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THE DETOUR IRREDUNDANT NUMBER OF A GRAPH

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Abstract: For two vertices u and v of a connected graph G, the set $I_D[u,v]$ consists of all those vertices lying on $u-v$ detour in G. Given a set S of vertices of G, the union of all sets $I_D[u,v]$ for $u,v \in S$ is denoted by $I_D[S]$. A detour convex set S satisfies $I_D[S] = S$. The detour convex hull $[S]_D$ of S is the smallest detour convex set containing S. The detour hull number $d_h(G)$ is the minimum cardinality among the subsets S of V with $[S]_D = V$. In this paper, we introduce and study the detour irredundant number of a graph. A set S of vertices of G is a detour irredundant set if $u \notin I_D[S-\{u\}]$ for all $u \in S$ and the maximum cardinality of a detour irredudant set is its detour irredundant number $dir(G)$ of G. We determine the detour irredundant number of certain standard classes of graphs. Certain general properties of these concepts are studied. We characterize the classes of graphs of order n for which $dir(G) = n$ or $dir(G) = n-1$ or $dir(G) = n-2$, respectvely. We prove that for any integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $d_h(G) = a$ and $dir(G) = b$. We also prove that for integers a, b and $k \geq 2$ with $a < b \leq 2a$, there exists a connected graph G with $rad_D(G) = a, diam_D(G) = b$ and $dir(G) = k$.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges.For basic definitions and terminologies, we refer to [1, 2]. For vertices x and y in a nontrivial connected graph G of order n, the *detour distance* $D(x, y)$ is the length of a longest $x-y$ path in G. An $x-y$ path of length $D(x, y)$ is an $x-y$ detour. It is known that the detour distance is a metric on the vertex set $V(G)$. The *detour eccentricity* $e_D(v)$ of a vertex v in G is the maximum detour distance from v to a vertex of G. The *detour radius*, $rad_D(G)$ of G is the minimum detour eccentricity among the vertices of G, while the *detour diameter*, $diam_D(G)$ of G is the maximum detour eccentricity among the vertices of G. The detour distance, detour center, Wiener, hyper-Wiener detour, edge detour and connected detour of a graph were studied in [3, 4, 5, 9, 10, 11].

The closed interval $I_D[x, y]$ consists of x, y, and all vertices lying in some $x-y$ detour of G; while for $S \subseteq V(G)$, $I_D[S]_G = \bigcup_{x,y\in S} I_D[x, y]$. A set S of vertices is a *detour convex set* if $I_D[S] = S$. The *detour convex hull* $[S]_D$ of S in G is the smallest detour convex set containing S. A set S of vertices of G is a *detour hull set* if $[S]_D = V(G)$ and a detour hull set of minimum cardinality is the *detour hull number* $d_h(G)$. The detour hull number of a graph was introduced and studied in [6]. These concepts have interesting applications in Channel Assignment Problem in radio technologies [6, 7, 8]. In [8], the distance martix and the detour matrix of a connected graph are used to discuss the applications of the graph parameters Wiener index, the detour index, the hyper-Wiener index and the hyper-detour index to a class of graphs viz. bridge and chain graphs, which in turn, capture different aspects of certain molecular graphs associated to the molecules arising in special situations of molecular problems in theoretical Chemistry.

A vertex v in a connected graph G is a *detour extreme vertex* if it is an initial or terminal vertex of any detour in G containing the vertex v.

If S is detour hull set of a connected graph G and $u, v \in S$, then each vertex of every $u-v$ detour of G belongs to $I_D[S]$. This gives the following observation.

Theorem 1. [6] *Let S be a detour hull set of a connected graph G and let $u, v \in S$. If $w \neq (u, v)$ lies on a $u-v$ detour in G, then $w \notin S$.*

The above observation motivate us to study a new type of sets, called detour irredundant sets, which generalizes minimum detour hull sets in a graph. In the next section, we introduce and study detour irredundant sets and the detour irredundant number of a graph. The detour irredundant number of certain standard classes of graphs are determined. Various characterization results are proved.

The following theorems are used in the sequel.

Theorem 2. Each detour extreme vertex of a nontrivial connected graph G belongs to every detour hull set of G.

2. Detour irredundant sets in graphs

Let S be a set of vertices in a connected graph G. A vertex v in S is called an *d-redundant vertex* of S, if $v \in I_D[S - \{v\}]$. The set of all d-redundant vertices of S is denoted by S^0. It can be observe that if $S^0 = \emptyset$, then $T^0 = \emptyset$ for any subset T of S. A set S of vertices is called a *detour irredundant set* or simply *d-irredundant set* if $S^0 = \emptyset$. An d-irredundant set of maximum cardinality is called a *maximum d-irredundant set* or a $dir - set$ of G. The cardinality of a $dir - set$ is the *d-irredundant number* $dir(G)$ of G. It follows from Theorem 1 that every minimum detour hull set of a connected graph G is an d-irredundant set in G and so we have that $2 \leq d_h(G) \leq dir(G) \leq n$, where n is the order of G. every $u - v$ detour of G belongs to $I_D[S]$. This gives the following observation.

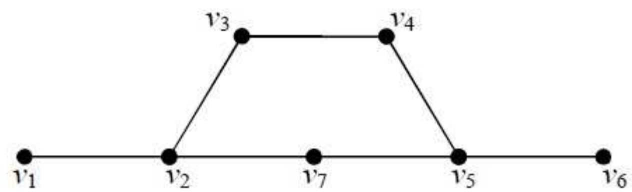

Figure 1: G

To illustrate these concepts, consider the graph G in Figure 1. Let $S = \{v_1, v_6\}$. Then $I_D[S] = V(G) - \{v_7\}$ and $I_D^2[S] = V(G)$. Thus S is a minimum detour hull set of G and so $d_h(G) = 2$. On the otherhand, let $T = \{v_1, v_6, v_7\}$. Then it is clear that $T^0 = \emptyset$ and so T is a d-irredundant set. It can be easily verified that any set with four or more vertices is not a d-irredundant set of G and so $dir(G) = |S| = 3$. Since the irredundant number of a disconnected graph is the sum of the irredundant numbers of its components, we are only concerned with connected graphs. The d-irredundant number of a graph has certain properties that are also possessed by the detour hull number of a graph. In [6], it was shown that if G is a connected graph of order $n \geq 2$ and detour diameter D, then $d_h(G) \leq n - D + 1$. The same result is also true for the d-irredundant number of a graph.

Theorem 3. *Let G be a connected graph of order n and detour diameter D. Then $dir(G) \leq n - D + 1$.*

Proof. Let S be any set of cardinality greater than $n - D + 1$. Let $P : u_0, u_1, \ldots, u_D = v$ be a detour diameteral path in G. Since $|S| > n - D + 1$, it follows that S contains at least three vertices from the diameteral path P, say, u_i, u_j and u_k with $0 \leq i < j < k \leq d$. This implies that $u_j \in I_D[u_i, u_k]$ or $u_i \in I_D[u_j, u_k]$ or $u_k \in I_D[u_i, u_j]$. Hence it follows that $S^0 \neq \emptyset$. Thus $dir(G) \leq n - D + 1$. □

We determine $dir(T)$ for T a tree.

Theorem 4. *For any tree T with k end vertices, $dir(T) = k$.*

Proof. Let S be a dir-set of T. Suppose that the set S contains a cut vertex, say, v of T. Let $C_1, C_2, \ldots, C_l (l \geq 2)$ be the components of $T - v$. It is clear that each component C_i of $T - v$ contains at least one end vertex, say, u_i of T . Since S is an irredundant set of T containing the cut vertex v, without loss of generality, we may assume that $C_1 \cap S \neq \phi$ and $C_i \cap S = \emptyset$ for all $i = 2, 3, \ldots, l$. First, we prove that $l = 2$. Otherwise, if $l \geq 3$, then the set $S' = (S - \{v\}) \cup \{u_2, u_3\}$ is an d-irredundant set in T with $|S'| = dir(G) + 1$. This is a contradiction. Hence $l = 2$. Now, let $S_1 = (S - \{v\}) \cup \{u_2\}$. Then S_1 is an d-irredundant set of cardinality $dir(G)$. Moreover, S_1 excludes the cut vertex v and includes a new end vertex u_2. We can continue this process until the resultant dir-set has no cut vertices. This is possible only when S has k vertices or less. Now, since the set of all end vertices of T is an d-irredundant set, the result follows. □

A *caterpillar* is a tree of order 3 or more, the removal of whose end-vertices produces a path.

Theorem 5. *For any non trivial tree T of order n and diameter D, $dir(T) = n - D + 1$ if and only if T is a caterpillar.*

Proof. Let T be any non trivial tree. Let u, v be two vertices in T such that $D(u, v) = D$; and let $P : u = v_0, v_1, \ldots, v_{D-1}, v_D = v$ be a diameteral

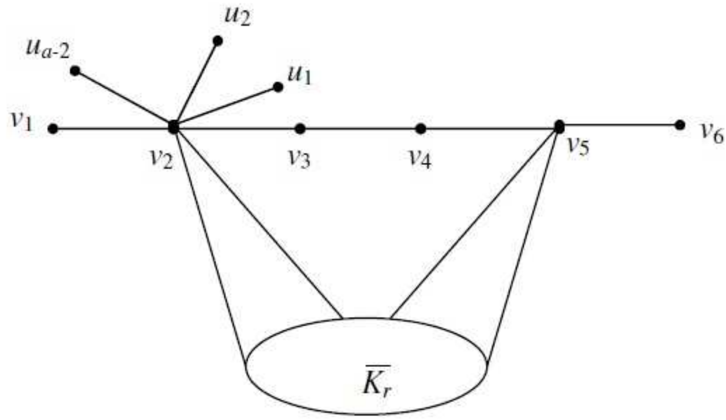

Figure 2: G

path. Let k be the number of end vertices of T and l the number of internal vertices of T other than $v_1, v_2, \ldots, v_{D-1}$. Then $D-1+l+k=n$. By Theorem 4, $dir(T)=k=n-D-l+1$. Hence $dir(T)=n-D+1$ if and only if $l=0$, if and only if all the internal vertices of T lie on the diameteral path P, if and only if T is a caterpillar. □

Theorem 6. *For every pair a, b of integers with $2 \leq a \leq b$, there exists a connected graph G such that $d_h(G)=a$ and $dir(G)=b$.*

Proof. If $a=b=k$, say, then $G=K_{1,k}$ has the desired properties. So, assume that $a<b$ and let $r=b-a$. Now, let H_1 be the graph obtained from the path $P_6 : v_1, v_2, v_3, v_4, v_6, v_6$ by adding $a-2$ new vertices $u_1, u_2, \ldots, u_{a-2}$ and joining each $u_i (1 \leq i \leq a-2)$ to the vertex v_2. Let H_2 be the totally disconnected graph $\overline{K_r}$ on r vertices such that both H_1 and H_2 are vertex disjoint. Now, let G be the graph obained from H_1 and H_2 by joining each vertex of H_2 to both v_2 and v_5. The graph G is shown in Figure 2. Now, let $S=\{v_1, v_6, u_1, u_2, \ldots, u_{b-a}\}$. Then $I_D[S]=V(G)-V(H_1)$ and $I_D^2[S]=V(G)$ and so it follows from Theorem 2 that S is a minimum hull set of G. Thus $h_d(G)=|S|=a$. On the otherhand, consider the set $T=S \cup V(H_2)$. Then $|T|=|S|+|V(H_2)|=a+b-a=b$. Since S is a minimum detour hull set, it follows from Theorem 1 that S is a d-irredundant set in G. Also, note that $I_D[x,y]=\{x, v_2, v_3, v_4, v_5, y\}$ for all $x, y \in V(H_2)$. Thus $V(H_2)^0=\emptyset$. One can also observe that $I_D[x,s] \cap T=\{x,s\}$. This shows that T is a d-irredundant set in G. Hence $dir(G) \geq |T|=a+b-a=b$. Now, it follows from Theorem

3, that $dir(G) \leq n - D + 1 = (a - 2 + 6 + b - a) - 5 + 1 = b$. Thus $dir(G) = b$ and so the result follows. □

Theorem 7. For each triple D, k and n of positive integers with $2 \leq k \leq n - D + 1$ and $D \geq 3$, there is a connected graph of order n, detour diameter D with $d_h(G) = k$ and $dir(G) = n - D + 1$.

Proof. Let G be the graph obtained from the cycle $C_D : u_1, u_2, \ldots, u_D, u_1$ of order D by (1) adding $k - 1$ new vertices $v_1, v_2, \ldots, v_{k-1}$ and joining each vertex $v_i (1 \leq i \leq k - 1)$ to u_1 and (2) adding $n - D - k + 1$ new vertices $w_1, w_2, \ldots, w_{n-D-k+1}$ and joining each vertex $w_i (1 \leq i \leq n - D - k + 1)$ to both u_1 and u_2. The graph G has order n and detour diameter D and is shown in Figure 3.

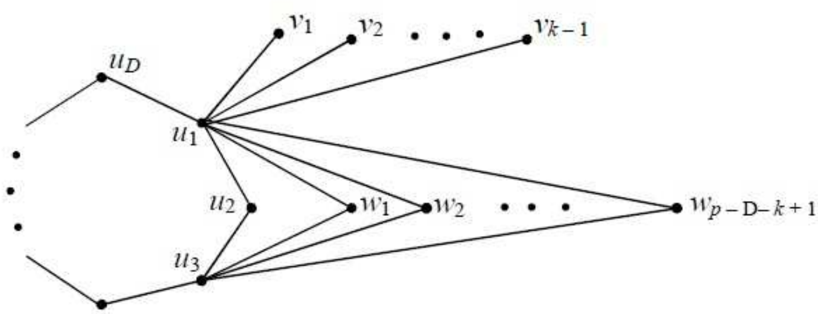

Figure 3: G

Next, we show that $d_h(G) = k$. Let $S = \{v_1, v_2, \ldots, v_{k-1}\}$ be the set of end vertices of G. It is clear that $I_D[S]_G = S \cup \{u_1\}$ and $I_D^2[S]_G = I_D[S]_G$. Thus $[S]_G^D = S \cup \{u_1\} \neq V(G)$ and so S is not a detour hull set of G. Since $I_D[S \cup \{u_D\}]_G = V(G)$, it follows from Theorem 2, that $S \cup \{u_D\}$ is a minimum detour hull set of G so that $d_h(G) = |S| + 1 = k$.Now, let $T = \{v_1, v_2, \ldots, v_{k-1}, w_1, w_2, \ldots, w_{n-D-k+1}, u_2\}$. We claim that T is d-irredundant set in G. For, it is obvious that $I_D[v_i, v_j] = \{v_i, u_1, v_j\}$; Also, since $D \geq 5$, it follows from the figure that $I_D[w_i, w_j] = \{w_i, u_3, u_4, \ldots, u_D, u_1, w_j\}$ and $I_D[v_i, w_j] = \{v_i, u_D, u_{D-1}, \ldots, u_3, w_j\}$. This shows that $T^0 = \emptyset$ and so T is a d-irreudndant set in G. Since $|T| = n - D + 1$, it follows from Theorem 3, that $dir(G) = |T| = n - D + 1$. Hence the Theorem. □

Remark 1. In a connected graph G, cut vertices do not belong to any minimm detour hull set of G. But cut vertices may belong to dir-sets of a graph.

For the graph G in Figure 5, the set $S = \{u_1, u_2\}$ is an *dir*-set containing the cut vertex u_1.

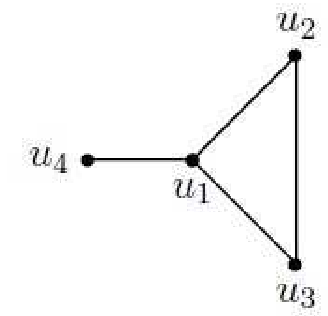

Figure 4: G

Theorem 8. *In a connected graph G, if a cut vertex v belongs to an dir-set in G, then $G - v$ has exactly two components and at least one of them is K_1.*

Proof. First, let S be an *dir*-set of G containing the cut vertex v. Suppose that $G-v$ has three components, say C_1, C_2 and C_3. Since S is an d-irredundant set containing the cut vertex v, it follows that S intersect with at most one of C_1, C_2 and C_3. Assume without loss of generality that $S \cap V(C_2) = \emptyset$ and $S \cap V(C_3) = \emptyset$. Choose vertices x and y in G such that $x \in V(C_2)$ and $y \in V(C_3)$. Then it is obvious that the set $T = (S - \{v\}) \cup \{x, y\}$ is a d-irredundant set in G. This is a contradiction to the maximality of S. Hence $G-v$ has exactly two components, say C_1 and C_2. Now, suppose that $C_1 \neq K_1$ and $C_2 \neq K_1$. Then as above, we have that $S \cap V(C_1) = \emptyset$ or $S \cap V(C_2) = \emptyset$. Since $|S| \geq 2$, we can assume that $S \cap V(C_1) \neq \emptyset$ and $S \cap V(C_2) = \emptyset$. Let x and y be any two distinct vertices in C_2. Then the set $T = (S - \{v\}) \cup \{x, y\}$ is a d-irredundant set in G, which is impossible. Hence either $C_1 = K_1$ or $C_2 = K_1$. $\square$

Chartrand, Escuadro and Zhang [3] proved that the detour radius and detour diameter of a connected graph G satisfy $rad_D(G) \leq diam_D(G) \leq 2\,rad_D(G)$. Also they proved that every pair a, b of positive integers can be realized as the detour radius and detour diameter respectively of some connected graph provided $a \leq b \leq 2a$. We extend this theorem so that the d-irredundant number can be prescribed as well when $a < b \leq 2a$.

Theorem 9. For positive integers a, b and $k \geq 2$ with $a < b \leq 2a$, there exists a connected graph G with $rad_D(G) = a, diam_D(G) = b$ and $dir(G) = k$.

Proof. If $a = 1$ and $b = 2$, then $G = K_{1,k}$ has the desired properties. So, let $a \geq 2$. Let K_a and K_{b-a} be complete graphs of order a and $b - a$ respectively such that both are vertex disjoint. Let H be the graph of order $b - 1$ obtained by identifying a vertex v of K_a and K_{b-a}. Let G be the graph obtained from H by adding k new vertices $u_1, u_2, \ldots, u_k$ and joining each $u_i (1 \leq i \leq k - 1)$ to a vertex $x \neq v$ of K_a; and if $b - a \geq 2$, then join the vertex u_k to a vertex $y \neq v$ of K_{b-a} otherwise join u_k to v.

Note that $D(v, z) = \begin{cases} a - 1 & \text{if } z \in V(K_a) \text{ and } z \neq v \\ a & z = u_i (1 \leq i \leq k - 1) \\ b - a - 1 & \text{if } z \in V(K_{b-a}) \text{ and } z \neq u_k \\ b - a & \text{if } x = u_k \end{cases}$

Since $b \leq 2a$, we have $b - a \leq a$. Thus it follows that $e_D(v) = a$. Similarly, it can easily seen that $e_D(u_i) = b$ for $i = 1, 2, \ldots, k$ and $e_D(x) = b - 1$ for all $x \neq v, u_i (1 \leq i \leq k)$. Hence it follows that $rad_D(G) = a$ and $daim_D(G) = b$. Now, we show that $dir(G) = k$. For, let $w_1, w_2 \in V(H)$, then it follows from the figure that: $I_D[w_1, w_2] = \begin{cases} V(H) & \text{if } w_1 \in V(K_a) \textit{ and } w_2 \in V(K_{b-a}) \\ V(K_a) & \text{if } w_1, w_2 \in V(K_a) \\ V(K_{b-a}) & \text{if } w_1, w_2 \in V(K_{b-a}) \end{cases}$

This shows that any three element subset of $V(H)$ is a d-redundant set of G. Now, let T be any set in G of cardinality $\geq k+1$. This implies that $|T \cap V(H)| \leq 2$. Hence without loss of generality, we may assume that there exists three distinct vertices, say w_1, w_2 and w_3 in T such that $w_1 = u_i$ and; w_2 and w_3 lies on two different components of $G - v$. This shows that either $w_2 \in I_D[u_i, w_3]$ or $w_3 \in I_D[u_i, w_2]$ and hence $T^0 \neq \emptyset$. Now, since $T' = \{u_1, u_2, \ldots, u_k\}$ is a d-irredundant set, we have that $dir(G) = k$. □

The following result follows immediately from Theorem 2.

Theorem 10. *Let G be a connected graph G of order $n \geq 2$. Then $dir(G) = n$ if and only if $G = K_2$.*

Theorem 11. *Let G be connected graph G or order $n \geq 2$.Then $dir(G) = n - 1$ if and only if $G = K_3$ or $G = K_{1,n-1}$.*

Proof. If $G = K_{1,n-1}$, then it follows fom Theorem 4, that $dir(G) = n - 1$. Conversely, let G be any connected graph such that $dir(G) = n - 1$. Then it follows from Theorem 3, that $dim_D(G) \leq n-(n-1)+1=2$. Thus any path in G has length at most 2. Hence to prove the result it is enough to prove that G has no cycles. On the otherhand, suppose that G has a cycle $C: v_1, v_2, \ldots, v_k, v_1$ of length k. Since any path in G has length at most 2, it follows that $k=3$. If $G=K_3$, then nothing to prove. So, let $G \neq K_3$ and hence there exists a vertex say, v_1 in C has degree ≥ 3. This shows that G contains a path of length of 3, which is impossible. Hence we have that either $G=K_3$ or $G=K_{1,n-1}$. □

A double star is a tree with diameter 3.

Theorem 12. *Let G be a connected graph of order $n \geq 5$. Then $dir(G) = n-2$ if and only if $G=K_{1,n-1}+e$ or G is a double star.*

Proof. If G is a double star, then G has exactly two cut vertices and $n-2$ end vertices. Hence from Theorem 4, that $dir(G)=n-2$. Now, if $G = K_{1,n-2}+e$. Then $diam_D(G)=3$ and hence it follows from Theorem 3, that $dir(G) \leq n-3+1=n-2$. Let v_1, v_2, v_3, v_4 be a detour diameteral path in $K_{1,n-1}+e$, then one can observe that $V(G)-\{v_2, v_3\}$ is a detour irredundant set of cardinality $n-2$ in G and so $dir(G)=n-2$. Conversely, let G a connected graph or order $n \geq 5$ and $dir(G)=n-2$. Then by Theorem 3, $dim_D(G) \leq n-(n-2)+1=3$. If $dim_D(G)=1$, then $G=K_2$ and hence $dir(G)=n$. Also, if $dim_D(G)=2$, then $G=K_3$ or $G=K_{1,n-1}$ and so $dir(G)=n-1$. This shows that $dim_D(G)=3$. First, consider the case that G is acyclic. Since $dim_D(G)=3$, we have that any path in G has length at most 3. This shows that G is a double star. Now, assume that G contains cycles. Then it is clear that $cir(G)=3$. Otherwise, if G contains a cycle of length ≥ 4, then $dim_D(G) \geq 4$, which is impossible. Hence $cir(G)=3$. Moreover, since $dim_D(G)=3$, we have that G is unicyclic. Now, let $C: v_1, v_2, v_3, v_1$ be the cycle in G Then exacly one vertex, say v in C has degree ≥ 3. Otherwise, G contains a path of length ≥ 4. This shows that $G=K_{1,n-1}+e$. □

We leave the following problem as open.

Problem: Characterize the classes of graphs G for which $dir(G)=d_h(G)$.

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