + M_{\varepsilon_*}]$ and therefore the operator $\Re$ defined with (6) and satisfying (12) maps $E^{\Phi}_{M_\varepsilon}$ into $E^{\Phi}_{M_\varepsilon}$ for every $\varepsilon \in (0, \varepsilon_*]$.

Let $\varepsilon \in (0, \varepsilon_*]$ be an arbitrary number. Then for every $G_{M_\varepsilon}, G^*_{M_\varepsilon} \in E^{\Phi}_{M_\varepsilon}$ and each $t \in [a + M^{\max}, a + M_{\varepsilon_*}]$ from (6) we obtain that

$$
\begin{aligned}
&|\Re_k g_k(t) - \Re_k g^*_k(t)| \leq \sum_{l=1}^{r} (\sum_{j=1}^{n} | \int_{-\tau}^{0} (g_j(t+\theta) - g^*_j(t+\theta)) d_\theta v^l_{kj}(t,\theta)|) \\
&+ \frac{1}{\Gamma(\alpha_k)} \int_a^t (t-s)^{\alpha_k - 1} \sum_{i=0}^{m} (\sum_{j=1}^{n} | \int_{-\sigma}^{0} (g_j(s+\theta) - g^*_j(s+\theta)) du^i_{kl}(s,\theta)|\, ds) \\
&\leq \frac{1}{\Gamma(\alpha_k)} \int_{a+M^{\max}}^{t} (t-s)^{\alpha_k - 1} \sum_{i=0}^{m} (\sum_{j=1}^{n} | \int_{-\sigma}^{0} (g_j(s+\theta) - g^*_j(s+\theta)) du^i_{kl}(s,\theta)|\, ds) \\
&\leq \frac{(t - (a + M^{\max}))^{\alpha_k} C^U}{\Gamma(1+\alpha_k)} \sum_{l=1}^{n} \sup_{t \in J_{a+M_{\varepsilon_*}}} |(g_l(t) - g^*_l(t)|
\end{aligned}
\tag{13}
$$

where $C^U = \max_{i \in \langle m \rangle_0} (\sup_{s \in J_{a+M_{\varepsilon_*}}} Var_{\theta \in [-\sigma, 0]} U^i(s,\theta))$.

If $\varepsilon_0 = \min((\frac{\Gamma(1+\alpha_k)}{2nC^U})^{\frac{1}{\alpha_k}}, \varepsilon_*)$ we have that $\frac{\varepsilon_0^{\alpha_k} C^U}{\Gamma(1+\alpha_k)} = \frac{1}{2n}$ and from (13) it follows that the operator $\Re$ is contractive in $E^{\Phi}_{M_{\varepsilon_0}}$.

Hence we prove that the solution $X^{\max}(t)$ has a continuation which contradicts with our assumption concerning $X^{\max}(t)$. Thus we proved that $M^{\max} = \tau$ and then the interval of existence of $X^{\max}(t)$ is $J_{a+M^{\max}} = J_{a+\tau}$.

Since $J_a = \bigcup_{j=1}^{\infty} [a + (j-1)\tau,\ a + j\tau]$ then according the principle of the mathematical induction we assume that that there exists a unique solution $X^j(t) = (x^j_1(t), ..., x^j_n(t))^T$ of IP (4), (3) with interval of existence $J_{a+j\tau}$, $j \in \mathbb{N}$ (Note that the existence of $X^1(t) = X^{\max}(t)$ is proved above). Then we can consider an auxiliary IP for the system (4) in the interval $J((j+1)\tau) = [a+j\tau, a+(j+1)\tau]$ with initial interval $[a+j\tau-h, a+j\tau]$, i.e. $X^{j+1}(t)$ satisfies the system (4) for $t \in (a+j\tau, a+(j+1)\tau]$ and $X^{j+1}(t) = X^j(t)$ when $t \in [a+j\tau-h, a+j\tau]$. As in the case $j=1$ introduce the metric space

$$E^{\Phi}_{(j+1)\tau} = \{G_{(j+1)\tau} : J_{a+(j+1)\tau} \to \mathbb{R}^n |\ G_{(j+1)\tau} = G|_{[a,a+(j+1)\tau]},\ G \in E^{\Phi}\}$$

with metric function

$$d^{\Phi}_{a+(j+1)\tau}(G_{(j+1)\tau}, G^*_{(j+1)\tau}) = \sum_{k=1}^{n} \sup_{t \in J_{a+(j+1)\tau}} |g_k(t) - g^*_k(t)|.$$

For $t \in (a+j\tau, a+(j+1)\tau)$ define the operators $\Re_k g_k(t)$, $k \in \langle n \rangle$ with (6) and the additional conditions

$$\begin{aligned} &\Re_k g_k(t) = x^j_k(t),\ t \in [a+j\tau-h, a+j\tau] \\ &\Re_k g_k(a+(j+1)\tau) = \lim_{t \to (a+(j+1)\tau)-0} \Re_k g_k(t) \end{aligned} \tag{14}$$

Then for each $t \in [a+j\tau, a+(j+1)\tau]$ we have that $t - \tau_l(t) \geq a$, $l \in \langle r \rangle$ and for arbitrary $G_{(j+1)\tau} \in E^{\Phi}_{(j+1)\tau}$ we have that the right side of (6) is a continuous function and hence the operator $\Re$ defined with (6) and satisfying (14) maps $E^{\Phi}_{(j+1)\tau}$ into $E^{\Phi}_{(j+1)\tau}$. According Theorem 5 there exists a unique local solution $X^{j+1}_{M^0}(t)$, such that $X^{j+1}_{M^0}(t) = X^j(t)$ for $t \in [a+j\tau-h, a+j\tau]$ and with interval of existence $[a+j\tau, a+j\tau+M^0]$,$M^0 \in (a+j\tau, a+(j+1)\tau]$.

Then analogical way as in the proof of point (ii2) above, we obtain that $M^0 = \tau$. Thus we proved that the IP (4), (3) has a unique solution in J_*. □

4. Applications

Let $s \in J_a$ be arbitrary fixed number, $J_s = [s, \infty)$ and consider the following matrix IP

$$\begin{aligned} &D^{\alpha}_{a+}(G(t,s) - \sum_{l=1}^{r} \int_{-\tau}^{0} [d_\theta V^l(t,\theta)]G(t+\theta,s)) \\ &= \sum_{i=0}^{m} \int_{-\sigma}^{0} [d_\theta U^i(t,\theta)]G(t+\theta,s), \quad t \in (s,\infty), \end{aligned} \tag{15}$$

$$C(t,s) = \begin{cases} I, & t = s \\ 0, & -\infty < t < s \end{cases} \tag{16}$$

Definition 7. For each $s \in J_a$ the matrix valued function $t \to G(t,s) = \{\gamma_{kj}(t,s)\}_{k,j=1}^{n}$ is called a solution of the IP (15), (16) for $t \in J_s$, if $G(\cdot,s) : J_s \to \mathbb{R}^{n\times n}$ is continuous in t on J_s and satisfies the matrix equation (15) for $t \in (s,\infty)$, as well as the initial condition (16) too.

Corollary 8. *Let the conditions (S) hold.*
Then for every $s \in J_a$ the IP (15), (16) *has a unique solution in J_s.*

Proof. Let $s \in J_a$ be an arbitrary fixed number and for $j \in \langle n\rangle$ consider the IP (2), (3) in J_s with initial condition $X^j(t,s) = I^j, t = s; X^j(t,s) = 0, s-h < t < s$ and I^j denotes the j-th column of the identity matrix I. Since the system (2) is homogeneous and for each $j \in \langle n\rangle$ the corresponding initial function is I^j for $t = s$ and zero for $t < s$, then we can conclude that all conditions of Theorem 6 are fulfilled. Then for each $j \in \langle n\rangle$ denote by $X^j(t,s) = (\gamma_{1j}(t,s), ..., \gamma_{nj}(t,s))^T$ the unique solution of the IP (2), (3) in J_s existing according Theorem 6. Then the matrix $G(t,s) = (X^1(t,s), ..., X^n(t,s))$, $G(\cdot,s) : [s,\infty) \to \mathbb{R}^{n\times m}$ is a unique solution of the IP (15), (16) in J_s. $\square$

Remark 9. Note that since for each $s \in J_a$ every column $X^j(t,a)$is a unique solution of (2) with initial condition $X^j(t,s) = I^j, t = s; X^j(t,s) = 0, s-h < t < s; j \in \langle n\rangle$ we can define $X^j(t,s) = 0, j \in \langle n\rangle$ for $t \in (-\infty, s-h)$ and then for each $s \in J_a$ the matrix $G(t,s)$ is prolonged as continuous in t on $(-\infty, s)$.

The matrix $G(t,s)$ will be called fundamental (or Cauchy) matrix for the system (2) for $t \in J_a$.

Let for a fixed $s \in [a-h,a]$ introduce the matrix valued function $(t \to Q(t,s)) : J_a \to \mathbb{R}^{n\times n}$ satisfying the system (15) for $t \in (a,\infty)$ and the initial condition

$$Q(t,s) = \begin{cases} I, & s \le t \le a \\ 0, & t \in (-\infty, s) \end{cases} \tag{17}$$

Note that $G(t,a) = Q(t,a)$ for $t \in J_a$.

For arbitrary initial function $\Phi \in \mathbf{C}_a^*$ introduce the vector function

$$X_\Phi(t) = \int\limits_{a-h}^{a} Q(t,s) d_s \Phi_h(s), t \in J_* \tag{18}$$

where $\Phi_h(t) \equiv \Phi(t)$ for $t \in (a-h,a]$ and $\Phi_h(a-h) = 0$.

Theorem 10. *Let the following conditions are fulfilled:*

1. *The conditions of Corollary 8 hold.*
2. *The kernels $V^l(t,\theta) \equiv V^l(\theta)$ for each $l \in \langle n \rangle$.*

Then the vector function $X_\Phi(t)$ defined by equality (18) *is a unique solution of the IP* (2), (3) *for $t \in J_a$.*

Proof. It is simply to see that for arbitrary $t \in [a-h, a]$ we have that

$$\begin{aligned} X_\Phi(t) &= \int\limits_{a-h}^{a} Q(t,s) d_s \Phi_h(s) = \int\limits_{a-h}^{a} I \, d_s \Phi_h(s) \\ &= \Phi_h(t) - \Phi_h(a-h) = \Phi(t), \end{aligned}$$

i.e. $X_\Phi(t)$ satisfies the initial condition (3).

Lemma 1 in [15] implies that the vector function $X_\Phi(t)$ defined by equality (18) is a continuous function for $t \in J_a$. Substituting in the left side of (2) the vector function $X_\Phi(t)$ we obtain

$$\begin{aligned} {}_C D^\alpha_{a+} X_\Phi(t) &= {}_C D^\alpha_{a+} \int\limits_{a-h}^{a} Q(t,s) d_s \Phi_h(s) \\ &= {}_{RL} D^\alpha_{a+} [\int\limits_{a-h}^{a} Q(t,s) d_s \Phi_h(s) - \int\limits_{a-h}^{a} Q(a,s) d_s \Phi_h(s)] \qquad (19) \\ &= {}_{RL} D^\alpha_{a+} \int\limits_{a-h}^{a} [Q(t,s) - Q(a,s)] d_s \Phi_h(s) \end{aligned}$$

Then from (19) using the Fubini theorem we have that

$$\begin{aligned} &{}_C D^\alpha_{a+} X_\Phi(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int\limits_{a}^{t} (t-\eta)^{-\alpha} (\int\limits_{a-h}^{a} [Q(\eta,s) - Q(a,s)] d_s \Phi_h(s)) d\eta \\ &= \int\limits_{a-h}^{a} (\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int\limits_{a}^{t} (t-\eta)^{-\alpha} [Q(\eta,s) - Q(a,s)] d\eta) d_s \Phi_h(s) \\ &= \int\limits_{a-h}^{a} {}_{RL} D^\alpha_{a+} [Q(t,s) - Q(a,s)] d_s \Phi_h(s) \end{aligned}$$

$$= \int\limits_{a-h}^{a} {}_{C}D_{a+}^{\alpha} Q(t,s) d_s \Phi_h(s). \tag{20}$$

For each $l \in \langle r \rangle$ applying the unsymmetric Fubini theorem (see [7]) we have that

$$\begin{aligned}
&D_{a+}^{\alpha} \int\limits_{-\tau}^{0} [d_\theta V^l(\theta)] X_\Phi(t+\theta) \\
&=_{RL} D_{a+}^{\alpha} \int\limits_{-\tau}^{0} [d_\theta V^l(\theta)](X_\Phi(t+\theta) - X_\Phi(a+\theta)) \\
&=_{RL} D_{a+}^{\alpha} \int\limits_{-\tau}^{0} [d_\theta V^l(\theta)](\int\limits_{a-h}^{a} [Q(t+\theta,s) - Q(a+\theta,s)] d_s\Phi_h(s)) \\
&=_{RL} D_{a+}^{\alpha} \int\limits_{a-h}^{a} (\int\limits_{-\tau}^{0} [d_\theta V^l(\theta)](Q(t+\theta,s) - Q(a+\theta,s))) d_s\Phi_h(s) \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int\limits_{a}^{t} (t-\eta)^{-\alpha} (\int\limits_{a-h}^{a} (\int\limits_{-\tau}^{0} [d_\theta V^l(\theta)](Q(\eta+\theta,s) - Q(a+\theta,s))) d_s\Phi_h(s)) d\eta \\
&= \int\limits_{a-h}^{a} (\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int\limits_{a}^{t} (t-\eta)^{-\alpha} (\int\limits_{-\tau}^{0} [d_\theta V^l(\theta)](Q(\eta+\theta,s) - Q(a+\theta,s))) d\eta) d_s\Phi_h(s) \\
&= \int\limits_{a-h}^{a} {}_{RL}D_{a+}^{\alpha} (\int\limits_{-\tau}^{0} [d_\theta V^l(\theta)](Q(t+\theta,s) - Q(a+\theta,s))) d_s\Phi_h(s) \\
&= \int\limits_{a-h}^{a} {}_{C}D_{a+}^{\alpha} (\int\limits_{-\tau}^{0} [d_\theta V^l(\theta)] Q(t+\theta,s)) d_s\Phi_h(s).
\end{aligned} \tag{21}$$

Then from (20) and (21) it follows that

$$\begin{aligned}
&{}_C D^{\alpha}_{a+}(X_{\Phi}(t) - \sum_{l=1}^{r} \int_{-\tau}^{0} [d_{\theta} V^{l}(\theta)] X_{\Phi}(t+\theta)) \\
&= \int_{a-h}^{a} {}_C D^{\alpha}_{a+}(Q(t,s) - \sum_{l=1}^{r} \int_{-\tau}^{0} [d_{\theta} V^{l}(\theta)] Q(t+\theta,s)) d_s \Phi_h(s).
\end{aligned} \tag{22}$$

In the right side of (2) for each $i \in \langle m \rangle$ applying the unsymmetric Fubini theorem (see [7]) we obtain

$$\begin{aligned}
&\int_{-\sigma}^{0} [d_{\theta} U^{i}(t,\theta)] X_{\Phi}(t+\theta) = \int_{-\sigma}^{0} [d_{\theta} U^{i}(t,\theta)] \int_{a-h}^{a} Q(t+\theta,s) d_s \Phi_h(s) = \\
&= \int_{a-h}^{a} (\int_{-\sigma}^{0} [d_{\theta} U^{i}(t,\theta)] G(t+\theta,s)) d_s \Phi_h(s)
\end{aligned} \tag{23}$$

From (22) and (23) it follows that

$$\begin{aligned}
&\int_{a-h}^{a} [{}_C D^{\alpha}_{a+}(Q(t,s) - \sum_{l=1}^{r} \int_{-\tau}^{0} [d_{\theta} V^{l}(t,\theta)] Q(t+\theta,s)) \\
&- \sum_{i=0}^{m} \int_{-\sigma}^{0} [d_{\theta} U^{i}(t,\theta)] Q(t+\theta,s)] d_s \Phi_h(s) = 0
\end{aligned}$$

and hence $X_{\Phi}(t)$ satisfies (2) for $t > a$. □

Remark 11. Note that the matrix valued function $(t \to Q(t,s)) : J_a \to \mathbb{R}^{n \times n}$ satisfying the system (15) for $t \in (a, \infty)$ and the initial condition (17) is not depending from the function $\Phi \in \mathbf{C}^*_a$ used in IP (2), (3) as initial function. Thus we can conclude that the initial function $\Phi \in \mathbf{C}^*_a$ used in (18) can have arbitrary finite number of jumps.

Corollary 12. *Let the following conditions are fulfilled:*

1. *The conditions of Theorem 10 hold.*
2. *The Lebesgue decomposition of the function $\Phi \in \mathbf{C}^*_a$ does not include a singular term.*

Then the vector function $X_\Phi(t)$ defined by equality (18) *has the representation*

$$\begin{aligned} X_\Phi(t) &= Q(t,a)(\Phi_h(a+) - \Phi_h(a-)) \\ &+ \sum_{s_q \in S^{\Phi_h} \setminus \{a\}} Q(t,s_q)(\Phi_h(s_q+) - \Phi_h(s_q-)) \\ &+ \int\limits_{a-h}^{a} Q(t,s)\tilde{\Phi}_h{}'(s)ds \end{aligned} \tag{24}$$

where the summation is over all discontinuity points $s_q \in S^{\Phi_h} \setminus \{a\}$ of the function $\Phi_h(t)$ and with $\tilde{\Phi}_h$ is denoted the absolutely continuous term in its Lebesgue decomposition.

Proof. Since $\Phi_h \in \mathbf{C}^*_a$ then (24) immediately follows from the Lebesgue decomposition of $\Phi_h(t)$ and (18). □

Remark 13. Our result improves the corresponding results even in the case of one constant delay which is considered in Theorem 5.3 in [25]. Actually their result is proved only for absolute continuous solutions, since he authors used the definition of Caputo derivative useful only in the case for absolute continuous functions. Moreover the fact that the solutions are absolute continuous is essentially used in the proofs in the whole their work and hence the definition cannot be simply replaced with the other one. In addition we point out that in [25] as a space of initial functions in the partial case of one constant delay τ and initial point $a = 0$ is used the space $C^1([-\tau,0],\mathbb{R}^n)$ for which obviously the inclusions $C^1([-\tau,0],\mathbb{R}^n) \subset \mathbf{C}^*_0 \subset PC([-\tau,0],\mathbb{R}^n) \cap BV([-\tau,0],\mathbb{R}^n)$ hold.

References

[1] B. Bonilla,M. Rivero, J. Trujillo, On systems of linear fractional differential equations with constant coefficients, *Appl. Math. Comput.* **187** No. 1 (2007), 68-78.

[2] D. Boyadzhiev, H. Kiskinov, M. Veselinova, A. Zahariev, Stability analysis of linear distributed order fractional systems with distributed delays, *Fract. Calc. Appl. Anal.* **20**, No. 4 (2017), 914-935; doi: 10.1515/fca-2017-0048.

[3] K. Diethelm, *The Analysis of Fractional Differential Equations, an Application-oriented Exposition Using Differential Operators of Caputo Type.* Lecture Notes in Mathematics, vol. 2004, Springer-Verlag, Berlin, (2010).

[4] M. Feckan, J. Wang, M. Pospisil, *Fractional-Order Equations and Inclusions*, De Gruyter,(2017).

[5] J. Hale, S. Lunel, *Introduction to Functional Differential Equations.* Springer-Verlag, New York, (1993).

[6] Z. Jiao, Y.Q. Chen, I. Podlubny, *Distributed-Order Dynamic Systems: Stability, Simulation, Applications and Perspectives.* Springer Brief, Springer-Verlag (2012).

[7] G. W. Johnson, An unsymmetric Fubini theorem, *The American Mathematical Monthly* **91** , No. 2 (1984), 131-133.

[8] T. Kaczorek, D. Idczak, Cauchy formula for the time-varying linear systems with Caputo derivative, *Fract. Calc. Appl. Anal.* **20**, No 2 (2017), 494-505; doi: 10.1515/fca-2017-0025.

[9] A. Kilbas, H. Srivastava, J. Trujillo, *Theory and Applications of Fractional Differential Equations.* Elsevier Science B.V, Amsterdam (2006).

[10] V. Kiryakova, *Generalized Fractional Calculus and Applications.* Longman Sci. Techn., Harlow, & John Wiley and Sons, New York (1994).

[11] H. Kiskinov, A. Zahariev, On fractional systems with Riemann-Liouville derivatives and distributed delays - Choice of initial conditions, existence and uniqueness of the solutions, *Eur. Phys. J. Special Topics* (2018), doi:10.1140/epjst/e2018-00077-9.

[12] V. Kolmanovskii, A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht / Boston / London, (1999).

[13] K. Li, J. Peng, Laplace transform and fractional differential equations, *Appl. Math. Lett.* **24**, No. 12 (2011), 2019-2023.

[14] S. D. Lin, C. H. Lu, Laplace transform for solving some families of fractional differential equations and its applications, *Adv. Difference Equ.* **137** (2013), 1-9.

[15] A. Myshkis, *Linear Differential Equations with Retarded Argument.* Nauka, Moscow (1972) (in Russian).

[16] Z. Odibat, Analytic study on linear systems of fractional differential equations, *Comput. Math. Appl.* **59**, No. 3 (2010), 1171-1183.

[17] I. Podlubny, *Fractional Differential Equation.* Academic Press, San Diego (1999).

[18] I. Stamova, G. Stamov, *Functional and Impulsive Differential Equations of Fractional Order, Qualitative analysis and applications.* Boca Raton, FL, CRC Press, (2017).

[19] M. Veselinova, H. Kiskinov, A. Zahariev, Stability analysis of linear fractional differential system with distributed delays. *AIP Conference Proceedings* **1690** (2015), Art. ID 040013; doi: 10.1063/1.4936720.

[20] M. Veselinova, H. Kiskinov, A. Zahariev, Stability analysis of neutral linear fractional system with distributed delays. *Filomat* **30**, No 3 (2016), 841–851; doi 10.2298/FIL1603841V.

[21] M. Veselinova, H. Kiskinov, A. Zahariev, Explicit conditions for stability of neutral linear fractional system with distributed delays. *AIP Conference Proceedings* **1789** (2016), Art. ID 040005; doi: 10.1063/1.4968458.

[22] M. Veselinova, H. Kiskinov, A. Zahariev, About stability conditions for retarded fractional differential systems with distributed delays, *Communications in Applied Analysis* **20** (2016), 325-334.

[23] J. Wei, The constant variation formulae for singular fractional differential systems with delay, *Computers and Mathematics with Applications* **59** (2010), 1184-1190.

[24] X. Zhang, Some results of linear fractional order time-delay system, *Appl. Math. Comput.* **197**, No. 1 (2008), 407-411.

[25] H. Zhang, J. Cao, W. Jiang, General solution of linear fractional neutral differential difference equations, *Discrete Dynamics in Nature and Society*, (2013), Art. ID 489521; doi: 10.1155/2013/489521.

[26] H. Zhang, D. Wu, Variation of constant formulae for time invariant and time varying Caputo fractional delay differential systems, *Journal of Mathematical Research with Applications* **34**, No. 5, (2014), 549-560; doi: 10.3770/j.issn:2095-2651.2014.05.006