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A MODEL FOR THE ADVERSE EFFECT OF TOXICANT ON REPRODUCTIVE HEALTH OF A SUBCLASS OF A BIOLOGICAL SPECIES

A.K. Agrawal
Department of Mathematics
Amity University
Lucknow Campus, U.P., INDIA

Abstract: We have proposed and analysed a non-linear mathematical model to study the effect of a toxicant on a biological species that can have adverse effect on reproductive health of a subclass of the species. Using stability theory, it has been shown that the density of biological species would settle down to its equilibrium level. It is also found that the population density of a subclass of those members of the population which is severely affected by the toxicant and not capable in further reproduction, increases as the emission rate of the toxicant increases.

AMS Subject Classification: 34C60, 34D20, 37N25
Key Words: mathematical model, reproduction failure, stability analysis, toxicant effect

1. Introduction

Many researchers use a well accepted tool, mathematical model, to monitor and understand the growth and survival of biological species in toxic environment such as effect of a single toxicant, simultaneous effect of two toxicants and more than two toxicants (Agrawal et al. 2000; Deluna and Hallam 1987; Dubey et al. 2010; Freedman and Shukla 1991; Hallam and Clark 1982; Hallam and Deluna 1984; Shukla and Agrawal 1999; Shukla et al. 2001; Shukla et al.

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2003; Shukla and Dubey 1996; Shukla et al. 2009). There are different types of situation considered for emission of toxicants in the environment such as toxicant emitted by external sources, biological species itself, other species and so forth. Specially, one phenomenon is common in all of these studies i.e. each individual of the species exhibit similar symptoms after-effect of toxicants. Further, some studies have been carried out by taking an observable fact that a subclass of biological species severely affected by toxicants and shows deformity (Agrawal and Shukla 2012; Agarwal et al. 2016; Kumar et al. 2016). They have shown that as the emission rate of toxicant in the environment increases, total density of species decreases and density of the subclass of species firstly increases than decreases as total density decreases. For highly emission rate, system becomes unstable.

In this paper, we have suggested a new model to predict the growth of a biological species in a toxic environment. A subclass of this biological species suffers from failure in reproduction. It is assumed here that the toxicant is being emitted in the environment by species itself.

2. Mathematical Model

Let us consider a logistically growing biological species affected by a toxicant which is emitted in the environment by the species itself. This toxicant affects a subclass of the biological population acutely showing failure in reproduction capability, etc. Keeping these aspects in view, the mathematical model is assumed which is based on a system of non linear differential equations,

$$\begin{aligned}
\frac{dN}{dt} &= r_0 N - r_1 U N - \frac{r_0 N^2}{K(T)} \\
\frac{dN_d}{dt} &= \rho r_1 U N - m_2 N_d - m_1 N_d \\
\frac{dT}{dt} &= \lambda N - \delta T - \alpha T N + \pi \nu N U \\
\frac{dU}{dt} &= \alpha T N - \beta U - \nu N U
\end{aligned} \tag{1}$$

$$N(0) \geq 0,\ N_d(0) \geq 0,\ T(0) \geq 0,\ U(0) \geq cN(0),\ c > 0,\ 0 \leq \pi \leq 1$$

Here $N(t)$ is the density of the biological species which is affected by the toxicant with environmental concentration $T(t)$, $N_d(t)$ is the density of the subclass of the population which is failure in reproduction after-effect of the toxicant. The toxicant is assumed to be emitted in to the environment by biological species itself with a rate of discharge of toxicant into the environment λN. $U(t)$ is the uptake concentration of toxicant by the species with density $N(t)$.

All the parameters which we use in the model (1) are positive constants. r_0 is the intrinsic growth rate of the population in the environment without pollutant, r_1 is the decreasing rate of the intrinsic growth rate associated with the uptake of the pollutant, ρ is the fraction of the subclass which is severely affected and failure in reproduction, m_1 and m_2 are the natural death rate coefficient and mortality rate coefficient due to high toxicity of the deformed population respectively, λ is the emission rate of toxicant by the biological species itself, δ is the natural depletion rate coefficient of $T(t)$, β is the natural depletion rate coefficient of $U(t)$, α is the rate of uptake of toxicant by the species, i.e. αTN, ν is the depletion rate coefficient of $U(t)$ due to decay of some members of N, i.e. νNU, π is the fraction of the depletion of $U(t)$ due to decay of some members of N which may reenter into the environment, i.e. $\pi\nu NU$. The constant $c \geq 0$ is the proportionality constant determining the measure of initial toxicant concentration in the population at $t = 0$.

In the model (1), the function $K(T)$ denotes the maximum population density which the environment can support and it decreases when T increases.

we can assume

$$K(0) = K_0 > 0, \qquad \frac{dK}{dt} < 0 \qquad \text{for} T > 0 \tag{2}$$

3. Mathematical Analysis

Here, the model (1) has two non-negative equilibrium points, $E_1 = (0,0,0,0)$ and $E_2 = (N^*, N_d^*, T^*, U^*)$. The existence of E_1 is obvious. We shall show the existence of E_2 as follows.

Here N^*, N_d^*, T^* and U^* are the positive solution of the following system of equations:

$$N = \frac{(r_0 - r_1 U)K(T)}{r_0} \tag{3a}$$

$$N_d = \frac{\rho r_1 U N}{m_1 + m_2} \tag{3b}$$

$$T = \frac{\lambda N(\beta + \nu N)}{f(N)} \qquad = g(N) \tag{3c}$$

$$U = \frac{\lambda\alpha N^2}{f(N)} \quad = h(N) \tag{3d}$$

$$\text{where} \quad f(N) = \delta\beta + (\delta\nu + \alpha\beta)N + \alpha\nu(1-\pi)N^2 \tag{3e}$$

we note that T and U increases as λ increases and N_d increases as U (or λ) increases and $K(T)$ is a decreasing function of T therefore $K(T)$ decreases as T (or λ) increases.

Let

$$F(N) = r_0N - (r_0 - r_1h(N))K(g(N)) \tag{3f}$$

we then note from (3f) that

$$F(0) < 0 \qquad \text{and} \qquad F(K_0) > 0$$

This guarantees the existence of a root of $F(N) = 0$ for $0 < N < K_0$ says N^*. Further, this root will be unique, if $F'(N) > 0$.

Here,

$$F'(N) = r_0 - (r_0 - r_1h(N))\frac{dK}{dT}\frac{dg}{dN} + r_1K(g(N))\frac{dh}{dN} \tag{4}$$

we note from eq.(2) that $K'(T) = \dfrac{dK}{dT} < 0$, therefore eq.(4) is satisfied if

$$\frac{dg}{dN} > 0 \qquad \text{and} \qquad \frac{dh}{dN} > 0$$

On computation from (3c) and (3d), we get that

$$\frac{dg}{dN} = \frac{\lambda}{f^2(N)}\left[\delta\beta^2 + 2\delta\beta\nu N + (\delta\nu^2 + \alpha\pi\nu\beta)N^2\right] > 0$$

and

$$\frac{dh}{dN} = \frac{\lambda\alpha N}{f^2(N)}\left[2\delta\beta + (\delta\nu + \alpha\beta)N\right] > 0$$

Hence, the condition (4) is automatically satisfied and the uniqueness of N^* is guaranteed without any condition.

Knowing the value of N^*, the values of N_d^*, T^* and U^* can be computed using equations (3a)-(3e).

3.1. Local Stability Analysis

To study the local stability behavior of the equilibria, we compute the variational matrices corresponding to the equilibria. Let M_1 and M_2 be the variational matrix corresponding to the equilibria E_1 and E_2, then we have,

$$M_1 = \begin{bmatrix} r_0 & 0 & 0 & 0 \\ 0 & -(m_1+m_2) & 0 & 0 \\ \lambda & 0 & -\delta & 0 \\ 0 & 0 & 0 & -\beta \end{bmatrix}$$

From M_1 we note that E_1 is a saddle point with unstable locally in the $N-$direction and with stable manifold locally in the $N_d - T - U$ space.

$$M_2 = \begin{bmatrix} -\dfrac{r_0N^*}{K(T^*)} & 0 & \dfrac{r_0N^{*2}}{K^2(T^*)}K'(T^*) & -r_1N^* \\ \rho r_1 U^* & -(m_1+m_2) & 0 & \rho r_1 N^* \\ \lambda - (\alpha T^* - \pi\nu U^*) & 0 & -(\delta+\alpha N^*) & \pi\nu N^* \\ (\alpha T^* - \nu U^*) & 0 & \alpha N^* & -(\beta+\nu N^*) \end{bmatrix}$$

The characteristic equation corresponding to M_2 can be written as follows:

$$p(x) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$$

where

$$a_1 = \frac{r_0N^*}{K(T^*)} + (\delta+\alpha N^*) + (\beta+\nu N^*) + (m_1+m_2)$$

$$\begin{aligned} a_2 = {} & \frac{r_0N^*}{K(T^*)}\{(\delta+\alpha N^*) + (\beta+\nu N^*) + (m_1+m_2)\} \\ & + (m_1+m_2)\{(\delta+\alpha N^*) + (\beta+\nu N^*)\} \\ & + \left\{\delta\beta + (\delta\nu+\alpha\beta)N^* + \alpha\nu(1-\pi)N^{*2}\right\} \\ & + \frac{r_0N^{*2}}{K^2(T^*)}K'(T^*)\{(\alpha T^* - \pi\nu U^*) - \lambda\} + r_1N^*(\alpha T^* - \nu U^*) \end{aligned}$$

$$\begin{aligned} a_3 = {} & \frac{r_0N^*}{K(T^*)}\left[(m_1+m_2)\{(\delta+\alpha N^*) + (\beta+\nu N^*)\}\right. \\ & \qquad \left. + \left\{\delta\beta + (\delta\nu+\alpha\beta)N^* + \alpha\nu(1-\pi)N^{*2}\right\}\right] \\ & + (m_1+m_2)\left\{\delta\beta + (\delta\nu+\alpha\beta)N^* + \alpha\nu(1-\pi)N^{*2}\right\} \\ & + \frac{r_0N^{*2}}{K^2(T^*)}K'(T^*)\left[\{(m_1+m_2) + (\beta+\nu N^*)\}\{(\alpha T^* - \pi\nu U^*) - \lambda\}\right. \\ & \qquad \left. -(\alpha T^* - \nu U^*)\pi\nu N^*\right] \\ & + r_1N^*\left[\{(m_1+m_2) + (\delta+\alpha N^*)\}(\alpha T^* - \nu U^*)\right. \\ & \qquad \left. - \{(\alpha T^* - \pi\nu U^*) - \lambda\}\alpha N^*\right] \end{aligned}$$

$$
\begin{aligned}
a_4 = & \frac{r_0 N^*}{K(T^*)}(m_1+m_2)\left\{\delta\beta + (\delta\nu+\alpha\beta)N^* + \alpha\nu(1-\pi)N^{*2}\right\} \\
& + \frac{r_0 N^{*2}}{K^2(T^*)}K'(T^*)(m_1+m_2) \\
& \quad \{\{(\alpha T^* - \pi\nu U^*) - \lambda\}(\beta+\nu N^*) - (\alpha T^* - \nu U^*)\pi\nu N^*\} \\
& + r_1 N^*(m_1+m_2)\{(\delta+\alpha N^*)(\alpha T^* - \nu U^*) - \{(\alpha T^* - \pi\nu U^*) - \lambda\}\alpha N^*\}
\end{aligned}
$$

Now, According to the Routh-Hurwitz Criteria, all the roots of the polynomial $p(x)$ are negative or negative real parts iff

$$a_i > 0 \qquad (i = 1, 2, 3, 4) \tag{5a}$$

$$a_1 a_2 > a_3 \tag{5b}$$

$$a_1 a_2 a_3 > a_3^2 + a_1^2 a_4 \tag{5c}$$

Thus, we are now able to state the following result.

Theorem 1. *The positive equilibrium point E_2 is locally asymptotically stable under conditions (5a)-(5c).*

Thus the solutions of the model (1) will be oscillatory (locally) and settling down to E_2 provided the above conditions (5a)-(5c) hold.

3.2. Global Stability Analysis

In the following theorem we show that E_2 is globally asymptotically stable under certain condition. To prove this theorem we give the following lemma which establishes the region of attraction for E_2.

Lemma 1. *The region*

$$\Omega = \{(N, N_d, T, U) :\ 0 \le N \le K_0,\ 0 \le N_d \le \frac{\rho r_1 \lambda K_0^2}{(m_1+m_2)\delta_m}, \quad 0 \le T(t) + U(t) \le \frac{\lambda K_0}{\delta_m}\}$$

attracts all solutions initiating in the interior of the positive orthant, where $\delta_m = min(\delta, \beta)$.

Proof: From the first equation of model (1), we have

$$\frac{dN}{dt} \le r_0 N - \frac{r_0 N^2}{K_0} = r_0\left(1 - \frac{N}{K_0}\right)N$$

Thus, $\limsup_{t\to\infty} N(t) \le K_0$

From the second equation of model (1), we have

$$\frac{dN_d}{dt} \le \frac{\rho r_1 \lambda K_0^2}{\delta_m} - (m_1 + m_2)N_d$$

Thus, $\limsup_{t\to\infty} N_d(t) \le \dfrac{\rho r_1 \lambda K_0^2}{(m_1+m_2)\delta_m}$

Also, adding the last two equations of model (1), we get

$$\frac{dT}{dt} + \frac{dU}{dt} = \lambda N - \delta T - \beta U - (1-\pi)\nu N U \le \lambda K_0 - \delta_m (T+U)$$

Thus, $\limsup_{t\to\infty}[T(t) + U(t)] = \dfrac{\lambda K_0}{\delta_m}$

proving the lemma.

The following theorem characterizes the global stability behavior of the equilibrium point E_2.

Theorem 2. *In addition to the assumption (2), let $K(T)$ satisfy inequalities hold in Ω:*

$$K_m \le K(T) \le K_0, \qquad |K'(T)| \le \kappa$$

where K_m and κ are positive constants. Then if the following inequalities hold in Ω:

$$[\rho r_1 U^*]^2 < \frac{2}{3}\frac{r_0}{K(T^*)}(m_1+m_2) \tag{6a}$$

$$\left[\lambda - \frac{r_0 K_0}{K_m^2}\kappa - (\alpha + \pi\nu)\frac{\lambda K_0}{\delta_m}\right]^2 < \frac{2}{3}\frac{r_0}{K(T^*)}(\delta + \alpha N^*) \tag{6b}$$

$$\left[(\alpha+\nu)\frac{\lambda K_0}{\delta_m} - r_1\right]^2 < \frac{4}{9}\frac{r_0}{K(T^*)}(\beta + \nu N^*) \tag{6c}$$

$$[\rho r_1 K_0]^2 < \frac{2}{3}(m_1+m_2)(\beta + \nu N^*) \tag{6d}$$

$$[(\alpha+\pi\nu)N^*]^2 < \frac{2}{3}(\delta + \alpha N^*)(\beta+\nu N^*) \tag{6e}$$

then E_2 is globally stable with respect to all solutions initiating in the interior of the positive orthant.

Proof: We consider the following positive definite function about E_2.

$$W(N,N_d,T,U)=\left\{N-N^*-N^*log\left(\frac{N}{N^*}\right)\right\}$$
$$+\frac{1}{2}(N_d-N_d^*)^2+\frac{1}{2}(T-T^*)^2+\frac{1}{2}(U-U^*)^2$$

Differentiating W with respect to t along the solution of (1), we get

$$\frac{dW}{dt}=(N-N^*)\left[r_0-r_1U-\frac{r_0N}{K(T)}\right]+(N_d-N_d^*)\left[\rho r_1UN-m_1N_d-m_2N_d\right]$$
$$+(T-T^*)\left[\lambda N-\delta T-\alpha TN+\pi\nu NU\right]$$
$$+(U-U^*)\left[-\beta U+\alpha TN-\nu NU\right]$$

using (3a)-(3e), we get after a little algebraic manipulations

$$\frac{dW}{dt}=-\frac{r_0}{K(T^*)}(N-N^*)^2-(m_1+m_2)(N_d-N_d^*)^2-(\delta+\alpha N^*)(T-T^*)^2$$
$$-(\beta+\nu N^*)(U-U^*)^2+(\rho r_1U^*)(N-N^*)(N_d-N_d^*)$$
$$+\{-r_0N\eta(T)+\lambda-(\alpha T-\pi\nu U)\}(N-N^*)(T-T^*)$$
$$+\{(\alpha T-\nu U)-r_1\}(N-N^*)(U-U^*)+(\rho r_1N)(N_d-N_d^*)(U-U^*)$$
$$+\{(\alpha+\pi\nu)N^*\}(T-T^*)(U-U^*)$$

where

$$\eta(T)=\begin{cases}\dfrac{\dfrac{1}{K(T)}-\dfrac{1}{K(T^*)}}{T-T^*}, & T\neq T^*\\[2ex] -\dfrac{K'(T^*)}{K^2(T^*)}, & T=T^*\end{cases}$$

Thus, $\frac{dW}{dt}$ can be written as sum of the quadratics

$$\frac{dW}{dt}=-\frac{1}{2}b_{11}(N-N^*)^2+b_{12}(N-N^*)(N_d-N_d^*)-\frac{1}{2}b_{22}(N_d-N_d^*)^2$$
$$-\frac{1}{2}b_{11}(N-N^*)^2+b_{13}(N-N^*)(T-T^*)-\frac{1}{2}b_{33}(T-T^*)^2$$
$$-\frac{1}{2}b_{11}(N-N^*)^2+b_{14}(N-N^*)(U-U^*)-\frac{1}{2}b_{44}(U-U^*)^2$$
$$-\frac{1}{2}b_{22}(N_d-N_d^*)^2+b_{24}(N_d-N_d^*)(U-U^*)-\frac{1}{2}b_{44}(U-U^*)^2$$
$$-\frac{1}{2}b_{33}(T-T^*)^2+b_{34}(T-T^*)(U-U^*)-\frac{1}{2}b_{44}(U-U^*)^2$$

where

$$b_{11} = \frac{2}{3}\frac{r_0}{K(T^*)}, \quad b_{22} = (m_1 + m_2), \quad b_{33} = (\delta + \alpha N^*), \quad b_{44} = \frac{2}{3}(\beta + \nu N^*),$$
$$b_{12} = \rho r_1 U^*, \quad b_{13} = [\lambda - (\alpha T - \pi\nu U) - r_0 N \eta(T)],$$
$$b_{14} = \{(\alpha T - \nu U) - r_1\}, \quad b_{24} = \rho r_1 N, \quad b_{34} = (\alpha + \pi\nu)N^*$$

Thus, $\frac{dW}{dt}$ will be negative definite provided

$$b_{12}^2 < b_{11}b_{22} \tag{7a}$$
$$b_{13}^2 < b_{11}b_{33} \tag{7b}$$
$$b_{14}^2 < b_{11}b_{44} \tag{7c}$$
$$b_{24}^2 < b_{22}b_{44} \tag{7d}$$
$$b_{34}^2 < b_{33}b_{44} \tag{7e}$$

we note that $(6a) \Rightarrow (7a)$, $(6b) \Rightarrow (7b)$, $(6c) \Rightarrow (7c)$, $(6d) \Rightarrow (7d)$, and $(6e) \Rightarrow (7e)$. Hence W is Lyapunov's function with respect to E_2 whose domain contains the region Ω and therefore E_2 is globally asymptotically stable. Hence the theorem.

4. Numerical Simulation

To give the better clarification of our analytical results, we present here numerical simulation of the mathematical model (1) with assuming

$$K(T) = K_0 - \frac{b_1 T}{1 + b_2 T} \tag{8}$$

and a set of parameters:

$$\begin{aligned} &r_0 = 0.0953,\ r_1 = 0.10,\ K_0 = 10,\ b_1 = 0.5,\ b_2 = 5, \\ &\rho = 0.5,\ m_1 = 0.02,\ m_2 = 0.03,\ \lambda = 0.5,\ \delta = 20, \\ &\alpha = 0.5,\ \nu = 0.8,\ \pi = 0.05,\ \beta = 18. \end{aligned} \tag{9}$$

Now with this choice of b_1 and b_2, we have $\frac{b_1 T}{1 + b_2 T} < 1$. Since $K_m \leq K(T) \leq K_0$, therefore we can choose K_m as $K_m = 5.0$.

We also note from equation (8) that $K'(T) = -\frac{b_1}{(1 + b_2 T)^2}$, therefore we can choose $\kappa = 1$

Now integrated model (1) using Runge-Kutta Method with the above parameter values, we found that the positive equilibrium E_2 exist and obtained as

$$N^* = 9.5735, \qquad N_d^* = 0.3459, \qquad T^* = 0.1937, \qquad U^* = 0.0361$$

Here, we note that for the choices mentioned above, locally stability conditions (5a)-(5c) and globally stability conditions (6a)-(6e) are satisfied. It is further noted that the density N of the species tends to settle to the carrying capacity with the time and would have settled to the original carrying capacity $K_0 = 10$. But in present case, it decreased to $N^* = 9.5735$ and a fraction of it has reproduction failure attaining its equilibrium $N_d^* = 0.3459$.

In Figure 1, we have shown the growth of biological species N and their subclass failure in reproduction N_d against time t.

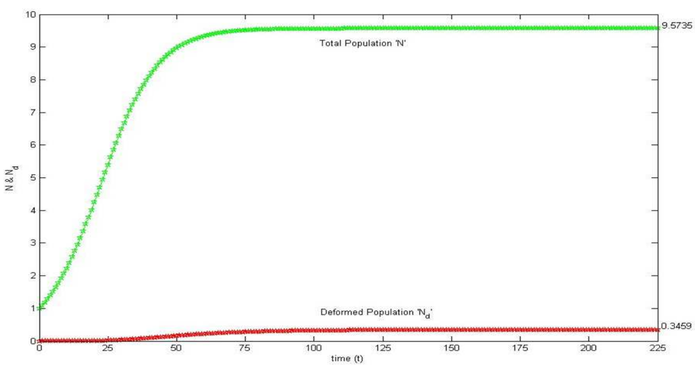

Figure 1: Growth of N and N_d with respect to time t.

In Figure 2, we have shown the variation of subclass which fails in reproduction of biological species N_d for different values of fraction of the subclass biological species which is severely affected ρ. This figure shows that if we increase the fraction of the subclass of the biological species which is highly effected and fails in further reproduction, increases.

In Figure 3, we have shown the variation of reproduction failure subclass N_d corresponding to the emission rate of the toxicant by the biological species into environment λ. In this figure, it can be seen that when we increase emission rate of the toxicant by the biological species into environment the density of reproduction failure subclass also increases, it is also seen that for a large

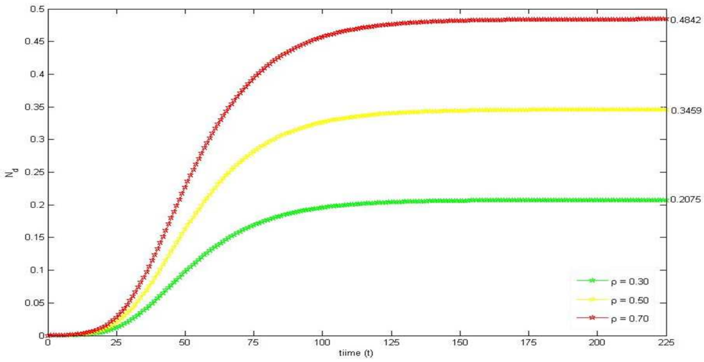

Figure 2: Growth of N_d for different values of ρ.

emission rate of the toxicant by the biological species, a large amount of the biological species gets severely affected and the globally stability conditions also fails, therefore we need to control the emission of toxicant from industrial plants, fuel combustion in motors vehicles, homes, etc.

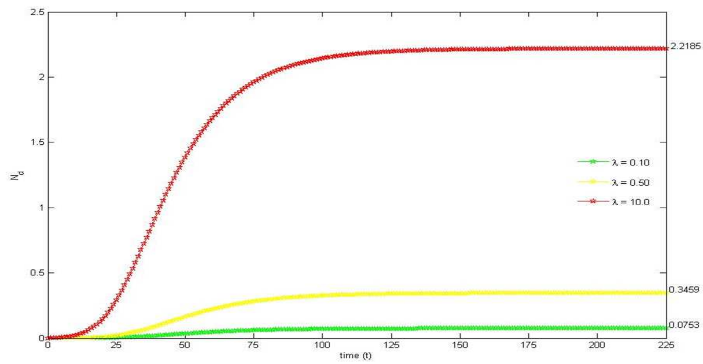

Figure 3: Growth of N_d for different values of λ.

In Fig. 4, we have shown the variation of reproduction failure subclass N_d for different values of the mortality rate coefficient of the severely affected population due to high toxicity, m_2. This figure shows that if we increase the mortality rate coefficient of the reproduction failure subclass due to high toxicity then the density of reproduction failure subclass decreases.

In Figure 5, we have shown the variation of reproduction failure subclass N_d

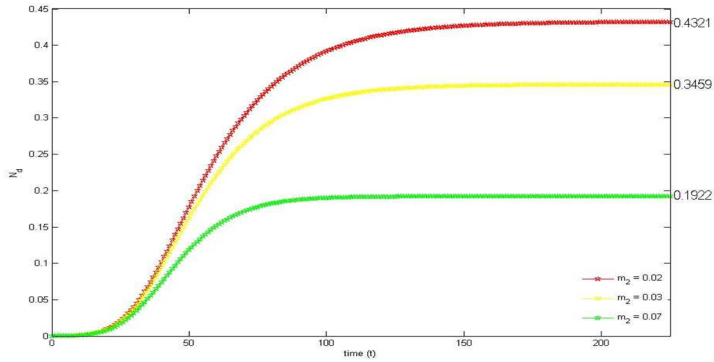

Figure 4: Growth of N_d for different values of m_2.

for different values of the decreasing rate of the intrinsic growth rate associated with the uptake of the pollutant, r_1. This figure shows that if we increase the decreasing rate of the intrinsic growth rate associated with the uptake of the pollutant, r_1 then the density of reproduction failure subclass increases.

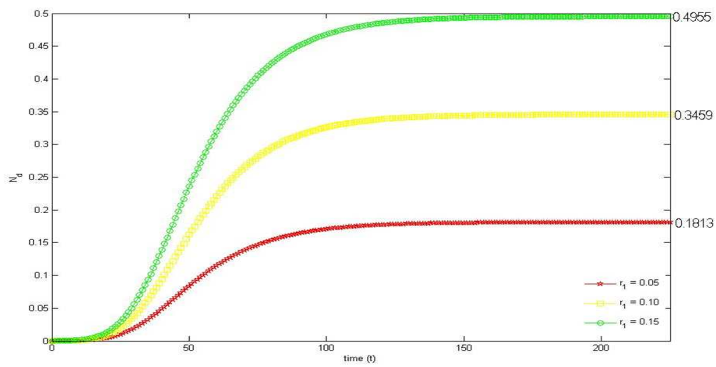

Figure 5: Growth of N_d for different values of r_1.

5. Conclusions

In this paper, we have proposed an analyzed a non-linear mathematical model for the effect of a toxicant on a biological species, some members of the species highly affected by the toxicant and lose their capability of reproduction, when toxicant is emitted in the environment by the biological species itself.

It has been shown that, the system has two equilibrium points namely $(0,0,0,0)$ and (N^*, N_d^*, T^*, U^*). The first equilibrium point i.e. $(0,0,0,0)$ is a saddle point. The second equilibrium point i.e. (N^*, N_d^*, T^*, U^*) is stable under certain conditions (5a)-(5c) and (6a)-(6e), which means that the biological population would settle down to its equilibrium level. It is also found that a subclass of biological species which is highly affected by the toxicant settles down to its equilibrium level but the number of members increases as the emission rate of the toxicant (i.e. λ) or rate of toxicant uptaken by the species increases (i.e. α). The analysis of model implies that by keeping the uptake rate of toxicant at low level, the density of reproduction failure subclass can be lowered down and it also suggests the need of a regulatory agency to control the emission of toxicant from industries and other manmade project.

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