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QUALITATIVE ANALYSIS AND HOMOTOPY BASED SOLUTION OF TWO SPECIES LOTKA-VOLTERRA MODEL

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Abstract: This paper presents a qualitative analysis and homotopy based solution of predator-prey Lotka-Volterra model. We established a mathematical model that shows the dynamics of a multi-species predator-prey interactions. Qualitative analysis and some qualitative information about the solution of the model have been carried out. The Homotopy Analysis Method (HAM) has been used to solve the model and the results have been compared with other numerical solution and are found to be in good agreement. Finally, various simulations are done to discuss the solution.

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1. Introduction

Mathematical models can be applied to understand the interactions between species and their environments in population ecology. With the use of mathematical models, dynamic processes such as predator-prey interactions, competition interactions, mutualism or symbiotic interactions, renewable resource

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management, ecological control of a pest, and harvesting etc can be investigated. The Lotka-Volterra equations model the dynamical behaviour of an arbitrary number of competitors [11]. The accurate solution of the Lotka-Volterra equations may become a difficult task when the equations are stiff(even with a small number of species), or when the number of species is large[18].The main objective of this study is to understand the dynamics of two species Lotka-Volterra model, present a semi-analytic solution and provide some qualitative analysis with qualitative information about the solution of the model. In general, we seek a detailed numerical, with semi-analytic solution of the model and characterize some aspects of the system behaviour such as equilibrium points, stability and phase plane analysis of the two species Lotka-Volterra model. In order solve the two species Lotka-Volterra model, we have applied, the homotopy analysis method(HAM).Variational iteration method(VIM) and Adomian decomposition method(ADM) have successfully been applied to solve Lotka-Volterra model in [4]. In [4], the author, compared the VIM solution with ADM and fourth order Runge-Kutta methods. The HAM results in this paper agree well with the solutions given by [4]. HAM was first proposed in 1992 by Liao Shijun in his Ph.D. dissertation and further modified in 1997 to introduce a non-zero auxiliary parameter referred to as the convergence-control parameter to construct a homotopy on a differential system. Homotopy Analysis method employs the concept of the homotopy from topology to generate a convergent series solution for nonlinear systems which is enabled by utilizing a homotopy-McLaurin series to deal with the nonlinearity in the system. The strength of the HAM to naturally exhibit convergence of the series solution is strange in most analytic and semi-analytic approaches to nonlinear PDEs [20].In [18] the authors, applied homotopy analysis method(HAM) to solve the Lotka-Volterra problem and obtained approximate solutions.They studied a different model, which is a typical prey-predator model.The authors in [19] investigated the numerical solution of fuzzy arbitrary order predator-prey equations using the homotopy perturbation method(HPM). In their work, they took fuzziness in the initial conditions to mean convex normalised fuzzy sets viz triangular fuzzy number. They found that their solution was exactly equal to the crisp solution obtained in other literature.The work presented in this paper is different from the existing work done. We have analysed the model by investigating the positivity of solutions, boundedness of the solution and carried out its stability analysis of the equilibrium points. We have also examined the phase plane analysis of the model and compared numerical results of ode45 with that of homotopy analysis method. The present paper is organized as follows: in Section 2, the model formulation and analysis has been studied. In Section 3, we present the homotopy analysis approach to nonlinear system, while, in Section 4, we include the solution of the Lotka-Volterra model by HAM and in Section 5, numerical results and discussion are carried out. In Section 6, we present the conclusion and finally, the references are given.

2. Model Formulation and Analysis

The multi-species Lotka-Volterra equations model explains the dynamic nature of an arbitrary number of competitors.We present below the general Lotka-Volterra model for m-species as

$$\frac{dP_i}{dt} = P_i(\beta_i + \sum_{j=1}^{n} \alpha_{ij} P_j) \tag{1}$$

where $i = 1, 2, ..., m$ and P_i is the density or biomass of species i. We describe the associated model parameters in Table 1

Table 1: Parameters in the model Equation 1

Parameters	Description
α_{ij}	the interaction coefficients between species i and j
β_i	the intrinsic rate of change of species i
n	the number of species

2.1. Single Specie Case

For the case of single specie, equation (1) becomes single specie competing for a given finite source of food

$$\frac{dP_1}{dt} = P_1(\beta_1 + \alpha_{11} P_1) \tag{2}$$

such that $\beta_1 > 0$, $\alpha_{11} < 0$, and $P_1 > 0$, where α_{11} represent the intra-specific interaction. Equation (2) above has a closed form solution

$$P_1(t) = \frac{\beta_1 e^{\beta_1 t}}{\frac{\beta_1 + \alpha_{11} P(0)}{P(0) - \alpha_{11} e^{\beta_1 t}}} \quad if \quad \beta_1 \neq 0 \tag{3}$$

$$P_1(t) = \frac{P_1(0)}{1 - \alpha_{11} P_1(0) t} \quad if \quad \beta_1 = 0 \tag{4}$$

where $P_1(0)$ is the initial condition.

2.2. Two Species Case

Here we construct and analyse the Lotka-Volterra equations for two species competing for a common ecological niche as

$$\begin{aligned} \frac{dP_1}{dt} &= P_1(\beta_1 + \alpha_{11}P_1 + \alpha_{12}P_2) \\ \frac{dP_2}{dt} &= P_2(\beta_2 + \alpha_{21}P_1 + \alpha_{22}P_2) \end{aligned} \quad (5)$$

subject to the initial conditions

$$P_1(0) = P_{1,0} > 0,\ P_2(0) = P_{2,0} > 0$$

where α_{11}, α_{12},α_{21}, α_{22}, β_1 and β_2 are constants.

We describe the associated model parameters in Table 2

Table 2: Parameters in the model Equation 5

Parameters	Description
α_{11}	the intra-specific interaction between members of the same species 1 and 1
α_{12}	the interaction coefficients between species 1 and 2
α_{21}	the interaction coefficients between species 2 and 1
α_{22}	the intra-specific interaction between members of the same species 2 and 2
β_1	the intrinsic rate of change of specie 1
β_2	the intrinsic rate of change of specie 2

2.3. Positivity of the Solution

Theorem 1.[1]:Suppose the initial data $P_1 \geq 0$, $P_2 \geq 0$,$P_3 \geq 0$,...,$P_n \geq 0$ then the solutions $(P_1(t), P_2(t), P_3(t), ..., P_n(t)) \in \mathbb{R}^n_+$ of the Lotka-Volterra model (1) are non-negative for all $t > 0$ and will remain positive for all time.

2.4. Boundedness of the Solution

Theorem 2.[1]:Every solution $P_i(t)$ of the Lotka-Volterra model (1) is bounded. Therefore,from (1)

$$\lim_{t\to\infty} \sup P_i(t) \leq \frac{\beta_i}{\alpha_{ij}}$$

Proof. To proof boundedness, we note that $0 < P_i \leq P_i(t)$. We present the Lotka-Volterra model (1) as

$$\frac{dP_i}{(}t)dt \quad = \quad P_i(\beta_i + \alpha_{ij}P_i) \leq P_i(\beta_i + \alpha_{ij}P_i) \quad (6)$$

Then the

$$\lim_{t\to\infty} \sup P_i(t) \leq \frac{\beta_i}{\alpha_{ij}} \tag{7}$$

Therefore all solutions of model equation (1) are bounded. The feasible region for the total population is
$\Delta = \{P_i(t)|P_i(t) \leq \frac{\beta_i}{\alpha_{ij}}, 0 \leq P_i \leq P_i(t) \ \ foranyi = 1, 2, ...n\}$ We define Δ as the positively invariant region with respect to the model equation (1) therefore the model equation (1) is mathematically and ecologically well posed in Δ.

Theorem 3.[1]:The region $\Delta \subset \mathbb{R}_+$ is positively-invariant for the Predator-prey model (1) with non-negative initial conditions in $\mathbb{R}^n_+$.

2.5. Model Analysis

We analyse the model equation (5) by finding the equilibrium solution of the system when we set the Left Hand Side (L.H.S) of the equation to zero

$$\begin{aligned} P_1(\beta_1 + \alpha_{11}P_1 + \alpha_{12}P_2) &= 0 \\ P_2(\beta_2 + \alpha_{21}P_1 + \alpha_{22}P_2) &= 0 \end{aligned} \tag{8}$$

2.6. Equilibrium Solution and Stability Analysis

The system of Equation (8) has four equilibrium solutions and to determine the stability of this equilibrium point we find the linearization of the system.

Theorem 4.[6]: Suppose λ_1 and λ_2 are the eigenvalues of the 2×2 matrix of the linear system $\dot{x} = Ax$, then the equilibrium point $(0, 0)$ is

1. asymptotically stable if both eigenvalues λ_1 and λ_2 of A are real and negative, or have negative real parts.
2. stable but not asymptotically stable if λ_1 and λ_2 are pure imaginary, that is, $\lambda_1, \lambda_2 = \pm bi$.
3. unstable if either λ_1 or λ_2 is real and positive, or has a positive real part.
4. saddle point if either λ_1 or λ_2 is positive and one is negative(are real and of opposite signs.)The saddle is always unstable.
5. unstable if either λ_1 or λ_2 is real and positive, or has a positive real part.

Next, we present the four equilibrium solution of equation (8) given by $\Omega_1 = (0,0)$, $\Omega_2 = (0, -\frac{\beta_2}{\alpha_{22}})$, $\Omega_3 = (-\frac{\beta_1}{\alpha_{11}}, 0)$,$\Omega_4 = (\frac{\beta_1\alpha_{22}-\alpha_{12}\beta_2}{\alpha_{21}\alpha_{12}-\alpha_{22}\alpha_{11}}, \frac{\beta_2\alpha_{22}-\alpha_{12}\beta_1}{\alpha_{21}\alpha_{12}-\alpha_{22}\alpha_{11}})$

Case(i) When $\Omega_1 = (0,0)$

It represents the trivial equilibrium state. Ecologically, it means the simultaneous extinction of both species.Hence for stability of this equilibrium point we find Jacobian matrix of the system at

$$J(P_1^*, P_2^*) = \begin{bmatrix} \beta_1 + 2\alpha_{11}P_1^* & \alpha_{12}P_1^* \\ \beta_2 + \alpha_{21}P_2^* & 2\alpha_{22}P_2^* \end{bmatrix} \tag{9}$$

At $(0,0)$,

$$J(0,0) = \begin{bmatrix} \beta_1 & 0 \\ \beta_2 & 0 \end{bmatrix} \tag{10}$$

and the eigenvalues are $\lambda_1 = \beta_1 > 0$ and $\lambda_2 = 0$. With parameter values are $\alpha_{11} = -0.0014$,$\alpha_{12} = -0.0012$,$\alpha_{21} = -0.00009$, $\alpha_{22} = -0.001$, $\beta_1 = 0.1$,$\beta_2 = 0.08$ the eigenvalues become $\lambda_1 = 0.1 > 0$, and $\lambda_2 = 0$, hence, the equilibrium point is unstable, of the Lotka-Volterra equation (5)

Case(ii) When $\Omega_2 = (0, -\frac{\beta_2}{\alpha_{22}})$

This stands for the non-trivial equilibrium state. Ecologically, this means that the P_1 species eventually disappears due to predation or competition while P_2 species persist.Therefore, for the stability of this equilibrium point we find Jacobian matrix of the system at $(0, -\frac{\beta_2}{\alpha_{22}})$,

$$J\left(0, -\frac{\beta_2}{\alpha_{22}}\right) = \begin{bmatrix} \beta_1 & 0 \\ \beta_2 - \frac{\alpha_{21}\beta_2}{\alpha_{22}} & -2\beta_2 \end{bmatrix} \tag{11}$$

with eigenvalues $\lambda_1 = \beta_1$ and $\lambda_2 = -2\beta_2$. The parameter values are $\alpha_{11} = -0.0014$,$\alpha_{12} = -0.0012$,$\alpha_{21} = -0.00009$, $\alpha_{22} = -0.001$, $\beta_1 = 0.1$,$\beta_2 = 0.08$ the eigenvalues become $\lambda_1 = 0.1 > 0$, and $\lambda_2 = -0.16 < 0$, hence, the equilibrium point is saddle point of the Lotka-Volterra equation (5).Hence, it is unstable.

Case(iii) When $\Omega_3 = (-\frac{\beta_1}{\alpha_{11}}, 0)$

This stands for the non-trivial equilibrium state. Ecologically, this means that the P_2 species eventually disappears due to predation or competition while P_1 species persist.For the stability of this equilibrium point we find Jacobian matrix of the system at $(-\frac{\beta_1}{\alpha_{11}}, 0)$,

$$J\left(-\frac{\beta_1}{\alpha_{11}}, 0\right) = \begin{bmatrix} -\beta_1 & \frac{-\alpha_{12}\beta_1}{\alpha_{11}} \\ \beta_2 & 0 \end{bmatrix} \tag{12}$$

with eigenvalues

$$\lambda_1 = \frac{-\beta_1\alpha_{11} + \sqrt{\beta_1^2\alpha_{11}^2 - 4\alpha_{11}\beta_2\alpha_{12}\beta_1}}{2\alpha_{11}}$$

and

$$\lambda_2 = \frac{-\beta_1\alpha_{11} - \sqrt{\beta_1^2\alpha_{11}^2 - 4\alpha_{11}\beta_2\alpha_{12}\beta_1}}{2\alpha_{11}}.$$

The parameter values are $\alpha_{11} = -0.0014$,$\alpha_{12} = -0.0012$,$\alpha_{21} = -0.00009$, $\alpha_{22} = -0.001$, $\beta_1 = 0.1$,$\beta_2 = 0.08$ the eigenvalues become $\lambda_1 = -0.05 + 0.0008077i$, and $\lambda_2 = -0.05 - 0.0008077i$, hence, the equilibrium point is asymptotically stable, of the Lotka-Volterra equation (5).

Case(iv) When $\Omega_4 = (\frac{\beta_1\alpha_{22}-\alpha_{12}\beta_2}{\alpha_{21}\alpha_{12}-\alpha_{22}\alpha_{11}}, \frac{\beta_2\alpha_{22}-\alpha_{12}\beta_1}{\alpha_{21}\alpha_{12}-\alpha_{22}\alpha_{11}})$

This stands for the non-zero constant predator-prey or two competing species population that can co-exist(Non-zero coexistence equilibrium solution). Hence ,we present the stability of this equilibrium point using the Jacobian matrix at $(\frac{\beta_1\alpha_{22}-\alpha_{12}\beta_2}{\alpha_{21}\alpha_{12}-\alpha_{22}\alpha_{11}}, \frac{\beta_2\alpha_{22}-\alpha_{12}\beta_1}{\alpha_{21}\alpha_{12}-\alpha_{22}\alpha_{11}})$,

$$J\left(\frac{\beta_1\alpha_{22}-\alpha_{12}\beta_2}{\alpha_{21}\alpha_{12}-\alpha_{22}\alpha_{11}}, \frac{\beta_2\alpha_{22}-\alpha_{12}\beta_1}{\alpha_{21}\alpha_{12}-\alpha_{22}\alpha_{11}}\right) = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{13}$$

where $A_{11} = \beta_1 + \frac{2\alpha_{11}(\beta_1\alpha_{22}-\beta_2\alpha_1 1)}{\alpha_{22}\alpha_{11}-\alpha_{21}\alpha_{12}}$
$A_{12} = \frac{\alpha_{12}(\beta_1\alpha_{22}-\beta_2\alpha_{12})}{\alpha_{22}\alpha_{11}-\alpha_{21}\alpha_{12}}$
$A_{21} = \beta_2 + \frac{\alpha_{12}(\beta_1\alpha_{22}-\beta_2\alpha_1 1)}{\alpha_{22}\alpha_{11}-\alpha_{21}\alpha_{12}}$
$A_{22} = \frac{2\alpha_{22}(\beta_1\alpha_{21}-\beta_2\alpha_1 1)}{\alpha_{22}\alpha_{11}-\alpha_{21}\alpha_{12}}$ and the eigenvalues are

$$\lambda_1 = \frac{-(A_{11}+A_{22}) + \sqrt{(A_{11}+A_{22})^2 - 4(A_{21}A_{12} - A_{11}A_{12})}}{2}$$

and

$$\lambda_2 = \frac{-(A_{11}+A_{22}) - \sqrt{(A_{11}+A_{22})^2 - 4(A_{21}A_{12} - A_{11}A_{12})}}{2}.$$

With parameters values $\alpha_{11} = -0.0014$,$\alpha_{12} = -0.0012$,$\alpha_{21} = -0.00009$, $\alpha_{22} = -0.001$, $\beta_1 = 0.1$,$\beta_2 = 0.08$ the eigenvalues become $\lambda_1 = 0.1348623158 > 0$, and $\lambda_2 = -0.1373623158 < 0$, hence, the equilibrium point is saddle point of the Lotka-Volterra equation (5)

Phase Plane Analysis of the two species Lotka-Volterra Model(5) A phase plane analysis can be used to predict the long term behaviour of a system. We define the phase plane method as a method that graphically determine the existence of limit cycles in the solutions of the differential equation. The solutions to the differential equation are a family of functions. Graphically, this can be plotted in the phase plane like a two-dimensional vector field. Vectors representing the derivatives of the points with respect to a parameter (say time t), that is $(\frac{dP_1}{dt}, \frac{dP_2}{dt})$, at representative points are drawn. Here, we consider the evolution of the two population $P_1(t)$ and $P_2(t)$ living in the same community. Assume that $P_1(t)$ is the prey and $P_2(t)$ is the predator or both are two competing species. Thus we present in figures (1) and (2) the phase portrait and its direction field to obtain some qualitative information about the solution of the system (5) without solving it.

Figure (1), shows the direction field for the two species Lotka-Volterra model (5) while figure (2) shows the phase portrait for the two species Lotka-Volterra model (5). Figures (1) and (2), show the qualitative information of the behaviour of the solution without solving the equation. The arrows in plot give tangent lines to the solution curves and indicate where solution is increasing or deceasing. From figures (1) and (2), it is obvious that the population increases with increase in P_1 and P_2 and will increase without bound.

3. Homotopy Analysis Approach

Homotopy Analysis Method is a very powerful analytical method for solving nonlinear problems. It has a convenient means to control and adjust the convergence region and rate of approximation series, when necessary. We consider the following nonlinear differential equation

$$A[v(t)] = 0 \tag{14}$$

where A is a nonlinear operator, t denotes the time, and $v(t)$ is an unknown function. By applying an embedding parameter $q \in [0,1]$ we formulate the zero-order deformation equations (Liao[14]) and [18]

$$(1-q)L[\varphi(t;q) - v_0(t)] = qh_1H(t)A[\varphi(t;q)] \tag{15}$$

where $h_1 \neq 0$ is an auxilliary parameter, $H(t)$ is an auxilliary function, L is an auxilliary linear operator, $v_0(t)$ is an initial guess of $v(t)$, $\varphi(t;q)$ is an unknown

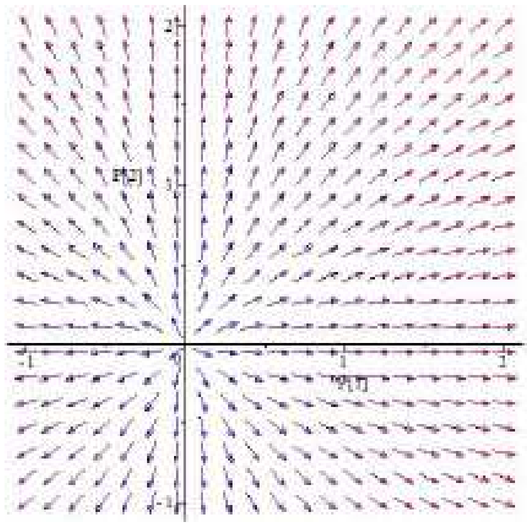

Figure 1: The direction field for the two species Lotka-Volterra model (5).

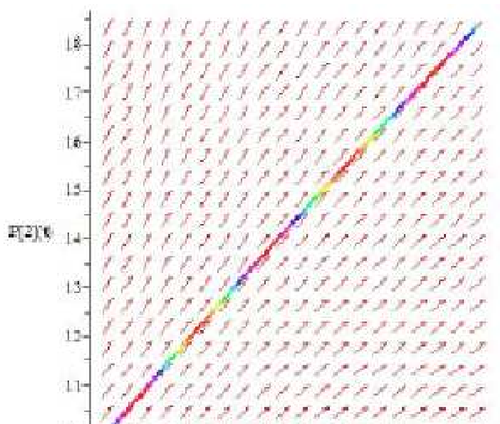

Figure 2: The phase portrait for the two species Lotka-Volterra model (5).

function. In the HAM based solution we have freedom to select h_1, $H(t)$, and L. It can easily be shown that when $q = 0$, we obtain

$$\varphi(t;0) = v_0(t) \tag{16}$$

and when $q = 1$, we obtain

$$\varphi(t;1) = v(t) \tag{17}$$

Hence, as q increases from 0 to 1, the solution $\varphi(t;q)$ varies continuously from the initial approximation $v_0(t)$ to the exact solution $v(t)$.Such a kind of continuous variation is called deformation in topology.Expanding $\varphi(t;p)$ by Taylor series with respect to q, we have

$$\varphi(t;q) = \varphi(t;0) + \Sigma_{\eta=1}^{\infty} v_\eta q^\eta \tag{18}$$

where

$$v_\eta(t) = \frac{1}{\eta!}\frac{\partial^m \varphi(t;q)}{\partial q^\eta} \mid q = 0 \tag{19}$$

is the deformation derivative. If the auxilliary nonlinear operator A, the initial approximation $v_0(t)$, the auxilliary parameter h_1 and the auxilliary function $H(t)$ are properly chosen so that

1. the solution $\varphi(t;q)$ of the zero-order deformation equation (15) exists for all $q \in [0,1]$
2. the deformation derivative (19) exists for all $\eta = 1, 2, ..$
3. the series (20) converges at $q = 1$.

Then, we have the series solution

$$v(t) = v_0(t) + \sum_{\eta=1}^{\infty} v_\eta(t) \tag{20}$$

which must be one of the solution of the original nonlinear equation (14) as proved by (Liao[13]). In short, we define the vector

$$\overrightarrow{v}_n(t) = \{v_0(t), v_1(t), ..., v_n(t)\} \tag{21}$$

By differentiating the zero-order equations (15) η-times with respect to the embedding parameter q, by setting $q = 0$ and finally dividing the resulting equation by $\eta!$, we have the η^{th} order deformation equation as

$$L[v_\eta(t) - \lambda_\eta v_{\eta-1}(t)] = h_1 H(t) P_\eta(\overrightarrow{v}_{\eta-1}(t)) \tag{22}$$

subject to initial condition

$$v_\eta(0) = 0 \tag{23}$$

where

$$P_\eta(\overrightarrow{v}_{\eta-1}(t)) = \frac{1}{(\eta-1)!}\frac{\partial^{\eta-1}A(t;q)}{\partial q^{\eta-1}} \mid q = 0 \tag{24}$$

and

$$\lambda_\eta = \begin{cases} 0 \; if \;\; \eta \leq 1, \\ \\ 1 \; if \;\; \eta > 1, \\ \end{cases} \tag{25}$$

Note that $v_\eta(t)$ for $\eta \geq 1$ can be obtained from equation (22) from uncoupled linear first order differential equation when solved subject to the initial condition (23). The solution of the equation of the system can be obtained easily by using the well known symbolic computation software such as MAPLE, MATLAB or MATHEMATICA.

4. Solution of the Lotka-Volterra model by Homotopy Analysis Method

To solve the model equation (5) by HAM, we consider the first equation in the model equation (5) and choose the linear operator

$$\begin{array}{rcl} L_1[P_1(t;q)] & = & \frac{dP_1(t;q)}{dt} \\ L_2[P_2(t;q)] & = & \frac{dP_2(t;q)}{dt} \end{array} \tag{26}$$

with the property that

$$\begin{array}{c} L_1[k_1] = 0 \\ L_2[k_2] = 0 \end{array} \tag{27}$$

where k_1 and k_2 are constants of integration.

Suppose $q \in [0,1]$ is an embedding parameter,h_1 and h_2 are auxilliary nonzero parameters and H_1 and H_2 are auxilliary functions, then the zeroth-order deformation equations are of the following form

$$\begin{array}{rcl} (1-q)L_1[P_1(t;q) - P_{1,0}(t)] & = & qh_1H_1(t)A_1[P_1(t;q), P_2(t;q)] \\ (1-q)L_2[P_2(t;q) - P_{2,0}(t)] & = & qh_2H_2(t)A_2[P_1(t;q), P_2(t;q)] \end{array} \tag{28}$$

subject to the initial conditions $P_1(0;q) = P_{1,0}$, $P_2(0;q) = P_{2,0}$

From the Lotka-Volterra equation (5) we define the following HAM nonlinear operators A_1 and A_2 as

$$\begin{aligned} A_1[P_1(t;q), P_2(t;q)] &= \tfrac{\partial P_1}{dt} - P_1(\beta_1 + \alpha_{11}P_1 + \alpha_{12}P_2) \\ A_2[P_1(t;q), P_2(t;q)] &= \tfrac{\partial P_2}{dt} - P_2(\beta_2 + \alpha_{21}P_1 + \alpha_{22}P_2) \end{aligned} \quad (29)$$

For $q = 0$ and $q = 1$ we have
$P_1(t;0) = P_{1,0}(t)$, $P_2(t;0) = P_{2,0}(t)$
$P_1(t;1) = P_{1,1}(t)$, $P_2(t;1) = P_{2,1}(t)$

As q increases from 0 to 1, $P_1(t;q)$ and $P_2(t;q)$ vary from $P_{1,0}(t)$ to $P_{1,1}(t)$ and $P_{2,1}(t)$.Using Taylor theorem, and $P_1(0;q) = P_{1,0}$, $P_2(0;q) = P_{2,0}$ we obtain

$$\begin{aligned} P_1(t;q) &= P_1(t;0) + \textstyle\sum_{\eta=1}^{\infty} P_{1,\eta}(t)q^{\eta} \\ P_2(t;q) &= P_2(t;0) + \textstyle\sum_{\eta=1}^{\infty} P_{2,\eta}(t)q^{\eta} \end{aligned} \quad (30)$$

where

$$\begin{aligned} P_{1,\eta}(t) &= \tfrac{1}{(\eta)!}\tfrac{\partial^{\eta} P_1(t;q)}{\partial q^{\eta}}|q = 0 \\ P_{2,\eta}(t) &= \tfrac{1}{(\eta)!}\tfrac{\partial^{\eta} P_2(t;q)}{\partial q^{\eta}}|q = 0 \end{aligned} \quad (31)$$

By Liao[14], the convergence of the series (30) strongly depend upon auxillary parameters h_1 and h_2. Assume that h_1 and h_2 are selected such that the series (30) are convergent at $q = 1$ then due to $P_1(t;0) = P_{1,0}(t)$, $P_2(t;0) = P_{2,0}(t)$ $P_1(t;1) = P_{1,1}(t)$, $P_2(t;1) = P_{2,1}(t)$, we have

$$\begin{aligned} P_1(t) &= P_1(t) + \textstyle\sum_{\eta=1}^{\infty} P_{1,\eta}(t) \\ P_2(t) &= P_2(t) + \textstyle\sum_{\eta=1}^{\infty} P_{2,\eta}(t) \end{aligned} \quad (32)$$

From the so-called ηth-order deformation equations (22) and (24), we differentiate equation (29) η times with respect to q divide by $\eta!$ and then set $q = 0$. The resulting deformation equation at the ηth-order are

$$\begin{aligned} L_1[P_{1,\eta}(t) - \lambda_{\eta}P_{1,\eta-1}(t)] &= h_1H_1(t)B_{1,\eta}(\overrightarrow{P}_{1,\eta-1}(t)) \\ L_2[P_{2,\eta}(t) - \lambda_{\eta}P_{2,\eta-1}(t)] &= h_2H_2(t)B_{2,\eta}(\overrightarrow{P}_{2,\eta-1}(t)) \end{aligned} \quad (33)$$

subject to the initial conditions $P_{1,\eta}(0) = P_{1,0}$, $P_{2,\eta}(0) = P_{2,0}$, where

$$B_{i,\eta}(\overrightarrow{P}_{\eta-1}(t)) = \frac{1}{(\eta-1)!}\frac{\partial^{\eta-1}A_i(t;q)}{\partial q^{\eta-1}} \mid q = 0 \ for \ all \ i = 1,2 \quad (34)$$

subject to initial conditions

$$P_{i,\eta}(0) = P_{i,0} \; for \; all \; i = 1, 2 \tag{35}$$

where

$$B_{1,\eta} = [\frac{dP_{1,(\eta-1)}(t)}{dt} - P_{1,(\eta-1)}(t)\beta_1 - \sum_{k=0}^{\eta-1} \alpha_{11} P_{1,k}(t) P_{1,\eta-1-k}(t) - \sum_{k=0}^{\eta-1} \alpha_{12} P_{1,k}(t) P_{2,\eta-1-k}(t)]dt$$

$$B_{2,\eta} = [\frac{dP_{2,(\eta-1)}(t)}{dt} - P_{2,(\eta-1)}(t)\beta_2 - \sum_{k=0}^{\eta-1} \alpha_{21} P_{2,k}(t) P_{1,\eta-1-k}(t) - \sum_{k=0}^{\eta-1} \alpha_{22} P_{2,k}(t) P_{2,\eta-1-k}(t)]dt.$$

Here, we take $h_i = -1$ and $H_i(t) = 1$ since HAM allows us to choose the control-convergence function $H_i(t)$ and the parameter h_i for all $i = 1, 2$. Finally, the HAM formulation for the η th- order deformation equation (32) for $\eta \geq 1$ yields

$$P_{1,\eta}(t) = \lambda_\eta P_{1,(\eta-1)}(t) - \int_\infty^t [\frac{dP_{1,(\eta-1)}(t)}{dt} - P_{1,(\eta-1)}(t)\beta_1 - \sum_{k=0}^{\eta-1} \alpha_{11} P_{1,k}(t) P_{1,\eta-1-k}(t) - \sum_{k=0}^{\eta-1} \alpha_{12} P_{1,k}(t) P_{2,\eta-1-k}(t)]dt, \quad \eta \geq 1 \tag{36}$$

$$P_{2,\eta}(t) = \lambda_\eta P_{2,(\eta-1)}(t) - \int_\infty^t [\frac{dP_{2,(\eta-1)}(t)}{dt} - P_{2,(\eta-1)}(t)\beta_2 - \sum_{k=0}^{\eta-1} \alpha_{21} P_{2,k}(t) P_{1,\eta-1-k}(t) - \sum_{k=0}^{\eta-1} \alpha_{22} P_{2,k}(t) P_{2,\eta-1-k}(t)]dt, \quad \eta \geq 1 \tag{37}$$

and

$$\lambda_\eta = \begin{cases} 0 \; if \; \eta \leq 1, \\ 1 \; if \; \eta > 1, \end{cases} \tag{38}$$

5. Numerical Results and Discussions

We authenticate the model using the parameters obtained from different literatures. Tables 3 and 4 below shows the details of the parameters and their values.The result of the numerical experiment carried out in this paper is similar to that of the ode-45.

Table 3: Parameters of the model Equation 5

Parameter	Symbol	Value	Source
the intraspecific interaction between members of the same species 1 and 1	α_{11}	-0.0014	[4]
the interaction coefficients between species 1 and 2	α_{12}	-0.0012	[4]
the interaction coefficients between species 1 and 2	α_{21}	-0.0009	[4]
the intraspecific interaction between members of the same species 2 and 2	α_{22}	-0.001	[4]
the intrinsic rate of change of species 1	β_1	0.1	[4]
the intrinsic rate of change of species 2	β_2	0.08	[4]

Table 4: Initial values of variables used in the simulations.

Variables	Symbol	Value	Source
Initial population of first specie	$P_{1,0}$	4	[4]
Initial population of second specie	$P_{2,0}$	10	[4]

The 7^{th} to 9^{th} approximations for P_1, P_2 are obtained and presented below:

7^{th} approximations:

$$P_{1,7}(t) = 4+0.3296t+0.01396352t^2+0.0004045745493t^3+0.00000898037635t^4+0.0000001627620074t^5+0.000000000068151057t^6+0.000000000003310090379t^7$$

$$P_{2,7}(t) = 10+0.664t+0.0217568t^2+0.0004657681067t^3+0.000007240649132t^4-0.000000000244143011 01t^6-0.0000002529052902t^7$$

8^{th} approximations:

$$P_{1,8}(t) = 4+0.3296t+0.01396352t^2+0.0004045745493t^3+0.00000898037635t^4+0.0000001627620074t^5+0.000000000068151057t^6+0.000000000000108647803t^7+0.0000000000003880151975t^8$$

$$P_{2,8}(t) = 10+0.664t+0.0217568t^2+0.0004657681067t^3+0.000007240649132t^4-0.00000000024414301101t^6-0.000000000004292445464t^7+0.000000000000000032038105t^8$$

9^{th} approximations:
$P_{1,9}(t) = 4+0.3296t+0.01396352t^2+0.0004045745493t^3+0.00000898037635t^4+0.0000001627620074t^5+0.0000000000681510\,57t^6+0.00000000000108647803t^7+0.0000000000000143378726t^8+0.000000000000004069677423t^9$
$P_{2,9}(t) = 10+0.664t+0.0217568t^2+0.0004657681067t^3+0.000007240649132t^4-0.0000000024414301101t^6-0.00000000002529052902t^7+0.00000002335150068t^8-0.000000003855233449t^9$

5.1. Homotopy Analysis Method for the Solution of the Two Species Lotka-Volterra Model

Table 5: Result of homotopy analysis method when t=0.2 and t=0.4

order	t=0.2		t=0.4	
	P_1	P_2	P_1	P_2
7th	4.066481792	10.13367401	4.134100288	10.2611109
8th	4.066481792	10.13367401	4.134100288	10.2611109
9th	4.066481792	10.13367401	4.134100288	10.2611109

Table 6: Result of homotopy analysis method when t=0.6 and t=0.8

order	t=0.6		t=0.8	
	P_1	P_2	P_1	P_2
7th	4.202875433	10.40633400	4.2728827530	10.54536579
8th	4.202875433	10.40633400	4.2728827530	10.54536579
9th	4.202875433	10.40633400	4.2728827530	10.54536579

Table 7: Result of homotopy analysis method when t=1.0

order	t=1.0	
	P_1	P_2
7th	4.343977247	10.68622981
8th	4.343977247	10.68622981
9th	4.343977247	10.68622981

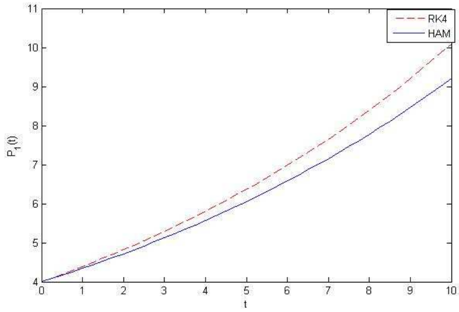

Figure 3: Plot of 9^{th} approximation for P_1 against time with HAM and ode45

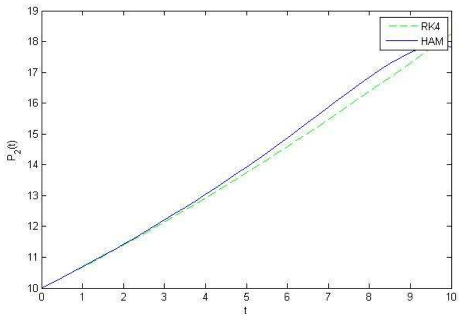

Figure 4: Plot of 9^{th} approximation for P_2 against time with HAM and ode45

5.2. Comparison of HAM and Ode45 Solution of the Two Species Lotka-Volterra Model

[t!] In this section, we have compared the solutions of HAM with the result of ode45. Figure (3), presents the plot of 9^{th} approximation for P_1 against time, while figure (4),shows the plot of 9^{th} approximation for P_2 against time with HAM and ode45. In figure (3), it is observed that P_1 (first specie population) is increasing with time as well as in figure (4), the population of the second specie P_2 is also increasing with time which agrees with the plot obtain by using ode45(for fourth order Runge-Kutta method) for P_1 and P_2. Figure (3) and (4) show that HAM is in good agreement with RK4. Hence,it is obvious that the analytical approximations to the solutions are reliable method and confirm the power and ability of the Homotopy Analysis method.

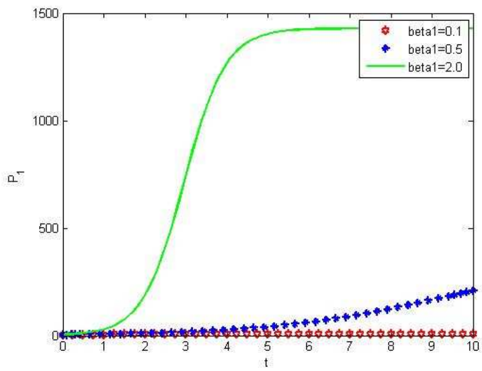

Figure 5: Variation of P_1 for different high values of β_1.

5.3. Variation of Different Values of β_1 and β_2 on the Dynamics of the Two Species Lotka-Volterra Model

In figures (5) and (7), the variations of P_1 for different high and low values of β_1 are shown while figures (6) and (8) give the variations of P_2 for different high and low values of β_2. In figure(5), it is observed that for different high values of β_1, the population of P_1 increases very slowly at $\beta_1 = 0.1$, while at $\beta_1 = 0.5$, the population of P_1 increases slowly. At $\beta_1 = 2$, we found that the population(P_1) increases and remains at constant state. Figure(6) shows that, as the population of P_2 increases with increase in β_2 while for low values of β_2, the population of P_2 increases at $\beta_2 = 0.08$ but decreases at $\beta_2 = 0.0001$ and $\beta_2 = 0.0005$ in figure(7). For low values of β_1, the population of P_1 also increases at $\beta_1 = 0.08$ but the population decreases at $\beta_1 = 0.0001$ and $\beta_1 = 0.0005$ in figure(8).

6. Conclusion

In this paper, a mathematical model of the dynamics of two species Lotka-Volterra model is proposed and its qualitative analysis has been revisited. Some qualitative information about the solution of the model is carried out by phase-plane analysis.Semi-analytical solution for the two species Lotka-Volterra model is obtained by HAM and these are compared with Runge-Kutta of fourth order and that of [4] which reveals that HAM is also a powerful, easy-to-use analytic method for nonlinear problems. HAM provides us a simple way to adjust and

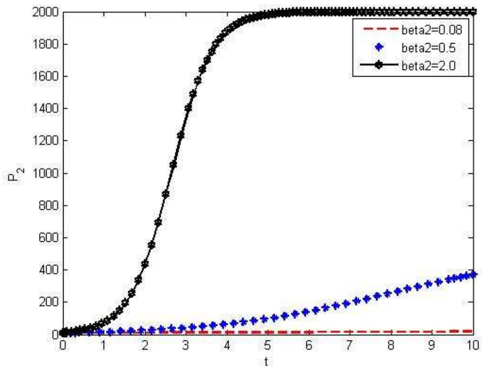

Figure 6: Variation of P_2 for different high values of β_2.

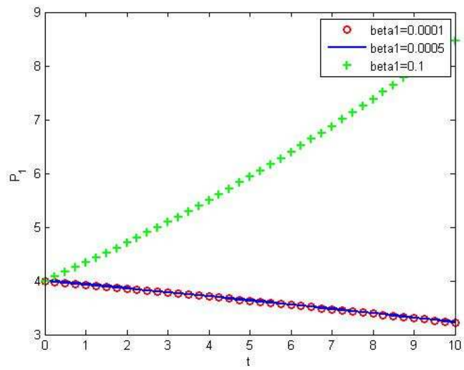

Figure 7: Variation of P_1 for different low values of β_1.

control the convergence region of the series solution by means of auxilliary parameters h_1 and h_2.

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Figure 8: Variation of P_2 for different low values of β_2.

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