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A NUMERICAL APPROACH FOR GENERALIZED VARIATIONAL-LIKE INEQUALITY PROBLEMS

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Abstract: A pair of generalized variational-like inequality problems $(GVLIP)$ and its corresponding dual variational-like inequality problems $(GDVLIP)$ are defined. The equivalence theorem of the problems $(GVLIP)$ and $(GDVLIP)$ is studied using the concept of affineness in the first quadrant of a unit circle via a transformation T^*. Again a pair of generalized complementarity-like problems and its dual complementarity problems are defined. The existence of their solutions are shown in the presence of the problems $(GVLIP)$ and $(GDVLIP)$. A numerical study of the problems $(GVLIP)$ has been undertaken in η-invex set using auxiliary principle technique.

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1. Introduction

Variational inequality problems was introduced by Stampacchia [7] in 1964. The theory of variational inequality problems is the best tool to study the exis-

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tence of solution of various inequality problems arises in Engineering branches, Applied physical and Applied mathematical branches, Finance, etc. The researchers have investigated various types of variational inequalities such as nonlinear variational inequality problems, variational-like inequality problems, generalized variational inequality problems, vector variational inequality problems, variational inequality problems etc. associated with various functions.

In our study we concentrate to form a generalized variational inequalities and to study their existence of the solutions under the affineness property.

1.1. The Problems

Let $\mathbb{X}$ be a separable Banach space and X be a topological vector space in $\mathbb{X} \setminus \{0\}$. Let for each $x, y \in V \subset X$, $(xy)^{-1} = (yx)^{-1}$. Let $K \subset X$ and $\eta : K \times K \to X$ be any map. Let $T : K \to L(X, \mathbb{R}^n)$ be any map where $L(X, \mathbb{R}^n)$ is the set of continuous linear functionals from X to the finite dimensional vector space $\mathbb{R}^n$. Let $\phi : L(X, \mathbb{R}^n) \to L(X, \mathbb{R}^n)$ be any map. Let a map $T^* : X \to L(X, \mathbb{R}^n)$ by the rule $T^*(x) = \phi_x(T(x)) = -T\left(x^{-1}\right)$ for all $0 \neq x \in X$, 0 otherwise. Consider $F^* : X \to \mathbb{R}^n$ is a mapping corresponds to the map $F : X \to \mathbb{R}^n$ by the rule $F^*(x) = F\left(x^{-1}\right)$ for all $0 \neq x \in X$, 0 otherwise. Let $\theta : X \times X \times X \to X$ be any mapping defined by $\theta(x, y; v) = (v)^{-1}\, \eta(x, y)$ for all $0 \neq x, y \in K$ and $v \in X$, 0 otherwise. Let W^* be any set defined by

$$\begin{aligned} W^* &= \left\{ f \in L(X, \mathbb{R}^n) : \left\langle f\left(u^{-1}\right), z^{-1} \right\rangle = \right. \\ &\quad \left. - \left\langle \phi_u\left(f\left(u\right)\right), z \right\rangle = - \left\langle f^*(u), z \right\rangle \ \forall\, z \in X \right\} \end{aligned}$$

where ϕ_u is called *function of relaxation*.

The generalized variational inequalities are defined as follows

(A) The generalized variational-like inequality problems $(GVLIP)$ is to find: $y \in K$ such that

$$\langle T^*(y), \theta(x, y; xy) \rangle \geq 0 \ \text{ for all } \ x \in K. \qquad (GVLIP)$$

(B) The generalized dual variational-like inequality problems $(GDVLIP)$ is to find: $y \in K$ such that

$$\langle T^*(x), \theta(y, x; xy) \rangle \leq 0 \ \text{ for all } \ x \in K. \qquad (GDVLIP)$$

(C) The generalized complementarity-like problems $(GCLP)$ is to find: $y \in K$ such that

$$\langle T^*(y), \theta(x, y; xy) \rangle = 0 \ \text{ for all } \ x \in K. \qquad (GCLP)$$

(D) The generalized dual complementarity-like problems $(GDCLP)$ is to find: $y \in K$ such that

$$\langle T^*(x), \theta(y, x; xy) \rangle = 0 \text{ for all } x \in K. \qquad (GDCLP)$$

Remark 1. The problem $(GVLIP)$ coincides with $(GVIP)$ and the problem $(GCLP)$ coincides with (GCP) if $n = 1$, $\eta(x, y) = x - y$, $xy = 1$, $\phi(x) = I_{X^*}$ (the identity map in X^*) and $K \subset X = \mathbb{R}$.

1.2. Preliminaries

The mathematical formulation of the variational inequality problem is defined as follows. Let K be nonempty closed and convex subset of a reflexive real Banach space X with dual X^* and $T : K \to X^*$ a nonlinear map. The pair $\langle f, x \rangle$ represent the value of $f \in X^*$ at $x \in X$.
(a) The *variational inequality problems* is to find $x_0 \in K$ such that for all $x \in K$,

$$\langle T(x_0), x - x_0 \rangle \geq 0. \qquad (VIP)$$

It is well known that the structure of convex set K formed by the line segment in the direction $v = x - y \in X$ as

$$K = \{y + t(x - y) \in K : x, y \in K, t \in [0, 1]\}.$$

To generalize the concept of convex function, Hanson ([5],1981) has developed the differentiable function by replacing the direction $x-y \in X$ with a bifunction $\eta(x, y) \in X$ where term invex was coined by Craven ([2], 1981). This concept brought the imagination to develop the $GVIP$ as follows
(c) The *generalized variational inequality problems* **(*VIP*)** is to find $y \in K$ such that for all $x \in K$,

$$\langle T(y), \eta(x, y) \rangle \geq 0. \qquad (GVIP)$$

(d) The *generalized complementarity problems* **(*VIP*)**[1] is to find $y \in K$ such that for all $x \in K$,

$$\langle T(y), \eta(x, y) \rangle = 0. \qquad (GCP)$$

For our need, we recall the concept of η-invex set as follows.

Definition 2. [1] Let $\eta : K \times K \to X$ be any map. The set K is said to be

(a) η-*invex* set at $y \in K$ if for all $x \in K$ and $t \in [0, 1]$,

$$y + t\eta(x, y) \in K,$$

(*b*) *η-invex* set on K if K is η-invex for each $y \in K$.

Note 3. The interior of invex combinations connecting $x, y \in K$ is the set

$$K(\eta) = \{y + t\eta(x, y) \in K : x, y \in K, t \in (0, 1)\} \subset K$$

so K is η-invex if the interior of K is $K(\eta)$, i.e., $\overset{\circ}{K} = K(\eta)$.

Definition 4. [1] The mapping $T : K \to X^*$ is said to be

(*i*) *η-monotone* at $y \in K$ if for all $x \in K(\eta)$,

$$\langle T(x), \eta(y, x)\rangle + \langle T(y), \eta(x, y)\rangle \leq 0.$$

(*ii*) *η-monotone* on K if for all $x, y \in K$,

$$\langle T(x), \eta(y, x)\rangle + \langle T(y), \eta(x, y)\rangle \leq 0.$$

For strictly η-monotonicity case, equality hold in the above equation for $x = y$ only.

2. Generalized Variational Inequality Problems

Throughout of this section, $\mathbb{X}$ is considered as a Banach space, X is a topological vector space in $\mathbb{X} \setminus \{0\}$, $K \subset X$ and $\eta : K \times K \to X$ is any map any $T : K \to L(X, \mathbb{R}^n)$ be any map.

2.1. Equivalence Theorem

The following theorem proves the equivalence between the problems $(GVLIP)$ and $(GDVLIP)$. For simplicity we write $\theta(x, y; v) = \theta(x, y)$ if $v = xy \in V = \{xy \in K : x, y \in K\}$.

Theorem 5. *Let K be a η-invex set. If the mapping $T^* : X \to L(X, \mathbb{R}^n)$ is η-monotone on $K(\eta)$ and $y \in K(\eta)$ solves the problem $(GVLIP)$, then $y \in K(\eta)$ solves the problem $(GDVLIP)$.*

Proof. Since $y \in K$ solves the problem $(GVLIP)$, we have

$$\langle T^*(y), \theta(x, y)\rangle \geq 0$$

for all $x \in K$. We have T is η-monotone on K, implying T^* is η-monotone at $y \in K$, i.e.,

$$\langle T^*(x), \theta(y,x)\rangle + \langle T^*(y), \theta(x,y)\rangle \leq 0$$

for all $x \in K(\eta)$, i.e.,

$$\begin{aligned} \langle T^*(x), \theta(y,x)\rangle &= -\langle T^*(y), \theta(x,y)\rangle \\ &\leq 0 \end{aligned}$$

for all $x \in K$, i.e., $y \in K$ solves the problem $(GDVLIP)$. This completes the proof. □

Definition 6. Let η is said to be *affine* on $K(\eta)$ if for all $x, y \in K$, $u \in K(\eta) \subset K$, η satisfies the either conditions (a) or (b) given as follows

(A) for all $t \in [0,1]$ satisfying the condition $t^2 + (1-t)^2 = 1$, i.e., $t(1-t) < \frac{1}{2}$,

(a) $\eta(y + t\eta(x,y), u) = t\eta(x,u) + (1-t)\eta(y,u)$,

(b) $\eta(u, y + t\eta(x,y)) = (1-t)\eta(u,y) + t\eta(u,x)$.

(B) for all $t \in [0,1]$ satisfying the condition $t^2 + \rho t(1-t) = 1$ for some relaxation $\rho \in \mathbb{R}$ to find $t < 1$,

(a) $\eta(y + t\eta(x,y), u) = t\eta(x,u) + (1-t)\eta(y,u)$,

(b) $\eta(u, y + t\eta(x,y)) = (1-t)\eta(u,x) + t\eta(u,y)$.

Note 7. It can be easily seen that $\theta(x,y;v)$ is affine on $K(\eta)$ if η is said to be affine on $K(\eta)$.

Definition 8. The mapping $T^* : K \to L(X, \mathbb{R}^n)$ is said to be

(i) *θ-monotone* at $y \in K(\eta)$ if for all $x \in K(\eta)$,

$$\langle T^*(x), \theta(y,x;v)\rangle + \langle T^*(y), \theta(x,y;v)\rangle \leq 0.$$

(ii) *θ-monotone* on $K(\eta)$ if for all $x, y \in K(\eta)$,

$$\langle T^*(x), \theta(y,x;v)\rangle + \langle T^*(y), \theta(x,y;v)\rangle \leq 0.$$

For *strictly θ-monotonicity* case, equality hold in the above equation for $x = y$ only. For *θ-dissipativeness* of T^*, the symbol $\leq$ will be replaced by $\geq$.

Note 9. T is η-monotone on $K(\eta)$ if and only if T^* is θ-monotone on $K(\eta)$ with $v = xy$.

Lemma 10. *Let $K \subset X$. Let $\eta : K \times K \to X$ be any mapping $\eta(x,x) = 0$ for all $x \in K$. If K is η-invex and η is affine on $K = K(\eta)$, then for any map $T \in W^*$, we have*

$$\begin{array}{ll} (a) & t^2 \langle T^*(u), \theta(x,y;v)\rangle + (1-t)^2 \langle T^*(u), \theta(y,x;v)\rangle = 0, \\ (b) & t^2 \langle T^*(x), \theta(x,y)\rangle + (1-t)^2 \langle T^*(x), \theta(y,x)\rangle = 0 \\ (c) & t^2 \langle T^*(y), \theta(x,y)\rangle + (1-t)^2 \langle T^*(y), \theta(x,y)\rangle = 0 \end{array}$$

$u \in \mathbb{I}(x,y)$ the invex path joining $x, y \in K$ contained in K, $v \in V$ and for all $t \in [0,1]$ satisfying the condition $t^2 + (1-t)^2 = 1$, i.e., $t(1-t) < \frac{1}{2}$.

Proof. Since K is η-invex and η is affine on $K = K(\eta)$, for all $x, y \in K$, we have $u = y + t\eta(x,y) \in K$ for all $t \in [0,1]$ satisfying the condition $t^2+(1-t)^2 = 1$. Before proving the theorem, we first show,

$$t^2\theta(x,y;v) + (1-t)^2\theta(y,x;v) = 0.$$

Since $\eta(x,x) = 0$ for all $x \in K$, we have

$$\begin{aligned} 0 = \eta(u,u) &= t\eta(x,u) + (1-t)\eta(y,u) \\ &= t\left[(1-t)\eta(x,x) + t\eta(x,y)\right] \\ &\quad (1-t)\left[t\eta(y,y) + (1-t)\eta(y,x)\right] \\ &= t^2\eta(x,y) + (1-t)^2\eta(y,x) \end{aligned}$$

for all $t \in [0,1]$ satisfying the condition $t^2 + (1-t)^2 = 1$. Therefore

$$t^2v^{-1}\eta(x,y) + (1-t)^2v^{-1}\eta(y,x) = 0,$$

i.e.,

$$t^2\theta(x,y;v) + (1-t)^2\theta(y,x;v) = 0,$$

for all $x, y \in K$, $v \in X$, $t \in [0,1]$ satisfying $t^2 + (1-t)^2 = 1$, i.e., $t(1-t) < \frac{1}{2}$. Hence

$$\begin{aligned} & t^2 \langle T^*(u), \theta(x,y;v)\rangle + (1-t)^2 \langle T^*(u), \theta(y,x;v)\rangle \\ &\quad = \left\langle T^*(u), t^2\theta(x,y;v) + (1-t)^2\theta(y,x;v)\right\rangle \\ &\quad = \langle T^*(u), 0\rangle = 0 \end{aligned}$$

for all $x, y \in K, u = y + t\eta(x,y) \in K$, $v \in X$ and $t \in [0,1]$ satisfying the condition $t^2 + (1-t)^2 = 1$, i.e., $t(1-t) < \frac{1}{2}$. This proves (a). Proof of (b) follows from (a) by choosing $u = x$ and $v = xy$. Proof of (c) follows from (a) by choosing $u = y$ and $v = xy$ This completes the proof. □

The following theorem establishes the existence of the problem $GVLIP$ in the presence of the problem $GDVLIP$.

Theorem 11. *Let $K \subset X$. Let $\eta : K \times K \to X$ be any mapping $\eta(x,x) = 0$ for all $x \in K$. Let K be η-invex on $K(\eta) \subset K$. For any T^* corresponds to T in W^*, assume that*

(a) η is affine on K,

(b) for any $T \in W^*$, $x, y \in K$, there exists a $u \in K(\eta)$ and for all $v \in V$,

$$\langle T^*(u), \eta(y,x;v)\rangle \le \langle T^*(x), \eta(y,x;v)\rangle$$

If $\overline{y} \in K(\eta)$ solves the problem $(GDVLIP)$, then $\overline{y} \in K(\eta)$ solves the problem $(GVLIP)$.

Proof. Since $y \in K(\eta)$ solves the problem $(GDVLIP)$, we have

$$\langle T^*(x), \theta(\overline{y}, x; v)\rangle \le 0$$

for all $x \in K(\eta)$ where T^* corresponds to T in W^*. Since η is affine on K, by Lemma 10, we have,

$$t^2 \langle T^*(u), \theta(x, \overline{y}; v)\rangle + (1-t)^2 \langle T^*(u), \theta(y, x; v)\rangle = 0$$

for all $x, y \in K$, $u \in \mathbb{I}(x,y) \subset K(\eta)$, $v \in V$ and $t \in [0,1]$ satisfying the condition $t^2 + (1-t)^2 = 1$, i.e., $t(1-t) < \frac{1}{2}$. Taking $y = \overline{y}$ to obtain

$$\begin{aligned} t^2 \langle T^*(u), \theta(x, \overline{y}; v)\rangle &= -(1-t)^2 \langle T^*(u), \theta(\overline{y}, x; v)\rangle \\ &\ge -(1-t)^2 \langle T^*(x), \theta(\overline{y}, x; v)\rangle \ge 0, \end{aligned}$$

i.e.,

$$\langle T^*(u), \theta(x, \overline{y}; v)\rangle \ge 0$$

for all $x, y \in K$, $u \in K(\eta)$, $v \in V$ and $t \in [0,1]$ satisfying the condition $t^2 + (1-t)^2 = 1$, i.e., $t(1-t) < \frac{1}{2}$. Choosing $u = \overline{y}$ and $v = x\overline{y}$ we have i.e.,

$$\langle T^*(\overline{y}), \theta(x, \overline{y}; v)\rangle \ge 0$$

for all $x \in K(\eta)$, i.e., $y \in K(\eta)$ solves the problem $(GVLIP)$. This completes the proof. □

Theorem 12. *Let $K \subset X$. Let $\eta : K \times K \to X$ be any mapping $\eta(x,x) = 0$ for all $x \in K$. Let K be harmonically η-invex on $K(\eta) \subset K$. For any T^* corresponds to T in W^*, assume that*

(a) *T^* is harmonically η-monotone on K,*

(b) *η is affine on K,*

(c) *for any $T \in W^*$, $x, y \in K$ and $u \in K(\eta)$,*

$$\langle T^*(u), \theta(y, x; v)\rangle \leq \langle T^*(x), \theta(y, x; v)\rangle$$

for all $v \in V = \{xy \in K : x, y \in K\}$.

then the problems $GVLIP$ and $GDVLIP$ are equivalent.

Proof. As T^* is θ-monotone on K, T is harmonically η-monotone on K. Thus all the conditions of Theorem 5 and Theorem 11 are satisfied. Therefore both the problems $GVLIP$ and $GDVLIP$ are equivalent. This completes the proof. □

2.2. Condition $C^\star$ and GCLP

For our need we define the following condition.

Definition 13. Let X be a topological vector space and K be a nonempty η-invex subset of $X \setminus \{0\}$ where $\eta : K \times K \to X$ is any vector function. The function θ is said to satisfy *condition* $C^\star$ associated with h on $K(\eta)$ if there exists a well defined function $h : K \times K \times K(\eta) \to K$ such that $h(x, y; y + \lambda\eta(x, y)) \in K$ for all $x, y \in K$, $t \in (0, 1)$ and satisfy the following conditions

(a) $\eta(x, y) + \eta(y, x) = 0$ for all $x, y \in K$,

(b) for all $x, y \in K$, $x_\lambda = h(x, y; y + \lambda\eta(x, y))$ and $\lambda \in (0, 1)$, we have

$$\theta(x_\lambda, y; x_\lambda y) = (1 - \lambda)\theta(x, y; xy),$$

(c) for all $x, y \in K$, $x_\lambda = h(x, y; y + \lambda\eta(x, y))$ and $\lambda \in (0, 1)$, we have

$$\theta(x_\lambda, x; x_\lambda x) = -\lambda\theta(x, y; xy),$$

(d) for all $x, y \in K$, $x_\lambda = h(y, x; x + \lambda\eta(y, x))$ and $\lambda \in (0, 1)$, we have

$$\theta(x_\lambda, y; x_\lambda y) = -\lambda\theta(y, x; xy),$$

(e) for all $x, y \in K$, $x_\lambda = h(y, x; x + \lambda\eta(y, x))$ and $\lambda \in (0, 1)$, we have

$$\theta(x_\lambda, x; x_\lambda x) = (1 - \lambda)\theta(y, x; xy),$$

Theorem 14. *Let $\overline{y} \in K$ solves the problems $GVLIP$ where θ satisfies condition C^* associated with h on $K(\eta)$, then $\overline{y} \in K$ solves the problems $GCLP$.*

Proof. Since $\overline{y} \in K$ solves the problems $GVLIP$, we have

$$\langle T^*(\overline{y}), \theta(x, \overline{y}; x\overline{y})\rangle \geq 0 \tag{1}$$

For some $y \in K$, replace x by $h(x, y; y + \lambda\eta(x, y))$ in (1) and use condition C^* to obtain

$$-\lambda \langle T^*(\overline{y}), \theta(x, y; xy)\rangle \geq 0,$$

i.e.,

$$\langle T^*(\overline{y}), \theta(x, y; x\overline{y})\rangle \leq 0,$$

for all $x \in K$ as $\lambda \in (0, 1)$. Taking $y = \overline{y}$, we have

$$\langle T^*(\overline{y}), \theta(x, \overline{y}; x\overline{y})\rangle \leq 0 \tag{2}$$

for all $x \in K$ which together with (1) gives

$$\langle T^*(\overline{y}), \theta(x, \overline{y}; x\overline{y})\rangle = 0$$

for all $x \in K$, i.e., $\overline{y} \in K$ solves the problems $GCLP$. This completes the proof. □

Theorem 15. *Let $\overline{y} \in K$ solves the problems $GDVLIP$ where θ satisfies condition C^* associated with h on $K(\eta)$, then $\overline{y} \in K$ solves the problems $GDCLP$.*

Proof. Since $\overline{y} \in K$ solves the problems $GDVLIP$, we have

$$\langle T^*(x), \theta(\overline{y}, x; x\overline{y})\rangle \leq 0 \tag{3}$$

Replacing x by $h(x, \overline{y}; \overline{y} + \lambda\eta(x, \overline{y}))$ in (3) and using condition C^* to obtain

$$-\lambda \langle T^*(x), \theta(y, x; xy)\rangle \leq 0,$$

for all $x \in K$. Taking $y = \overline{y}$, we have

$$\langle T^*(x), \theta(\overline{y}, x; x\overline{y})\rangle \geq 0 \tag{4}$$

for all $x \in K$ which together with (3) gives

$$\langle T^*(x), \theta(\overline{y}, x; x\overline{y})\rangle = 0$$

for all $x \in K$, i.e., $\overline{y} \in K$ solves the problems $GDCLP$. This completes the proof. □

Theorem 16. *Let $\overline{y} \in K$ solves the problems $GDVLIP$. Let T^* satisfies the condition*

$$\langle T^*(x_\lambda), \theta(y, x_\lambda; x_\lambda y)\rangle \geq \min\{-\langle T^*(x), \theta(y, x; xy)\rangle, \langle T^*(y), \theta(x, y; xy)\rangle\}$$

for all $x, y \in K$ and $x_\lambda = h(x, y; y + \lambda\eta(x, y)) \in K$, then $\overline{y} \in K$ solves the problems $GDCLP$.

Proof. Since $\overline{y} \in K$ solves the problems $GDVLIP$, we have

$$\langle T^*(x), \theta(\overline{y}, x; x\overline{y})\rangle \leq 0 \tag{5}$$

Replacing x by $h(x, \overline{y}; \overline{y} + \lambda\eta(x, \overline{y}))$ in (3) and using condition C^* to obtain

$$\langle T^*(x_\lambda), \theta(\overline{y}, x_\lambda; x_\lambda\overline{y})\rangle \leq 0,$$

for all $\lambda \in (0, 1)$. Using the condition of T^* to obtain

$$-\langle T^*(x), \theta(\overline{y}, x; x\overline{y})\rangle \leq \langle T^*(x_\lambda), \theta(\overline{y}, x_\lambda; x_\lambda\overline{y})\rangle \leq 0,$$

i.e.,

$$\langle T^*(x), \theta(\overline{y}, x; x\overline{y})\rangle \geq 0$$

for all $x \in K$ which together with (5) gives

$$\langle T^*(x), \theta(\overline{y}, x; x\overline{y})\rangle = 0$$

for all $x \in K$, i.e., $\overline{y} \in K$ solves the problems $GDCLP$. This completes the proof. □

3. Auxiliary Principle Technique and Solution of *GVLIP*

In this section we develop the numerical analysis of the solution of $GVLIP$ using auxiliary principle technique developed by authors Glowinski et al [4].

For existence of the numerical approach the concepts of η-strongly monotone property and η-Lipschitz property of T^* are defined as follows

Definition 17. *The mapping T^* is*

1. *η-Lipschitz continuous on $K(\eta)$ if there exists a constant $\beta > 0$ such that for all $x, y \in K$,*

$$\|T^*(x) - T^*(y)\| \leq \lambda \|\eta(x, y)\| \tag{1}$$

for all $v \in V$,

2. *strongly θ-monotone on $K(\eta)$ if there exists a constant $\alpha > 0$ such that for all $x, y \in K$,*

$$\langle T^*(x) - T^*(y), \theta(x, y; v)\rangle \geq \sigma \left\|\eta\left(x, y\right)\right\|^2 \tag{2}$$

for all $v \in V = \{v \in X : v = xy, x, y \in K\}$ where the term xy is to be understand as the product component wise, i.e., $xy = \{x_1y_1, x_2y_2, \cdots\}$.

Let $\mathbb{X}$ be a separable Hilbert space and X be a topological vector space in $\mathbb{X} \setminus \{0\}$. Let K be η-invex on $K(\eta)$. Let T^* be η-Lipschitz continuous on $K(\eta)$ with Lipschitz constant $\lambda > 0$ and strongly θ monotone associated with the constant $\sigma > 0$ satisfy the condition

$$0 < \rho < \frac{2\sigma}{\lambda^2}$$

for some $\rho > 0$. Let $y \in K$ solves the problem $GVLIP$, i.e.,

$$\langle T^*(y), \theta(x, y; xy)\rangle \geq 0 \;\text{ for all }\; x \in K. \tag{3}$$

The auxiliary equation: For a given solution $\overline{y} \in K$ of the problem $GVLIP$, the auxiliary equation is the problem to find $w \in K$ such that

$$\langle \rho T^*(\overline{y}) + w - \overline{y}, \theta(x, w; xw)\rangle \geq 0 \tag{4}$$

for all $x \in K$. Equation (4) coincides with (3) as $y \to w$. Now we show the mapping $y \mapsto w$ satisfying equation (4) has a fixed point belongs to K and solves (3), i.e., for a suitable choice $\rho > 0$, one has

$$\|w_1 - w_2\| \leq \kappa \|y_1 - y_2\|$$

with $\kappa \in (0, 1)$ is independent of y_1 and y_2.

Numerical approach: Assume that $\eta(x, y) \approx P(x) - P(y)$ with $\|\eta(x, y)\| \leq \|P(x) - P(y)\|$ where P is a linear projective expansive map and for each $xy \in V$, we have $0 < \left\|(xy)^{-1}\right\| \leq 1$, then $\theta(x, y; xy) = (xy)^{-1}\eta(x, y) \approx (xy)^{-1}\left(P(x) - P(y)\right)$ for all $x, y \in K$ and satisfies the condition $\theta(y, x; xy) = -\theta(x, y; xy)$ for all $x, y \in K$. Let $w = f(y)$ such that $w_i = f(y_i)$ for $i = 1, 2$. For $w_1 \neq w_2 \in K$ be two solutions of (4) corresponding to $u_1 \neq u_2 \in K$ then

$$\langle \rho T^*(y_1) + w_1 - y_1, \theta(x, w_1; xw_1)\rangle \geq 0 \tag{5}$$

and

$$\langle \rho T^*(y_2) + w_2 - y_2, \theta(x, w_2; xw_2)\rangle \geq 0 \tag{6}$$

for all $x \in K$. Letting $x = w_2$ in (5), $x = w_1$ in (6) and subtracting them to have

$$\begin{aligned}
&\langle w_2 - w_1, \theta(w_2, w_1; w_1w_2)\rangle \\
&\quad \le -\rho \left\langle (T^*(y_2) - T^*(y_1), \theta(w_2, w_1; w_1w_2)\right\rangle \\
&\quad\quad + \langle y_2 - y_1, \theta(w_2, w_1; w_1w_2)\rangle \\
&\quad = \langle y_2 - y_1 - \rho\left(T^*(y_2) - T^*(y_1)\right), \theta(w_2, w_1; w_1w_2)\rangle
\end{aligned}$$

from which it follows that

$$\|w_2 - w_1\| \le \|y_2 - y_1 - \rho\left(T^*(y_2) - T^*(y_1)\right\|. \tag{7}$$

Taking $y_2 = y_1 + \eta(y_2, y_1)$ and assuming

$$p = (y_1y_2)^{-1}(y_2 - y_1) - \rho\left(T^*(y_2) - T^*(y_1)\right),$$

we obtain

$$\begin{aligned}
\|p\|^2 &= \langle p, p\rangle \\
&= \Big\langle (y_1y_2)^{-1}(y_2 - y_1) - \rho\left(T^*(y_2) - T^*(y_1)\right), \\
&\qquad\qquad (y_1y_2)^{-1}(y_2 - y_1) - \rho\left(T^*(y_2) - T^*(y_1)\right)\Big\rangle \\
&\le \left\|(y_1y_2)^{-1}\right\| \|y_2 - y_1\|^2 - 2\rho\langle T^*(y_2) - T^*(y_1), (y_1y_2)^{-1}(y_2 - y_1)\rangle \\
&\quad + \rho^2 \|T^*(y_2) - T^*(y_1)\|^2 \\
&\le \left\|(y_1y_2)^{-1}\right\| \|y_2 - y_1\|^2 - 2\rho\langle T^*(y_2) - T^*(y_1), \theta(y_2, y_1; y_1y_2)\rangle \\
&\quad + \rho^2 \|T^*(y_2) - T^*(y_1)\|^2 \\
&\le \|\eta(y_2, y_1)\|^2 - 2\rho\sigma \|\eta(y_2, y_1)\|^2 + \rho^2\beta \|\eta(y_2, y_1)\|^2 \\
&= \left(1 - 2\rho\sigma + \rho^2\lambda\right) \|\eta(y_2, y_1)\|^2 \\
&\le \left(1 - 2\rho\sigma + \rho^2\lambda\right) \|P(y_2) - P(y_1)\|^2 \\
&\le \left(1 - 2\rho\sigma + \rho^2\lambda\right) \|y_2 - y_1\|^2
\end{aligned}$$

which follows that

$$\|w_2 - w_1\| \le \kappa \|y_2 - y_1\|$$

where $\kappa = \sqrt{1 - 2\rho\sigma + \rho^2\lambda} < 1$. By contraction principle f has a fixed point at $y \in K$, i.e., $w = f(\overline{y}) = \overline{y}$. Hence $\overline{y}$ solves the problem $GVLIP$, i.e.,

$$\langle T^*(\overline{y}), \theta(x, \overline{y}; x\overline{y})\rangle \ge 0$$

for all $x \in K$.

Conclusions

The generalized variational-like inequality problems and its corresponding dual variational-like inequality problems are defined. The affineness of a bifunction in the first quadrant of a circle is defined and used to show the existence of equivalence between the problems $GVLIP$ and $GDVLIP$ via a transformation T^* without help of pre-invexity, hemicontinuity property of T^*. The existence of the complementarity problems $GCLP$ $(GDCLP)$ is shown in the presence of the problems $GVLIP$ $(GDVLIP)$. The numerical approach of the problems $GVLIP$ is studied in η-invex set with the help of a projective map.

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