COMPUTATION OF TOPOLOGICAL INDICES
OF WINDMILL GRAPH

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Abstract: In this paper, we compute $ABC$ index, $ABC_4$ index, Sum connectivity index,
Randic connectivity index, $GA$ index and $GA_5$ index of windmill graph.

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Key Words: $ABC$ index, $ABC_4$ index, Sum connectivity index, Randic connectivity index,
$GA$ index and $GA_5$ index

1. Introduction

The Windmill graph is the graph obtained by taking $n$ copies of the complete
A topological index $\text{Top}(G)$ of a graph $G$, is a number with the property that for every graph $H$ isomorphic to $G$, $\text{Top}(H) = \text{Top}(G)$. So, a topological index is a real number derived from the structure of a graph, which is invariant under graph isomorphism. Topological indices are numerical parameters of a graph which characterize its topology and are usually graph invariants. Also topological indices are the molecular descriptors that describes the structures of chemical compounds and it help us to predict certain physico-chemical properties like boiling point, enthalpy of vaporization, stability etc.

All graphs considered in this paper are finite, connected, loop less and without multiple edges. Let $G = (V, E)$ be a graph with $n$ vertices and $m$ edges. The degree of a vertex $u \in V(G)$ is denoted by $d_u$ and is the number of vertices that are adjacent to $u$. The edge connecting the vertices $u$ and $v$ is denoted by $uv$.

Using these terminologies, certain topological indices are defined in the following manner.

The atom-bond connectivity index, $ABC$ index is one of the degree based molecular descriptor, which was introduced by Estrada et al. [5] in late 1990’s. Some upper bounds for the atom-bond connectivity index of graphs can be found in [2], The atom-bond connectivity index of chemical bicyclic graphs, connected graphs can be seen in [3, 15]. For further results on $ABC$ index of trees see the papers [8, 10, 14, 16] and the references cited there in.
Definition 1.1. Let $G = (V, E)$ be a molecular graph and $d_u$ is the degree of the vertex $u$, then $ABC$ index of $G$ is defined as,

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_ud_v}}.$$ 

The fourth atom bond connectivity index, $ABC_4(G)$ index was introduced by M. Ghorbani et al. [9] in 2010. Further studies on $ABC_4(G)$ index can be found in [6, 7].

Definition 1.2. Let $G$ be a graph, then its fourth $ABC$ index is defined as, $ABC_4(G) = \sum_{uv \in E(G)} \sqrt{\frac{S_u + S_v - 2}{S_uS_v}}$, where $S_u$ is sum of the degrees of all neighbors of vertex $u$ in $G$. In other words, $S_u = \sum_{uv \in E(G)} d_v$. Similarly for $S_v$.

The first and oldest degree based topological index is Randic index [12] denoted by $\chi(G)$ and was introduced by Milan Randic in 1975. It provides a quantitative assessment of branching of molecules.

Definition 1.3. For the graph $G$ Randic index is defined as,

$$\chi(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}}.$$ 

Sum connectivity index belongs to a family of Randic like indices and it was introduced by Zhou and Trinajstic [18]. Further studies on Sum connectivity index can be found in [19, 20].

Definition 1.4. For a simple connected graph $G$, its sum connectivity index $S(G)$ is defined as, $S(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u + d_v}}$.

The Geometric-arithmetic index, $GA(G)$ index of a graph $G$ was introduced by D. Vukicevic et.al[13]. Further studies on $GA$ index can be found in [1, 4, 17]

Definition 1.5. Let $G$ be a graph and $e = uv$ be an edge of $G$ then,

$$GA(G) = \sum_{e=uv \in E(G)} \frac{2\sqrt{d_u d_v}}{d_ud_v}.$$ 

The fifth Geometric-arithmetic index, $GA_5(G)$ was introduced by A.Graovac et al [11] in 2011.
Definition 1.6. For a Graph $G$, the fifth Geometric-arithmetic index is defined as $GA_5(G) = \sum_{uv \in E(G)} \frac{2\sqrt{S_u S_v}}{S_u + S_v}$, Where $S_u$ is the sum of the degrees of all neighbors of the vertex $u$ in $G$, similarly $S_v$.

2. Main results and Discussion

Theorem 2.1. The Atom bond connectivity index of Windmill graph is

$$ABC(W_m^{(n)}) = \frac{n(m-2)^{\frac{3}{2}}}{\sqrt{2}} + \sqrt{n[(m-1)(n+1) - 2]}.$$ 

Proof. Consider the Windmill graph $W_m^{(n)}$. We partition the edges of $W_m^{(n)}$ into edges of the type $E_{(d_u,d_v)}$ where $uv$ is an edge and $d_u$ is the degree of the vertex $u$. In $W_m^{(n)}$, we get edges of the type $E_{(m-1,m-1)}$ and $E_{(m-1,(m-1)n]}$. Edges of the type $E_{(m-1,m-1)}$ and $E_{(m-1,(m-1)n]}$ are colored in black and red respectively as shown in the Figure [4]. The number of edges of these types are given in the table 1.

![Figure 4](windmill_graph.png)

Table 1: Edge partition based on degrees of end vertices of each edge

<table>
<thead>
<tr>
<th>Edges of the type $E_{(d_u,d_v)}$</th>
<th>Number of edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{(m-1,m-1)}$</td>
<td>$(m-1)(m-2)n$</td>
</tr>
<tr>
<td>$E_{(m-1,(m-1)n]}$</td>
<td>$(m-1)n$</td>
</tr>
</tbody>
</table>
We know that $ABC(G) = \sum_{uv \in E(G)} \frac{\sqrt{d_u + d_v - 2}}{d_ud_v}$

i.e., $ABC(W_m^{(n)}) = |E_{(m-1),(m-1)}| \sum_{uv \in E_{(m-1),(m-1)}(G)} \frac{\sqrt{d_u + d_v - 2}}{d_ud_v} + |E_{(m-1),(m-1)n}| \sum_{uv \in E_{(m-1),(m-1)n}(G)} \frac{\sqrt{d_u + d_v - 2}}{d_ud_v}$

$$= \frac{(m-1)(m-2)n}{2} \sqrt{\frac{m-1 + m-1 - 2}{(m-1)(m-1)}} + (m-1)n \sqrt{\frac{(m-1) + (m-1)n - 2}{(m-1)(m-1)n}}$$

[ From table(1) and figure(4) ]

$$= \frac{(m-1)(m-2)n}{2} \sqrt{\frac{2m-4}{(m-1)^2}} + (m-1)n \sqrt{\frac{(m-1) + (m-1)n - 2}{(m-1)(m-1)n}}$$

$\therefore ABC(W_m^{(n)}) = \frac{n(m-2)^\frac{3}{2}}{\sqrt{2}} + \sqrt{n[(m-1)(n+1) - 2]}.$

**Theorem 2.2.** The Randic Index of Windmill graph is $\chi(W_m^{(n)}) = \frac{(m-2)n + 2\sqrt{n}}{2}$.

**Proof.** We know that $\chi(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_ud_v}}$

i.e., $\chi(W_m^{(n)}) = |E_{(m-1),(m-1)}| \sum_{uv \in E_{(m-1),(m-1)}(G)} \frac{1}{\sqrt{d_ud_v}} + |E_{(m-1),(m-1)n}| \sum_{uv \in E_{(m-1),(m-1)n}(G)} \frac{1}{\sqrt{d_ud_v}}$

$$= \frac{(m-1)(m-2)n}{2} \cdot \frac{1}{\sqrt{(m-1)(m-1)}} + (m-1)n \frac{1}{\sqrt{(m-1)(m-1)n}}$$
[ From table(1) and figure(4) ]

\[ \chi(W_{n}^{(m)}) = \frac{(m - 2)n + 2\sqrt{n}}{2}. \]

**Theorem 2.3.** The Geometric-arithmetic index of Windmill graph is

\[ GA(W_{m}^{(n)}) = \frac{(m - 1)(m - 2)n}{2} + \frac{2(m - 1)n^{\frac{3}{2}}}{(n + 1)}. \]

Proof. 

\[ GA(G) = \sum_{uv \in E(G)} \frac{2\sqrt{d_{u}d_{v}}}{d_{u} + d_{v}} \]

\[ GA(W_{m}^{(n)}) = |E_{(m-1),(m-1)}| \sum_{uv \in E_{(m-1),(m-1)}(G)} \frac{2\sqrt{d_{u}d_{v}}}{d_{u} + d_{v}} + \]

\[ |E_{(m-1),(m-1)n}| \sum_{uv \in E_{(m-1),(m-1)n}(G)} \frac{2\sqrt{d_{u}d_{v}}}{d_{u} + d_{v}} \]

[ From table(1) and figure(4) ]

\[ = \frac{(m - 1)(m - 2)n}{2} \left[ \frac{2\sqrt{(m - 1)(m - 1)}}{m - 1 + (m - 1)} \right] + (m - 1)n \left[ \frac{2\sqrt{(m - 1)(m - 1)n}}{m - 1 + (m - 1)n} \right] \]

\[ = \frac{(m - 1)(m - 2)n}{2(m - 1)} + \frac{2(m - 1)n(m - 1)\sqrt{n}}{(m - 1)(n + 1)} \]

\[ \therefore GA(W_{m}^{(n)}) = \frac{(m - 1)(m - 2)n}{2} + \frac{2(m - 1)n^{\frac{3}{2}}}{(n + 1)}. \]}

**Theorem 2.4.** The sum connectivity index of Windmill graph is

\[ S(W_{m}^{(n)}) = n(\sqrt{m - 1}) \left[ \frac{(m - 2)}{2^{\frac{3}{2}}} + \frac{1}{\sqrt{n + 1}} \right]. \]

Proof. 

\[ S(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_{u} + d_{v}}} \]

\[ S(W_{m}^{(n)}) = |E_{(m-1),(m-1)}| \sum_{uv \in E_{(m-1),(m-1)}(G)} \frac{1}{\sqrt{d_{u} + d_{v}}} + \]

\[ |E_{(m-1),(m-1)n}| \sum_{uv \in E_{(m-1),(m-1)n}(G)} \frac{1}{\sqrt{d_{u} + d_{v}}} \]

[ From table(1) and figure(4) ]
\[
\frac{(m - 1)(m - 2)n}{2\sqrt{2m - 2}} + \frac{(m - 1)n}{\sqrt{mn - n + m - 1}}
= n(\sqrt{m - 1}) \left[ \frac{(m - 2)}{2^\frac{1}{3}} + \frac{1}{\sqrt{n + 1}} \right].
\]

**Theorem 2.5.** The fourth atom bond connectivity index of Windmill graph is
\[
ABC_4(D_m^{(n)}) = \frac{n(m - 2)\sqrt{m^2 + mn - 3m - n + 1}}{(m + n - 2)\sqrt{2}} + \frac{n[(m - 1)(mn + m - 2) - 2]}{(m - 1)(m + n - 2)}.
\]

**Proof.** We partition the edges of \( W_m^{(n)} \) into edges of the type \( E^*_{(S_u, S_v)} \) where \( uv \) is an edge and \( S_u \) is the sum of the degrees of all neighbors of vertex \( u \) in \( G \). In other words, \( S_u = \sum_{uv \in E(G)} d_u \). Similarly for \( S_v \). If \( n \geq 2 \), in \( W_m^{(n)} \) we get edges of the type \( E^*_{(m - 1)n + (m - 1)(m - 2), (m - 1)n + (m - 1)(m - 2)} \) and \( E^*_{(m - 1)n + (m - 1)(m - 2), (m - 1)^2 n} \). Edges of the type
\[
E^*_{(m - 1)n + (m - 1)(m - 2), (m - 1)n + (m - 1)(m - 2)}
\]
and
\[
E^*_{(m - 1)n + (m - 1)(m - 2), (m - 1)^2 n}
\]
are colored in black and red respectively as shown in the Figure [5]. The number of edges of these types are given in the table 2.
Table 2

<table>
<thead>
<tr>
<th>Edges of the type</th>
<th>Number of edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E^*_{(m-1)n+(m-1)(m-2),(m-1)n+(m-1)(m-2)}$</td>
<td>$\frac{(m-1)(m-2)n}{2}$</td>
</tr>
<tr>
<td>$E^*_{(m-1)n+(m-1)(m-2),(m-1)^2n}$</td>
<td>$(m-1)n$</td>
</tr>
</tbody>
</table>

We know that $ABC_4(G) = \sum_{uv \in E(G)} \sqrt{\frac{S_u + S_v - 2}{S_u S_v}}$

i.e., $ABC_4(W_m^n) = |E^*_{(m-1)n+(m-1)(m-2),(m-1)n+(m-1)(m-2)}|$

$$\sum_{uv \in E^*_{(m-1)n+(m-1)(m-2),(m-1)n+(m-1)(m-2)}(G)} \sqrt{\frac{S_u + S_v - 2}{S_u S_v}}$$

$$+ |E^*_{(m-1)n+(m-1)(m-2),(m-1)^2n}| \sum_{uv \in E^*_{(m-1)n+(m-1)(m-2),(m-1)^2n}(G)} \sqrt{\frac{S_u + S_v - 2}{S_u S_v}}$$

[ From table(2) and figure(5) ]

$$= \frac{(m-1)(m-2)n}{2} \sqrt{\frac{2[(m-1)n + (m-1)(m-2)] - 2}{[(m-1)n + (m-1)(m-2)]^2}}$$

$$+ (m-1)n \sqrt{\frac{(m-1)n + (m-1)(m-2) + (m-1)^2n - 2}{[(m-1)n + (m-1)(m-2)][(m-1)^2n]}}$$

$$\therefore ABC_4(W_m^n) = \frac{n(m-2)\sqrt{m^2 + mn - 3m - n + 1}}{(m+n-2)\sqrt{2}} + \frac{\sqrt{n[(m-1)(mn + m - 2) - 2]}}{(m-1)(m+n-2)}.$$

Theorem 2.6. The fifth Geometric-arithmetic index of Windmill graph is $GA_5(W_m^n) = \frac{m^2n - 3mn + 2n}{2} + \sqrt{[(m-1)n]^3(m+n-2)}$.

Proof. We know that $GA_5(G) = \sum_{uv \in E(G)} \frac{2\sqrt{S_u S_v}}{(S_u + S_v)}$

$GA_5(W_m^n) = |E^*_{(m-1)n+(m-1)(m-2),(m-1)n+(m-1)(m-2)}|$

$$\sum_{uv \in E^*_{(m-1)n+(m-1)(m-2),(m-1)n+(m-1)(m-2)}(G)} \frac{2\sqrt{S_u S_v}}{(S_u + S_v)}.$$


\[
\left| E^*_n \right| \sum_{uv \in E^*_n} \frac{2\sqrt{S_u S_v}}{(S_u + S_v)} 
\]

\[
= \frac{(m - 1)(m - 2)n}{2} \left[ 2\sqrt{\left[ (m - 1)n + (m - 1)(m - 2) \right]^2} \right] + \frac{(m - 1)n}{2} \left[ 2\sqrt{\left[ (m - 1)n + (m - 1)(m - 2) \right]\left[ (m - 1)^2 n \right]} \right] 
\]

\[
= \frac{m^2 n - 3mn + 2n}{2} + \frac{2n(m - 1)\sqrt{n(m - 1)(m + n - 2)}}{mn + m - 2} 
\]

**Conclusion:**
The general formula for \( ABC \) index, \( ABC_4 \) index, Randic connectivity index, Sum connectivity index, \( GA \) index and \( GA_5 \) index of Windmill graph is computed here analytically.

**References**


