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TOTALLY θ-SG-CONTINUOUS FUNCTIONS

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Abstract: In this paper, we introduce and study a new class of functions called totally θ-sg-continuous functions as a new generalization of total continuity utilizing θ-sg-open sets and obtain their characterizations and properties. Some θ-sg- separation axioms by using the θ-sg-open sets are defined. Furthermore, we introduce some types of TSGO-compact spaces and investigate some relations among them. Some properties of their spaces are given.

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1. Introduction

General topology plays a very important role in applied sciences and in all

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branches of mathematics. An important concept in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized closed sets. Recently, M. Caldas and S. Jafari [3] introduced the notion of θ-sg-closed set using semi-θ-closure operator. They also defined semi-θ-T_0 space and some other spaces. In 1980, Jain [4] introduced totally continuous functions. T.M. Nour [7] introduced the concept of totally semi-continuous functions as a generalization of totally continuous functions and several properties of totally semi-continuous were obtained.

In this paper, we will continue the study of related functions with θ-sg-open and θ-sg-closed sets. We define some θ-sg- separation axioms by using the θ-sg-open sets. We also define semi-θ-T_1 and semi-θ-T_2 spaces, and investigate the relations with θ-sg-separation axioms. Moreover, we introduce and characterize the concept of totally θ-sg-continuous functions. The notion of totally θ-sg-continuity is a weaker form of totally semi-continuity. In addition, we introduce some types of TSGO-compact spaces and investigate some relations among them. Some properties of their spaces are given.

2. Preliminaries

Throughout this paper $(X,\ \tau)$, $(Y,\ \sigma)$ (or simply X, Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X the closure and interior of A with respect to τ are denoted by $Cl(A)$ and $Int(A)$ respectively.

Definition 1. A subset A of a space X is called:
(1) semi-open [5] if $A \subset Cl(Int(A))$.
(2) semi-closed [2] if $Int(Cl(Int(A))) \subset A$.

Definition 2. We denote by $SO(X)$ the set of all semi-open sets in (X,τ). The semi-closure of A is defined as the intersection of all semi-closed sets containing A and it is denoted by $sCl(A)$ [2].

Definition 3. [6] A point $x \in X$ is called a semi-θ-cluster point of A if $sCl(U) \cap A \neq \phi$, for each semi-open set U containing x. The set of all semi-θ-cluster point of A is called semi-θ-closure of A and is denoted by $sCl_\theta(A)$. A subset A is called semi-θ-closed set if $sCl_\theta(A) = A$. The complement of a semi-θ-closed set is semi-θ-open.

Definition 4. [3] A subset A of X is θ-semigeneralized closed (briefly, θ-sg-closed) set if $sCl_\theta(A) \subset U$ whenever $A \subset U$ and U is semi-open in X. The complement of θ-sg-closed set is θ-semigeneralized open (briefly,θ-sg-open). The family of all θ-sg-closed sets of X is denoted by $\theta\mathrm{SG}C(X,\tau)$ and θ-sg-open sets by $\theta\mathrm{SG}O(X,\tau)$.

Lemma 5. *[3] Every semi-θ-closed set is θ-sg-closed.*

3. θ-sg-Closures

In this section, the notion of θ-sg-closure is defined and some of its basic properties are studied.

Definition 6. The intersection of all θ-sg-closed sets containing a set A is called θ-sg-closure of A and is denoted by $\theta\mathrm{sg}Cl(A)$.

Definition 7. The union of all θ-sg-open sets contained in A is called θ-sg-interior of A and is denoted by $\theta\mathrm{sg}Int(A)$.

Remark 8. For any subset A of a topological space X, we have
(i) If A is θ-sg-closed, then $\theta\mathrm{sg}Cl(A) = A$.
(ii) If A is θ-sg-open, then $\theta\mathrm{sg}Int(A) = A$.

Definition 9. Let A be a subset of X. Then $\theta\mathrm{sg}Cl(A) - \theta\mathrm{sg}Int(A)$ is called θsg-frontier of A and is denoted by $\theta\mathrm{sg}Fr(A)$

Definition 10. For any subset A of space X, $A - \theta\mathrm{sg}Int(A)$ is defined as θ-sg-border of A and is denoted by $\theta\mathrm{sg}bd(A)$.

Theorem 11. *For a subset A of space X the following results hold*
(i) $\theta sgbd(A) \subset \theta sgFr(A)$.
(ii) $\theta sgCl(A) = \theta sgInt(A) \cup \theta sgFr(A)$.
(iii) $\theta sgInt(A) \cap \theta sgFr(A) = \phi$.
(iv) $\theta sgFr(A) = \theta sgCl(A) \cap \theta sgCl(X - A)$.
(v) $\theta sgFr(A) = \theta sgFr(X - A)$.
(vi) $\theta sgFr(A)$ is θ-sg-closed set.
(vii) $\theta sgInt(A) = A - \theta sgFr(A)$.
(viii) If A is θ-sg-open as well as θ-sg-closed, then $\theta sgFr(A) = \phi$.
(ix) $\theta sgFr(\theta sgInt(A)) \subset \theta sgFr(A)$.
(x) $\theta sgInt(A) \cup \theta sgInt(X - A) = X - \theta sgFr(A)$.
(xi) $\theta sgFr(\theta sgCl(A)) \subset \theta sgFr(A)$.
(xii) $\theta sgCl(A) = A \cup \theta sgFr(A)$.

Theorem 12. *Let A be subset of X and $x \in X$. If $x \in \theta sgCl(A)$, then $V \cap A \neq \phi$ for every θ-sg-open set V containing x.*

Proof. Suppose that there exists a θ-sg-open set V containing x such that $V \cap A = \phi$. Since $A \subset X - V, \theta \text{sg}Cl(A) \subset X - V$ and then $x \notin \theta\text{sg}Cl(A)$, a contradiction.

Theorem 13. *Let A and B be subsets of X. Then we have*
(i) $\theta sgCl(\phi) = \phi$ *and* $\theta sgCl(X) = X$.
(ii) If $A \subset B$, *then* $\theta sgCl(A) \subset \theta sgCl(B)$.
(iii) $\theta sgCl(A) = \theta sgCl(\theta sgCl(A))$.
(iv) $\theta sgCl(A \cup B) \supset \theta sgCl(A) \cup \theta sgCl(B)$.
(v) $\theta sgCl(A \cap B) \subset \theta sgCl(A) \cap \theta sgCl(B)$.

4. θ-sg-Separation Axioms

Definition 14. A topological space X is said to be θ-sg-T_0 if for each pair of distinct points x and y of X, there exists a θ-sg-open set containing one point but not the other.

Definition 15. [3] A topological space X is said to be semi-θ-T_0 if for each pair of distinct points x and y of X, there exists a semi-θ-open set containing one point but not the other.

Remark 16. Every semi-θ-T_0 space is θ-sg-T_0 space. But the converse need not be true in general.

Example 17. Consider the space $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. Then $SO(X) = \tau$. Therefore, θSGO(X,τ) = all subsets of X, and the set $\{a\}$ is the only semi-θ-open in X. Hence the space (X, τ) is θ-sg-T_0, but it is not semi-θ-T_0 space since every semi-θ-open set containing the point b contains the point c also.

Theorem 18. A topological space X is a θ-sg-T_0 if and only if θ-sg-closures of distinct points are distinct.

Proof. Let x and y be distinct points of X. Since X is θ-sg-T_0 space, there exists a θ-sg-open set G such that $x \in G$ and $y \notin G$. Consequently, $X - G$ is a θ-sg-closed set containing y but not x. But $\theta\text{sg}Cl\{y\}$ is the intersection of all θ-sg-closed sets containing y. Hence $y \in \theta\text{sg}Cl\{y\}$ but $x \notin \theta\text{sg}Cl\{y\}$ as $x \notin X - G$. Therefore, $\theta\text{sg}Cl\{x\} \neq \theta\text{sg}Cl\{y\}$

Conversely, let $\theta\text{sg}Cl\{x\} \neq \theta\text{sg}Cl\{y\}$ for $x \neq y$. Then there exists at least one point $z \in X$ such that $z \in \theta\text{sg}Cl\{x\}$ but $z \notin \theta\text{sg}Cl\{y\}$. We claim $x \notin$

θsg$Cl\{y\}$, because if $x \in \theta$sg$Cl\{y\}$, then $\{x\} \subset \theta$sg$Cl\{y\}$ implies θsg$Cl\{x\} \subset \theta$sg$Cl\{y\}$. So $z \in \theta$sg$Cl\{y\}$, which is a contradiction. Hence $x \notin \theta$sg$Cl\{y\}$, which implies $x \in X - \theta$sg$Cl\{y\}$, which is a θ-sg-open set containing x but not y. Hence X is θ-sg-T_0 space.

Definition 19. [3] A function $f : X \to Y$ is said to be θ-semigeneralized irresolute (briefly, θ-sg-irresolute) if $f^{-1}(F)$ is θ-sg-closed in X for every θ-sg-closed set F of Y.

Theorem 20. *If $f : X \to Y$ is injective θ-sg-irresolute function and Y is θ-sg-T_0, then X is θ-sg-T_0.*

Proof. Suppose Y is θ-sg-T_0 space. Let x and y be any two distinct point in space X. Since f is injective $f(x)$ and $f(y)$ are distinct points in Y. Since Y is θ-sg-T_0 space, there exists a θ-sg-open set G in Y containing $f(x)$ but not $f(y)$. Again since f is θ-sg-irresolute, $f^{-1(}G)$ is θ-sg-open set in X containing x but not y. Therefore, X is θ-sg-T_0 space.

Theorem 21. *If $f : X \to Y$ is injective, θ-sg-continuous and Y is T_0, then X is θ-sg-T_0 space.*

Proof. Suppose Y is T_0 space and let x and y be any two distinct points in X. Since f is injective $f(x)$ and $f(y)$ are distinct points in Y. Since Y is T_0, there exists an open set G in Y containing $f(x)$ but not $f(y)$. Again since f is θ-sg- continuous , $f^{-1}(G)$ is θ-sg-open set in X containing x but not y. Therefore, X is θ-sg-T_0 space.

Definition 22. A topological space X is said to be θ-sg-T_1 if for any pair of distinct points x and y, there exist θ-sg-open sets G and H such that $x \in G$, $y \notin G$ and $x \notin H$, $y \in H$.

Remark 23. Every θ-sg-T_1 space is θ-sg-T_0 space. But the converse need not be true in general.

Example 24. Let the space $X = \{a, b, c\}$ with $\tau = \{\phi, X, \{a, b\}\}$. Then $SO(X) = \tau$. Therefore, θSGO(X,τ) = $\{\phi, X$,{a},{b},{a, b}$\}$. Hence the space (X, τ) is θ-sg-T_0, but it is not θ-sg-T_1 space since for the points $a \neq c$ in X, then there is no two disjoint open sets G and H such that $a \in G$ and $c \in H$.

Definition 25. A topological space X is said to be semi-θ-T_1 if for any pair of distinct points x and y, there exist semi-θ-open sets G and H such that $x \in G$, $y \notin G$ and $x \notin H$, $y \in H$.

Remark 26. Every semi-θ-T_1 space is θ-sg-T_1 space. But the converse need not be true in general.

Example 27. The space (X, τ) as defined in Example 17 is θ-sg-T_1, but it is not semi-θ-T_1 space since for the points b and c in X such that $b \neq c$, then there is no two semi-θ-open sets G and H such that $b \in G$, $c \notin G$ and $b \notin H$, $c \in H$.

Theorem 28. *A topological space X is θ-sg-T_1 if every singleton subset $\{x\}$ of X is θ-sg-closed set.*

Proof. Suppose $\{x\}$ is θ-sg-closed set for every $x \in X$. Let x and $y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in X - \{x\}$. Hence $X - \{x\}$ is θ-sg-open set containing y but not x. Similarly, $X - \{y\}$ is θ-sg-open set containing x but not y. Therefore, X is θ-sg-T_1 space.

Theorem 29. *If $f : X \to Y$ is θ-sg-irresolute injective function and Y is θ-sg-T_1 space, then X is θ-sg-T_1 space.*

Proof. Let x_1, x_2 be pair of distinct points in X. Since f is injective there exist distinct points y_1, y_2 of Y such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since Y is θ-sg-T_1 space there exist θ-sg-open sets U and V in Y such that $y_1 \in U$, $y_2 \notin U$ and $y_1 \notin V$, $y_2 \in V$. That is $x_1 \in f^{-1}(U), x_1 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V), x_2 \notin f^{-1}(U)$. Since f is θ-sg-irresolute $f^{-1}(U)$, $f^{-1}(V)$ are θ-sg-open sets in X. Thus, for two distinct points x_1, x_2 of X there exist θ-sg-open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1 \in f^{-1}(U)$, $x_1 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V)$, $x_2 \notin f^{-1}(U)$. Therefore, X is θ-sg-T_1 space.

Theorem 30. *If $f : X \to Y$ is θ-sg-continuous injection and Y is T_1, then X is θ-sg-T_1 space.*

Proof. Let $f : X \to Y$ be θ-sg-continuous injection and Y be T_1. For any two distinct points x_1, x_2 of X there exist distinct points y_1, y_2 of Y such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since Y is T_1-space there exist open sets U and V in Y such that $y_1 \in U$, $y_2 \notin U$ and $y_1 \notin V$, $y_2 \in V$. That is $x_1 \in f^{-1}(U), x_1 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V), x_2 \notin f^{-1}(U)$. Since f is θ-sg-continuous $f^{-1}(U)$, $f^{-1}(V)$ are θ-sg-open sets in X. Thus, for two distinct points x_1, x_2 of X there exist θ-sg-open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1 \in f^{-1}(U), x_1 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V), x_2 \notin f^{-1}(U)$. Therefore, X is θ-sg-T_1 space.

Definition 31. A topological space X is said to be θ-sg-T_2 if for any pair of distinct points x and y, there exist disjoint θ-sg-open sets G and H such that $x \in G$ and $y \in H$.
Clearly every θ-sg-T_2 space is θ-sg-T_1 space.

Definition 32. A topological space X is said to be semi-θ-T_2 if for any pair of distinct points x and y, there exist disjoint semi-θ-open sets G and H such that $x \in G$ and $y \in H$.

Remark 33. Every semi-θ-T_2 space is θ-sg-T_2 space. But the converse need not be true in general.

Example 34. It is clear from Example 17 that the space (X, τ) is θ-sg-T_2, but it is not semi-θ-T_2 space since for the points $b \neq c$ in X, then there is no two disjoint semi-θ-open sets G and H such that $b \in G$ and $c \in H$.

Theorem 35. *If $f : X \to Y$ is θ-sg-continuous injection and Y is T_2, then X is θ-sg-T_2 space.*

Proof. Let $f : X \to Y$ be θ-sg-continuous injection and Y be T_2. For any two distinct points x_1, x_2 of X there exist distinct points y_1, y_2 of Y such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since Y is T_2 space there exist disjoint open sets U and V in Y such that $y_1 \in U$ and $y_2 \in V$. That is $x_1 \in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$. Since f is θ-sg-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are θ-sg-open sets in X. Further f is injective, $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\phi) = \phi$. Thus, for two disjoint points x_1, x_2 of X there exist disjoint θ-sg-open sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1 \in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$. Therefore, X is θ-sg-T_2 space.

5. Totally θ-sg-Continuous Functions

In this section, the notion of a new class of functions called totally θ-sg-continuous functions is introduced and obtained some of their characterizations and properties. Also, the relationships with some other related functions are discussed.

Definition 36. A function $f : X \to Y$ is said to be totally θ-sg-continuous if $f^{-1}(V)$ is θ-sg-clopen in X for every open set V of Y.

Definition 37. [5] A function $f : X \to Y$ is said to be totally semi-continuous if $f^{-1}(V)$ is semi-clopen in X for every open set V of Y.

Remark 38. Every totally semi-continuous function is totally θ-sg-continuous. But the converse need not to be true from the following example.

Example 39. Let $X = \{a, b, c\}$ with $\tau = \{\phi$,X,{a},{b, c}}, and $Y = \{1, 2, 3\}$ with $\rho = \{\phi, Y, \{1, 2\}, \{3\}\}$. Then $SO(X) = \tau$. Thus, θSGO(X,τ) = all subsets of X. Let $f \to (X, \tau) \to (Y, \rho)$ be a function defined as follows:

$$f(a) = 1, f(b) = 2 \text{ and } f(c) = 3$$

Then f is totally θ-sg-continuous, but it is not totally semi-continuous since $\{3\}$ is an open set in (Y, ρ), but $f^{-1}(\{3\}) = \{c\}$ and $\{c\}$ is not semi-clopen set in (X, τ)

Definition 40. [3] A function $f : X \to Y$ is said to be θ-semigeneralized continuous (briefly, θ-sg-continuous) if $f^{-1}(F)$ is θ-sg-closed in X for every semi-closed set F of Y.

Remark 41. The relation between totally θ-sg-continuous function and θ-sg-continuous function are independent as shown by the following two examples.

Example 42. Let $X = \{a, b, c\}$ with $\tau = \{\phi, X,\{a\},\{a, b\},\{a, c\}\}$, and $Y = \{1, 2, 3\}$ with $\rho = \{\phi, Y,\{1\},\{1, 2\},\{1, 3\}\}$. Then $SO(X) = \tau$ and $SO(Y) = \rho$. Thus, θSGO(X,τ) $= \{\phi, X, \{a\}\}$. Let $f \to (X, \tau) \to (Y, \rho)$ be a function defined as follows:

$$f(a) = 1 \text{ and } f(b) = f(c) = 2$$

Hence f is θ-sg-continuous, but it is not totally θ-sg-continuous since $\{1\}$ is an open set in (Y, ρ), but $f^{-1}(\{1\}) = \{a\}$ and $\{a\}$ is not θ-sg-clopen set in (X, τ).

Example 43. Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{a\}\}$. Then $SO(X) = \{\phi, X,\{a\},\{a, b\},\{a, c\}\}$. Thus, θSGO(X,τ) $= \tau$. Define $f \to (X, \tau) \to (X, \tau)$ be a function as follows:

$$f(a) = f(b) = b \text{ and } f(c) = c$$

Hence f is totally θ-sg-continuous, but it is not θ-sg-continuous since the set $\{c\}$ is semi-closed in (X, τ), but $f^{-1}(\{c\}) = \{c\}$ and $\{c\}$ is not θ-sg-closed set in (X, τ).

Definition 44. [3] A topological space X is said to be TSGO-connected if X cannot be written as the union of two non-empty disjoint θ-sg-open sets.

Theorem 45. *If f is totally θ-sg-continuous function from a TSGO-connected space X onto any space Y, then Y is an indiscrete space.*

Proof. Let us suppose that Y is not indiscrete. Let A be a proper non-empty subset of Y. Then $f^{-1}(A)$ is non-empty θ-sg-clopen subset of X, which is contradiction to the fact that X is TSGO-connected space. Therefore, Y is indiscrete.

Theorem 46. *Let X be TSGO-connected and Y be T_1. If $f : X \to Y$ is totally θ-sg-continuous, then f is constant.*

Proof. Suppose X is TSGO-connected and f is totally θ-sg-continuous. Since Y is T_1-space, $\Lambda = \left\{f^{-1}(y) : y \in Y\right\}$ is a disjoint θ-sg-clopen partition of

X. If $|\Lambda| \geq 2$, then there exist a proper θ-sg-clopen set W for some $U \in \Lambda$ in TSGO-connected space X. This is contradiction to the fact that X is TSGO-connected. Therefore, $|\Lambda| = 1$. This shows that f is constant.

Definition 47. Let X be a topological space. We define an equivalence relation on X by setting $x \approx y$ if there is a TSGO-connected subset of X containing both x and y. The equivalence classes are called the θ-sg-separation of X or θ-sg-component of X.

Theorem 48. *Let $f : X \to Y$ be a totally θ-sg-continuous function from a topological space X into T_1 space Y. Then f is constant on each θ-sg-component of X.*

Proof. The proof follows from the definition and theorem above.

Theorem 49. *The set of all points x of X at which $f : X \to Y$ is not totally θ-sg-continuous is identical with the union of θ-sg-frontier of the inverse images of closed sets of Y containing $f(x)$.*

Proof. Assume that f is not totally θ-sg-continuous at $x \in X$. Then there exists an open set V of Y containing $f(x)$ such that $f(U)$ is not contained in V for each θ-sg-clopen set U containing x. This implies $U \cap (X - f^{-1}(V)) \neq \phi$ for every $U \in \theta\text{SGO}(X, x)$. Therefore, $x \in \theta\text{sg}Cl(X - f^{-1}(V))$. However, since $x \in f^{-1}(V) \subset \theta\text{sg}Cl(f^{-1}(V))$, $x \in \theta\text{sg}Cl(f^{-1}(V)) \cap \theta\text{sg}Cl(X - f^{-1}(V))$. This shows that, $x \in \theta\text{sg}Fr(f^{-1}(V))$.

Conversely, suppose $x \in \theta\text{sg}Fr(f^{-1}(V))$ for some open set V of Y containing $f(x)$ and f is totally θsg-continuous at $x \in X$. Then there exists $U \in \theta\text{SGO}(X, x)$ such that $f(U) \subset V$. This implies $x \in U \subset f^{-1}(V)$. Therefore, $x \in \theta\text{sg}Int(f^{-1}(V)) \subset X - \theta\text{sg}Fr(f^{-1}(F))$. This is contradiction to the fact that, $x \in \theta\text{sg}Fr(f^{-1}(F))$. Therefore, f is not totally θ-sg-continuous.

Definition 50. A filter base Ω in a space X is said to be θ-sg-co-convergent to a point x in X if for any $U \in \theta\text{SGO}(X)$ containing x, there exists $B \in \Omega$ such that $B \subset U$.

Theorem 51. *If a function $f : X \to Y$ is totally θ-sg-continuous, then for each point $x \in X$ and each filter base Ω in X θ-sg-co-converging to x, the filter base $f(\Omega)$ is convergent to $f(x)$.*

Proof. Let $x \in X$ and Ω be any filter base in θ-sg-co-converging to x. Since f is totally θ-sg-continuous, then for any open set V of Y containing $f(x)$, there exists $U \in \theta\text{SGO}(X)$ containing x such that $f(U) \subset V$. Since Ω is θ-sg-co-converging to x, there exists $B \in \Omega$ such that $B \subset U$. This means that $f(B) \subset V$ and therefore, the filter base $f(\Omega)$ is convergent to $f(x)$.

6. TSGO-COMPACT SPACES

Definition 52. A space X is said to be
(i)TSGO-co-compact if every θ-sg-clopen cover of X has finite subcover.
(ii)TSGO-co-compact relative to X if every cover of A by θ-sg-clopen sets of X has finite subcover.
(iii)TSGO-compact if every cover of X by θ-sg-open sets (= θ-sg-open cover) has a finite subcover.

Definition 53. A subset A of a space X is called TSGO-compact relative to X if every collection $\{U_i : i \in I\}$ of θ-sg-open subsets of X such that $A \subset \bigcup\{U_i : i \in I\}$, there exists a finite subset I_o of I such that $A \subset \bigcup\{U_i : i \in I_o\}$.

Definition 54. A subset A of space X is called TSGO-compact if A is TSGO-compact as a subspace of X.

Theorem 55.
(i) Every θ-sg-closed subset of TSGO-compact space is TSGO-compact relative to X.
(ii) The surjective θ-sg-continuous image of a TSGO-compact space is compact.
(iii) If $f : X \to Y$ is θ-sg-irresolute and a subset A of X TSGO-compact relative to X, then its image f(A) is TSGO-compact relative to Y.

Proof. (i) Let A be a θ-sg-closed subset of a TSGO-compact space X. Let $\{U_i : i \in I\}$ be a cover of A by θ-sg-open subsets of X. So $A \subset \bigcup\{U_i : i \in I\}$ and then $(X - A) \cup (\bigcup\{U_i : i \in I\}) = X$. Since X is TSGO-compact, there exists a finite subset I_o of I such that $(X - A) \cup (\bigcup\{U_i : i \in I_o\}) = X$. Then $A \subset \bigcup\{U_i : i \in I_o\}$. Hence A is TSGO-compact relative to X.
(ii) Let X be a TSGO-compact space and $f : X \to Y$ be surjective θ-sg-continuous function. Let $\{U_i : i \in I\}$ be a cover of X by open sets. Then $\left\{f^{-1}(U_i) : i \in I\right\}$ is a cover of X by θ-sg-open sets, since f is θ-sg-continuous. By TSGO-compactness of X, there is finite subset I_o of I such that $X = \bigcup\left\{f^{-1}(U_i) : i \in I_o\right\}$. Since f is surjective, $Y = \bigcup\left\{f^{-1}(U_i) : i \in I_o\right\}$ and hence Y is compact.
(iii) is similar to (ii).

Theorem 56. *If a function $f : X \to Y$ is totally θ-sg-continuous and B is TSGO-co-compact relative to X, then f(B) is compact in Y.*

Proof. Let $\{H_i : i \in I\}$ be any cover of $f(B)$ by open sets of Y.For each $x \in B$, there exists $i_x \in I$ such that $f(x) \in (H_\alpha)_x$ and there exists $U_x \in \theta\mathrm{SGO}(X)$ containing x such that $f(U_x) \subset (H_\alpha)_x$. Since the family $U_x : x \in B$ is a cover of B by θ-sg-clopen sets of X, there exists a finite subset B_0 of B such that $B \subset \bigcup\{U_x : x \in B_0\}$. Therefore, we get $f(B) \subset \bigcup\{f(U_x) : x \in B_0\} \subset \bigcap\{(H_\alpha)_x : x \in B_0\}$ and hence $f(B)$ is compact.

Theorem 57. *For a function $f : X \to Y$ the following are equivalent*
(i) f is totally θ-sg-continuous.
(ii) For every open set V of Y, $f^{-1}(V)$ is θ-sg-clopen in X.
(iii) For every closed set F of Y, $f^{-1}(F)$ is θ-sg-clopen in X.

Proof.(i)⇒(ii) Let V be an open set in Y and let $x \in f^{-1}(V)$. Then $f(x) \in V$. By (i), there exists a θ-sg-clopen set U_x in X such that $x \in U_x \subset f^{-1}(V)$. This implies $f^{-1}(V)$ is θ-sg-clopen nhd of x. Since x is arbitrary, $f^{-1}(V)$ is θ-sg-clopen nhd of each of its points. This implies $f^{-1}(V)$ is θ-sg-clopen in X.
(ii)⇒(i) Let H be any open set in Y containing $f(x)$, so that $x \in f^{-1}(H)$. Also by (ii), $f^{-1}(H)$ is θ-sg-clopen in X. Set $G = f^{-1}(H)$, then G is θ-sg-clopen set in X containing x and $f(G) = f(f^{-1}(H)) \subset H$. This implies f is totally θ-sg-continuous at $x \in X$. Since x is arbitrary, it follows that f is totally θ-sg-continuous at every point x of X. Therefore, f is totally θ-sg-continuous.
(ii)⇒(iii) Let F be a closed set in Y. Then $Y - F$ is an open set in Y. By (ii), $f^{-1}(Y - F) = X - f^{-1}(F)$ is θ-sg-clopen in X. This implies $f^{-1}(F)$ is θ-sg-clopen in X.
(iii)⇒(ii) Let F be an open set in Y, then $Y - F$ is a closed set in Y. By (iii), $f^{-1}(Y - F) = X - f^{-1}(F)$ is θ-sg-clopen in X. This implies $f^{-1}(F)$ is θ-sg-clopen in X.

Theorem 58. *If $f : X \to Y$ is a totally θ-sg-continuous injection function and Y is T_0, then X is θ-sg-T_2.*

Proof. Let x and y be any two distinct points in X. Since f is injective $f(x)$ and $f(y)$ are distinct points in Y. Since Y is T_0, there exists an open set U containing say $f(x)$ but not $f(y)$, which implies, $x \in f^{-1}(U)$ and $y \notin f^{-1}(U)$. Since f is totally θ-sg-continuous, $f^{-1}(U)$ is a θ-sg-clopen subset of X. Thus for two distinct points x and y of X, there exist two disjoint θ-sg-clopen subsets of X such that $x \in f^{-1}(U)$ and $y \notin X - f^{-1}(U)$. Hence X is θ-sg-T_2.

Theorem 59. *Let $f : X \to Y$ be totally θ-sg-continuous injection function and Y is T_1-space. If A is a TSGO-connected subset of X, then $f(A)$ is a single point.*

Proof. Let A be a θ-sg-connected subset of X such that $f(A)$ is not a single point. Let x and y be distinct points of subset A in X. Since f is injective $f(x)$ and $f(y)$ are distinct points of $f(A)$ in Y. Since Y is T_1-space, there exist open sets U and V such that $f(x) \in U$, $f(y) \notin U$ and $f(y) \in V$, $f(x) \notin V$. This implies $x \in f^{-1}(U)$, $y \notin f^{-1}(U)$ and $y \in f^{-1}(V)$, $x \notin f^{-1}(V)$. Since f is totally θ-sg-continuous, implies $f^{-1}(U)$ and $f^{-1}(V)$ are proper θ-sg-clopen sets in A. This is contradiction to the fact that, A is TSGO-connected subset of X. Therefore, $f(A)$ is a single point.

Definition 60. Let X be a topological space. Then the set of all points y in X such that x and y cannot be separated by a θ-sg-separation of X is said to be quasi-θ-sg-component of X.

Theorem 61. *If $f : X \to Y$ is totally θ-sg-continuous injection function from a topological space X in to T_1 space Y, then f is constant on each quasi-θ-sg-component of X.*

Proof. Let x and y be two points of X that lie in the same quasi-θ-sg-component of X. Then $f(x)$ and $f(y)$ are elements of Y. Assume $f(x) = \alpha \neq \beta = f(y)$. Since Y is T_1, $\{\alpha\}$ is closed set in Y and so $Y - \{\alpha\}$ is an open set in Y. Since f is totally θ-sg-continuous, $f^{-1}(\{\alpha\})$ and $f^{-1}(Y - \{\alpha\})$ are disjoint θ-sg-clopen subsets of X. Further, $x \in f^{-1}(\{\alpha\})$ and $y \in f^{-1}(Y - \{\alpha\})$, which is a contradiction to the fact that y belongs to the quasi-θ-sg-component of x and hence y must belong to every θ-sg-clopen set containing x. This proves the result.

Definition 62. A collection $\{A_i : i \in I\}$ of θ-sg-clopen sets in a topological space X is called θ-sg-clopen cover of a subset A in X if $A \subset \bigcup_{i\in I} A_i$.

Definition 63. A space X is said to be
(i) Mildly TSGO-compact if every cover of X by θ-sg-clopen sets has a finite subcover.
(ii) Mildly countably TSGO-compact if every countable cover of X by θ-sg-clopen sets has a finite subcover.
(iii) Mildly θ-sg-Lindelöf if every cover of X by θ-sg-clopen sets has a countable subcover.
(iv) Strongly S-closed [5] if every closed cover of X has a finite subcover.

Theorem 64. *If $f : X \to Y$ is totally θ-sg-continuous surjection and X is mildly TSGO-compact, then Y is strongly S-closed.*

Proof. Let $\{V_\alpha : \alpha \in I\}$ be any closed cover of Y. Since f is totally θ-sg-continuous, imply $\left\{f^{-1}(V_\alpha) : \alpha \in I\right\}$ is θ-sg-clopen cover of X. Since X is mildly TSGO-compact, there exists a finite subset I_0 of I such that $X = \cup\left\{f^{-1}(V_\alpha) : \alpha \in I_0\right\}$. This implies, $Y = \cup\{V_\alpha : \alpha \in I_0\}$, which is finite subcover of Y. Therefore, Y is strongly S-closed.

Theorem 65. *Let $f : X \to Y$ be a totally θ-sg-continuous surjection. Then, the following properties hold.*

(i) If X is mildly TSGO-compact, then Y is compact.
(ii) If X is mildly countably TSGO-compact, then Y is countably compact.
(iii) If X is mildly θ-sg-Lindelöf, then Y is Lindelöf.

Proof. (i) Let $\{V_\alpha : \alpha \in I\}$ be any open cover of Y. Since f is totally θ-sg-continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is θ-sg-clopen cover of X. Since X is mildly TSGO-compact, there exists a finite subset I_0 of I such that

$$X = \cup\left\{f^{-1}(V_\alpha) : \alpha \in I_0\right\}.$$

This implies, $Y = \cup\{V_\alpha : \alpha \in I_0\}$, which is finite subcover of Y. Therefore, Y is compact.
(ii) Let $\{V_\alpha : \alpha \in I\}$ be any countable open cover of Y. Since f is totally θ-sg-continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is countable θ-sg-clopen cover of X. Since X is mildly countably TSGO-compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. This implies, $Y = \cup\{V_\alpha : \alpha \in I_0\}$, which is finite subcover of Y. Therefore, Y is countably compact.
(iii) Let $\{V_\alpha : \alpha \in I\}$ be any open cover of Y. Since f is totally θ-sg-continuous, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is θ-sg-clopen cover of X. Since X is mildly θ-sg-Lindelöf, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. This implies, $Y = \cup\{V_\alpha : \alpha \in I_0\}$, which is finite subcover of Y. Therefore, Y is Lindelöf.

Definition 66. A space X is said to be
(i) θ-sg-co-T_1 if for each pair of distinct points x and y of X there exist θ-sg-clopen sets G and H such that $x \in G$, $y \notin G$ and $y \in H$, $x \notin H$.
(ii) θ-sg-co-T_2 if for each pair of distinct points x and y of X, there exist disjoint θ-sg-clopen sets G and H such that $x \in G$ and $y \in H$.

Theorem 67. *A topological space X is θ-sg-co-T_1 if every singleton subset $\{x\}$ of X is θ-sg-clopen set.*

Proof. Suppose $\{x\}$ is θ-sg-clopen set for every $x \in X$. Let x and $y \in X$ with $x \neq y$. Now $x \neq y$ implies $y \in X - \{x\}$. Hence $X - \{x\}$ is θ-sg-clopen set containing y but not x. Similarly, $X - \{y\}$ is θ-sg-clopen set containing x but not y. Therefore, X is θ-sg-co-T_1 space.

Theorem 68. *If $f : X \to Y$ is totally θ-sg-continuous injection and Y is T_1, then X is θ-sg-co-T_1 space.*

Proof. Let $f : X \to Y$ is totally θ-sg-continuous injection and Y is T_1. For any two distinct points x_1, x_2 of X there exist distinct points y_1, y_2 of Y such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since Y is T_1-space there exist open sets U and V in Y such that $y_1 \in U$, $y_2 \notin U$ and $y_1 \notin V$, $y_2 \in V$. That is

$x_1 \in f^{-1}(U), x_1 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V), x_2 \notin f^{-1}(U)$. Since f is totally θ-sg-continuous $f^{-1}(U)$, $f^{-1}(V)$ are θ-sg-clopen sets in X. Thus, for two distinct points x_1, x_2 of X there exist θ-sg-clopen sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1 \in f^{-1}(U), x_1 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V), x_2 \notin f^{-1}(U)$. Therefore, X is θ-sg-co-T_1 space.

Theorem 69. *If $f : X \to Y$ is totally θ-sg-continuous injection and Y is T_2, then X is θ-sg-co-T_2 space.*

Proof. Let $f : X \to Y$ is totally θsg-continuous injection and Y is T_2. For any two distinct points x_1, x_2 of X there exist distinct points y_1, y_2 of Y such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since Y is T_2 space there exist disjoint open sets U and V in Y such that $y_1 \in U$ and $y_2 \in V$. That is $x_1 \in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$. Since f is totally θ-sg-continuous $f^{-1}(U)$, $f^{-1}(V)$ are θ-sg-clopen sets in X. Further f is injective, $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\phi) = \phi$. Thus, for two disjoint points x_1, x_2 of X there exist distinct θ-sg-clopen sets $f^{-1}(U)$ and $f^{-1}(V)$ such that $x_1 \in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$. Therefore, X is θ-sg-co-T_2 space.

Definition 70. A topological space X is said to be
(i) θ-sg-co-regular if for each closed set F and each point $x \notin F$, there exist disjoint θ-sg-clopen sets U and V such that $x \in U$ and $F \subset V$.
(ii) strongly θ-sg-co-regular if for every θ-sg-clopen set F and a point $x \notin F$, there exist disjoint open sets U and V such that $x \in U$ and $F \subset V$.
(iii) θ-sg-co-normal if for each pair of disjoint closed sets A and B, there exists a pair of disjoint θ-sg-clopen sets U and V in X such that $A \subset U$ and $B \subset V$.
(iv) strongly θ-sg-co-normal if for each pair of disjoint θ-sg-clopen sets A and B, there exists a pair of disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.

Theorem 71. *If $f : X \to Y$ is totally θ-sg-continuous, closed, injection and Y is regular, then X is θ-sg-co-regular.*

Proof. Let F be a closed set in X and $x \notin F$. Since f is closed injection $f(F)$ is closed set in Y such that $f(x) \notin f(F)$. Now Y is regular, there exist disjoint open sets G and H such that $f(x) \in G$ and $f(F) \subset H$. This implies $x \in f^{-1}(G)$ and $F \subset f^{-1}(H)$. Since f is totally θ-sg-continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are θ-sg-clopen sets in X. Further $f^{-1}(G) \cap f^{-1}(H) = \phi$. Hence X is θ-sg-co-regular.

Theorem 72. *If $f : X \to Y$ is totally θ-sg-continuous, open, injection and X is strongly θ-sg-co-regular, then Y is regular.*

Proof. Let F be a closed set in Y and $y \notin F$. Take $y = f(x)$ for some $x \in X$, then $x \notin f^{-1}(F)$. Since f is totally θ-sg-continuous, $f^{-1}(F)$ is θ-sg-clopen set in X, not containing x. Since X is strongly θ-sg-co-regular, there exist disjoint open sets U and V such that $x \in U$ and $f^{-1}(F) \subset V$. That is $f(x) \in f(U)$ and $F \subset f(V)$. Since f is open injective, $f(U)$ and $f(V)$ are disjoint open sets in Y. Therefore, Y is regular.

Theorem 73. *If $f : X \to Y$ is totally θ-sg-continuous, closed, injection and Y is normal, then X is θ-sg-co-normal.*

Proof. Let E and F be disjoint closed sets in Y. Since f closed injection $f(E)$ and $f(F)$ are disjoint closed sets in Y. Now Y is normal, there exist disjoint open sets G and H such that $f(E) \subset G$ and $f(F) \subset H$. This implies $E \subset f^{-1}(G)$ and $F \subset f^{-1}(H)$. Since f is totally θ-sg-continuous $f^{-1}(G)$ and $f^{-1}(H)$ are θ-sg-clopen sets in X. Further $f^{-1}(G) \cap f^{-1}(H) = \phi$. Hence X is θ-sg-co-normal.

Theorem 74. *If $f : X \to Y$ is totally θ-sg-continuous, open, injection and X is strongly θ-sg-co-normal, then Y is normal.*

Proof. Let E and F be disjoint closed set in Y. Since f is totally θ-sg-continuous, $f^{-1}(E)$ and $f^{-1}(F)$ are θ-sg-clopen sets in X. Since X is strongly θ-sg-co-normal, there exist disjoint open sets U and V such that $f^{-1}(E) \subset U$ and $f^{-1}(F) \subset V$. That is $E \subset f(U)$ and $F \subset f(V)$. Since f is open injective, $f(U)$ and $f(V)$ are disjoint open sets in Y. Therefore, Y is normal.

Definition 75. A function $f : X \to Y$ is said to be:
(i) totally θ-sg-irresolute if the preimage of a θ-sg-clopen subset of Y is a θ-sg-clopen subset of X.
(ii) totally pre-θ-sg-clopen if the image of every θ-sg-clopen subset of X is θ-sg-clopen subset of Y.

Theorem 76. *Let $f : X \to Y$ be surjective totally θ-sg-irresolute and totally pre-θ-sg-clopen and $g : Y \to Z$ be any function. Then $g \circ f : X \to Z$ is totally θ-sg-continuous if and only if g is totally θ-sg-continuous.*

Proof. The 'if' part is obvious. To prove 'only if' part, let $g \circ f : X \to Z$ be totally θ-sg-continuous and let V be an open subset of Z. Then $(g \circ f)^{-1}$(V) is a θ-sg-clopen subset of X, that is $f^{-1}(g^{-1}(V))$ is θ-sg-clopen. Since f is totally pre-θ-sg-clopen, $f(f^{-1}(g^{-1}(V)))$ is a θ-sg-clopen subset of Y. So, $g^{-1}(V)$ is θ-sg-clopen in Y. Hence g is totally θ-sg-continuous.

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