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BALL REMOTALLY IN TENSOR PRODUCT SPACES

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Abstract: In the paper, we study Banach spaces X, Y with subspaces H, G whose unit ball $H \otimes G$ is remotal in tensor product space $X \otimes Y$.

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1. Introduction

Farthest points and remotal sets in normed spaces and metric spaces, has been studied by many authors, e.g., [1, 5, 6, 7, 9, 12]. The study of remotal sets is a little more difficult and less more developed than that of proximinal sets. While best approximation has applications in many branches of mathematics, remotal sets and farthest points have applications in the study of geometry of Banach spaces. The study of metric properties of the unit ball B_Y of a proper subspace Y of a Banach space$X$ has been developed in the last decade. Since the closed ball B_X is always a remotal set in any Banach space X, so it is natural to ask what happens in case of B_Y for a subspace Y? Bandyopadhayay and his assistances replied to this question in the classical Banach spaces c_0 and l_∞ and the space $C(K)$ of all scalar-valued continuous functions on K, where K is a compact Hausdorff space, e.g., [2, 4]. Khalil and Matar [8] introduced and discussed strongly remotel sets in Banach spaces; In this paper, we prove

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some stability results on strongly remotel sets analogous to strongly proximinal sets(see [3]). Also, we study when is the closed unit ball $X \otimes G$ remotal in $X \otimes Y$. The structure of the paper is as follows. In section 2, we shall give some preliminary results on tensor product spaces. In section 3, we introduce the concept of strong and absolutely remotality of sets in Banach spaces and presented some results. In section 4, we develop the theory of ball remotaly in tensor product Banach spaces example of vector valued continuous, Bochner p-integrable functions and reflexive spaces.

2. Preliminary

Let X, Y be Banach spaces with their duals X^* and Y^*, respectively. We recall (see [10, 13]) that the uncompleted tensor product X and Y is the set of all formal expressions $\sum_{i=1}^{n} x_i \otimes y_i$, where $x_i \in X$ and $y_i \in Y$ and $n \in \mathbb{N}$. We regard such an expression as defining an operator $A : X^* \to Y$, given by

$$A(\phi) = \sum_{i=1}^{n} \phi(x_i) y_i \quad \phi \in X^*. \tag{2.1}$$

Amongst all these formal expressions we introduce the relation

$$z = \sum_{i=1}^{n} x_i \otimes y_i \sim \sum_{i=1}^{m} a_i \otimes b_i,$$

if both expressions define the same operator from X^* to Y. This relation is an equivalence relation on the set of all such formal expressions. We shall denote the set of all such equivalence classes by $X \otimes Y$. We recall (see [10, 13]) that it is possible to construct various norms on $X \otimes Y$ using the norms in X and Y. The most obvious way to introduce a norm which is independent of the representation of the equivalence classes is to assign to $\sum_{i=1}^{n} x_i \otimes y_i$ the norm it receives when regarded as an operator from X^* to Y. we define the injective norm ϵ of z by

$$\epsilon(z) = \sup\{\sum_{i=1}^{n} |\phi(x_i)||\psi(y_i)|, \phi \in B_{X^*} \ and\ \psi \in B_{Y^*}\}, \tag{2.2}$$

and the projective norm π of z by

$$\pi(z) = \inf\{\sum_{i=1}^{n} \|x_i\|\|y_i\|,\ z = \sum_{i=1}^{n} x_i \otimes y_i\}. \tag{2.3}$$

Let $1 \leq p \leq \infty$, the p-nuclear norm of z is defined by the following equation, in which $p^{-1} + q^{-1} = 1$,

$$\alpha_p(z) = \inf\{(\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}} \mu(y_1, ...y_n) : z = \sum_{i=1}^{n} x_i \otimes y_i\},$$

where $\mu(y_1, ...y_n) = \sup\{(\sum_{i=1}^{n} |\psi(y_i)|^q)^{\frac{1}{q}}, \psi \in B_{Y^*}\}$.

In general, the space $X \otimes Y$ equipped with above norm is not complect. We denote the completion $X \otimes Y$ with respect to α by $X \otimes_\alpha Y$. Let T be a compact Hasdroff space, we denoted by $C(T)$ the Banach space of all continuous functions from T to R, and $C(T, X)$ the Banach space of all continuous functions f from T to normed space X equipped with norm defined by

$$\|f\| = \sup_{t \in T} \|f(t)\|.$$

We denoted by $L^p(S, X)$, the Banach space of all p-Bochner integrable functions defined on a measurer space S which taking values in Banach space X.

Lemma 2.1. [10] For any compact Hausdorff space T and any Banach space X, we have

$$C(T) \otimes_\epsilon X \simeq C(T, X).$$

Lemma 2.2. [10] Let S and X be σ-finite measure spaces, then

$$L^1(S) \otimes_\pi X \simeq L^1(S \times X).$$

Lemma 2.3. [10] Let S be a finite measure space and X be a Banach space such that

$$L^p(S) \otimes_{\alpha_p} X \simeq X \otimes_{\alpha_p} L^p(S),$$

where $1 \leq p < \infty$. Then $L^p(S) \otimes_{\alpha_p} X \simeq L^p(S, X)$.

Let X, Y be Banach spaces, we denote by $L(X, Y)$ the vector space of all bounded linear maps from X to Y. This is a Banach space when endowed with the operator norm. The set of all compact operators from X to Y is denoted by $K(X, Y)$. We denote by $FR(X, Y)$ the set of finite-rank operators from X to Y.

Lemma 2.4. [10] Let X and Y be Banach spaces. Then

$$(X \otimes_\pi Y)^* = L(X, Y^*).$$

Definition 2.5. [10] A Banach space X has approximation property if for each compact subset K of X and every $\varepsilon > 0$ there exists a finite rank operator $S : X \to X$ such that $\|x - Sx\| < \varepsilon$ for every $x \in K$.

Lemma 2.6. [10] Let X and Y be Banach space. Then

$$X^* \otimes_\epsilon Y^* \simeq \overline{FR}(X, Y^*).$$

Moreover if X^* has approximation property. Then

$$X^* \otimes_\epsilon Y^* \simeq K(X, Y^*).$$

3. Uniquely and Strongly Remotal Sets in Banach Spaces

In this section, we will present the definition of the uniquely remotal sets and some properties related to this concept.

Let B be a bounded subset of a normed space X, the set of all farthest points from x to B is denoted by $\mathbf{F}_B(x)$, defined by

$$\mathbf{F}_B(x) := \{y \in B | \ \|x - y\| = \mathbf{S}_B(x) := \sup_{g \in B} \|x - g\|\}.$$

Note that this set may be empty and set-valued farthest point map $F_B : X \to B$, $x \to \mathbf{F}_B(x)$, is upper semi- continues but not necessarily a continuous one. Let $R(B) = \{x \in X : \ \mathbf{F}_B(x) \neq \emptyset\}$. The set B is said to remotal if and only if $R(B) = X$, and densely remotal if $R(B)$ is norm dense in X.

Definition 3.1. [2] Let W be a closed bounded subset of X, we say W be ball remotal if its closed ball B_W is remotal.

Definition 3.2. Let B be a closed subset of X and $x \in X$. We say that B is uniquely remotal for x if every maximizing sequence $\{z_n\} \subseteq B$ for x is convergent. Recall that a sequence $\{s_n\}$ in B said to be a maximizing sequence in B for x if $\|x - s_n\| \to \mathbf{S}_B(x)$.

Theorem 3.3. *Let K be a closed subset of a Banach space X and $x \in X$. If K is uniquely remotal for x then F_K is single-valued and continuous at x i.e., if $x_n \to x$ in norm and $z_n \in F_K(x_n)$, then $z_n \to z$.*

Proof. If K is uniquely remotal, it is trivally $F_K(x)$ is singleton, we say z. Observe that if $x_n \to x$ and $z_n \in F_K(x_n)$, then we have

$$\|x_n - z\| \leq \|x_n - z_n\| \leq \|x_n - x\| + \|x - z_n\|.$$

By taking limits we get

$$\|x - z\| \leq \lim_n \|x - z_n\|.$$

Thus $\{z_n\}$ is a maximizing sequence for x. Since K is uniquely remotal therefore $z_n \to z$. □

Definition 3.4. Let B be a bounded subset of a normed linear space. B is called strongly remotal in X if there exist a constant $k_x > 0$ and an element $g_0 \in B$ such that

$$\|x - g_0\| \geq \|x - g\| + k_x\|g - g_0\|, (\forall g \in B).$$

Corollary 3.5. *Let B be a strongly remotal subset of X. Then $\mathbb{F}_B(x)$ is uniq for $x \in X$.*

Proof. Let g_0 be a farthest point to x from B such that for every $g \in B$, there exist a constant $k_x > 0$ such that

$$\|x - g_0\| \geq \|x - g\| + k_x\|g - g_0\|.$$

Now if possible, assume g_1 be another farthest point to x from B. Then $||x - g_1|| = ||x - g_0||$. Thus

$$||x - g_1|| = ||x - g_0|| \quad \geq \quad ||x - g_1|| + k_x||g_1 - g_0||$$

Therefore $k_x||g_1 - g_0|| = 0$ and thus $g_1 = g_0$. □

Theorem 3.6. *Let B be a strongly remotal subset of X. Then the farthest point map from X to B is continuous.*

Proof. Let (x_n) be a sequence in X such that $x_n \to x$. We claim that $F_B(x_n) \to F_B(x)$. Since B is strongly remotal, there exist $k_x > 0$ such that for every $g \in B$,

$$\|x - F_B(x)\| \geq \|x - g\| + k_x\|g - F_B(x)\|.$$

Take $g = F_B(x_n)$. Hence

$$\|x - F_B(x)\| \geq \|x - F_B(x_n)\| + k_x\|F_B(x) - F_B(x_n)\|.$$

This implies

$$\|F_B(x) - F_B(x_n)\| \leq \frac{1}{k_x}[\|x - F_B(x)\| - \|x - F_B(x_n)\|]. \tag{3.1}$$

Now

$$\|x_n - F_B(x)\| \leq \|x_n - F_B(x_n)\| \leq \|x_n - x\| + \|x - F_B(x_n)\|.$$

By taking limits we get

$$\|x - F_B(x)\| \leq \lim_n \|x - F_B(x_n)\|. \tag{3.2}$$

Also

$$\|x - F_B(x_n)\| \leq \|x - x_n + x_n - F_B(x_n)\| \leq \|x_n - x\| + \|x_n - F_B(x_n)\|.$$

By taking limits we get

$$\lim_n \|x - F_B(x_n)\| \leq \lim_n \|x_n - F_B(x_n)\|.$$

By Proposition 1.1 [11] we have $\lim_n \|x_n - F_B(x_n)\| = \|x - F_B(x)\|$. This implies

$$\lim_n \|x - F_B(x_n)\| \leq \|x - F_B(x)\|. \tag{3.3}$$

It follows from (3.2) and (3.3), $\lim_n \|x - F_B(x_n)\| = \|x - F_B(x)\|$.

From (3.1) we get $\lim_n \|F_B(x) - F_B(x_n)\| = 0$. This completes the proof. □

Let C a bounded subset of X. For any $\delta > 0$ we set

$$F_C(x, \delta) = \{z \in C : \|x - z\| > \mathbf{S}_C(x) - \delta\}. \tag{3.4}$$

Definition 3.7. A bounded subset C of X is called absolutely remotal at x if for each $\varepsilon > 0$ there exist $\delta > 0$ such that

$$\sup\{d(z, F_C(x)) : z \in F_C(x, \delta)\} < \varepsilon. \tag{3.5}$$

If C is absolutely remotal at each $x \in X$, then we say C is absolutely remotal.

Definition 3.8. Let W be a closed subspace of X, we say W is ball absolutely remotal if B_W is absolutely remotal.

Definition 3.9. Let K be a closed subset of X and $x \in X$. We say that B is remotatively compact for x if every maximizing sequence $\{z_n\} \subseteq B$ for x has a convergent subsequence.

Note that if B be a remotatively compact then B is remotal. And none of the implications can be reversed.

Example 3.1. Let $X = l_\infty$, $Y = c$. Then Y is ball remotal set in X. For $x = (1, 1, 1...) \in l_\infty$ the sequence $y_n = (-1, -1, -1, ..., -1, 0, 0, ...)$ is maxmizing, but y_n has no-convergent subsequence. Thus remotatively does not imply remotatively compactness.

Theorem 3.10. *Let K be a closed subset of Banach space X and $x \in X$. Then K is remotatively compact for x if and only if K is absolutely remotal for x and $F_K(x)$ is compact.*

Proof. If K is remotatively compact for x, then it is trivial $F_K(x)$ is compact. Suppose that K is not absolutely remotal for x, then there exist a neighborhood V of 0 and a maximizing sequence $\{z_n\} \subseteq K$ with $z_n \notin F_K(x) + V$ for all n. Since K is remotatively compact for x, there is a subsequence $\{z_{n_k}\}$ such that $z_{n_k} \to z_0 \in K$. Then $z_0 \in F_K(x)$ and so, $z_n \in \{z_0\} + V \subseteq F_K(x) + V$ for some n. It is a contradiction.
Conversely, suppose K is strongly remotaal for x and $F_K(x)$ is compact, but K is not remotatively compact for x. Then there is a maximizing sequence $\{z_n\} \subseteq K$ such that no subsequence is convergent. It follows that for any $z \in F_K(x)$, there is a neighborhood U_z of z and $N_z \in N$ such that for all $n \geq N_z, z_n \notin U_z$. Since $F_K(x)$ is compact, there is a neighbourhood V of 0 and $N_0 \in N$ such that for all $n \geq N_0, z_n \notin F_K(x) + V$. Since K is absolutely remotall for x, there exists $\delta > 0$ such that $F_K(x, \delta) \subseteq F_K(x) + V$. Note that for any maximizing sequence $z_n \subseteq K, z_n \in F_K(x, \delta)$ eventually. This is a contradiction. □

Theorem 3.11. *Let Y be a finite-dimensional subspace of X. Then Y is a absolutely ball remotal subspace. Morover ball remotatively compact.*

Proof. Let $x \in X$ and $\varepsilon > 0$ be given. Since B_Y is compact, there exists $y \in B_Y$ such that $\|x - y\| = \mathbf{S}_{B_Y}(x)$, which implies remotalality of B_Y at x. To prove the absolutely ball remotality of B_Y, we need to show that for each $\varepsilon > 0$ and $x \in X$, there exists $\delta > 0$ such that $F_{B_Y}(x, \delta) \subset F_{B_Y}(x) + \varepsilon B_Y$. If not, for every n, there exists $y_n \in F_{B_Y}(x, \frac{1}{n})$ such that $d(y_n, F_{B_Y}(x)) > \varepsilon$. As Y is finite dimensional, y_n has a subsequence converging to some $y \in B_Y$. Then $\|x - y\| = \mathbf{S}_{B_Y}(x)$, which implies $y \in F_{B_Y}(x)$. But $d(y, F_{B_Y}(x)) \leq \varepsilon$, which is a contradiction. Hence Y is a absolutely ball remotal subspace in X. □

Theorem 3.12. *Let K be a closed subset of Banach space X and $x \in X$. Then the following are equivalent:*

(a) *K is uniquely remotal for x.*

(b) *K is remolatively compact and $F_K(x)$ is a singleton.*

(c) *K is absolutely remotal for x and $F_K(x)$ is a singleton.*

Proof. By Theorem 4.10 it suffices to show $(b) \Rightarrow (a)$. Suppose K is remotatively compact and $F_K(x) = \{z_0\}$. Let V be a neighbourhood of 0. Since K is remotatively compact for x, there exists $\delta > 0$ such that $F_K(x, \delta) \subseteq z_0 + V$. Thus for any maximizing sequence $\{z_n\} \subseteq K$, $z_n \in F_K(x, \delta) \subseteq z_0 + V$ for sufficiently large n. Hence $z_n \to z_0$. □

Corollary 3.13. *Let K be a closed absolutely subset of Banach space X, in addition it be a strongly remotal, then K is uniquely remotal.*

Proof. This follows by Corollary 3.5 and Theorem 3.12. □

4. Ball remotality in tensor product space

Theorem 4.1. *Let G be a closed subspace of Banach space X and T be a compact space. If there exists a continuous farthest point map of X onto B_G then $C(T) \otimes_\epsilon G$ is ball remotal in $C(T) \otimes_\epsilon X$.*

Proof. It follows from Lemma 2.1 that $C(T) \otimes_\epsilon X$ is isometric with $C(T, X)$. Let A be continues farthest point map from X onto B_G, We define $A' : C(T, X) \to B_{C(T,G)}$ by the equation $A'(f) = Af$ for $f \in C(T, X)$. It is elementary to prove that A' is a continuous farthest point map. Since

$$\|Af\| = \sup_{t \in T} \|Af(t)\| = \sup_{t \in T} \|A(f(t))\| \leq \sup_{x \in X} \|A(x)\| \leq 1,$$

thus $Af \in B_{C(T,G)}$. Now for $g \in B_{C(T,G)}$ and $t \in T$ we have

$$\|f(t) - Af(t)\| \geq \|f(t) - g(t)\|.$$

Hence

$$\|f - Af\| \geq \|f - g\|,$$

Thus $C(T, G)$ and in consequence $C(T) \otimes_\epsilon G$ are ball remotal. □

Theorem 4.2. *Let G be a closed subspace of $C(S)$ and T be a compact space. If $G \otimes_\epsilon C(T)$ is ball remotal in $C(S \times T)$, then G is ball remotal in $C(S)$.*

Proof. Assume that $G \otimes_\epsilon C(T)$ is ball remotal. Let x be any element of $C(S)$. Put $x'(s, t) = x(s)$ for all $(s, t) \in S \times T$. Note that for any $g \in B_G$,

$$\mathbf{S}_{G \otimes_\epsilon C(T)}(x') \geq \|x' - g \otimes_\epsilon 1\| = \|x - g\|,$$

whence

$$\mathbf{S}_{G\otimes_\epsilon C(T)}(x') \geq \mathbf{S}_G(x). \tag{4.1}$$

Let v be a farthest point to x' from $B_{G\otimes_\epsilon C(T)}$. Select $t \in T$ so that $||x' - v|| = \sup |x'(s,t) - v(s,t)|$. Put $g_0(s) = v(s,t)$. Since

$$\|g_0\| = \sup_{s\in S} \|g_0(s)\| \leq \sup_{s\in S, t\in T} \|v(s,t)\| \leq 1,$$

Then $g_0 \in B_G$, and g_0 is a farthest point to x since by (4.1)

$$\begin{aligned} \|x - g_0\| = \sup_s |x(s) - g_0(s)| &= \sup_s |x'(s,t) - v(s,t)| \\ &= ||x' - v|| = \mathbf{S}_{G\otimes_\epsilon C(T)}(x') \\ &\geq \mathbf{S}_G(x). \end{aligned}$$

Thus g_0 is farthest point to x from ball G. □

Let X and Y are non-empty sets and the function $f : X \times Y \to \mathbb{R}$ has a bounded range, then we have

$$\sup_{x\in X,\, y\in Y} f(x,y) = \sup_{x\in X} \sup_{y\in Y} f(x,y) = \sup_{y\in Y} \sup_{x\in X} f(x,y), \tag{4.2}$$

or, equivalently

$$\sup_{x\in X,\, y\in Y} f(x,y) = \sup_{x\in X} \sup_{y\in Y} f_x(y) = \sup_{y\in Y} \sup_{x\in X} f_x(y), \tag{4.3}$$

where f_x is defined on Y by $f_x(y) = f(x,y)$ for all $y \in Y$.

Let Y be a subset of a Banach space X. Notice that for each $x \in X$, we have

$$\mathbf{S}_{B_Y}(x) \leq \mathbf{S}_{B_X}(x) = \|x\| + 1.$$

Definition 4.3. [2] A subspace Y is called a $(*)$-subset of X if

$$\mathbf{S}_{B_Y}(x) = \mathbf{S}_{B_X}(x) = \|x\| + 1.$$

Theorem 4.4. *Let T be a compact Hausdorff space, X a Banach space and G a subspace of X. Then G is a $(*)$-subspace of X if and only if $C(T)\otimes_\epsilon G$ is a $(*)$-subspace of $C(T)\otimes_\epsilon X$.*

Proof. Assumes that $C(T)\otimes_\epsilon G$ is a $(*)$-subspace of $C(T)\otimes_\epsilon X$. Put $f = 1\otimes_\epsilon x$ and $g_0 \in \mathbf{F}_{B_{C(T)\otimes_\epsilon G}}(f)$. Let $t_0 \in T$ such that $f-g$ attain its norm at t_0, then

$$\begin{aligned} \|x-g_0(t_0)\| = \|1\otimes_\epsilon x(t_0) - g_0(t_0)\| &= \|1\otimes_\epsilon x - g_0\| \\ &= \sup_{g\in B_{C(T)\otimes_\epsilon G}} \|1\otimes_\epsilon x - g\| \\ &= \|1\otimes_\epsilon x\| + 1 \\ &= \|x\|+1, \end{aligned}$$

Conversely, let G is a $(*)$-subspace of X,

$$\begin{aligned} \|f-g_0\| = \sup_{g\in B_{C(T)\otimes_\epsilon G}} \|f-g\| &= \sup_{g\in B_{C(T)\otimes_\epsilon G}} \sup_{t\in T} \|f(t)-g(t)\| \\ &= \sup_{t\in T} \sup_{g\in B_{C(T)\otimes_\epsilon G}} \|f(t)-g(t)\| \\ &= \sup_{t\in T}(\|f(t)\|+1) = \|f\|+1. \end{aligned}$$

□

Theorem 4.5. *Let G be a subspace of X and (I,μ) be the unit interval with the Lebesgue measure and $1\le p<\infty$. If there exist a continuous farthest point map from X onto B_G, then $L^p(I)\otimes_{\alpha_p} B_G$ is remotal in $L^p(I)\otimes_{\alpha_p} X$.*

Proof. By Lemma 2.3, $L^p(I)\otimes_{\alpha_p} X \simeq L^p(I,X)$. Let F_G be continuous farthest point map from X onto B_G and $f\in L^p(I,X)$. We define

$$A : L^p(I,X) \to L^p(I,B_G); \quad f \hookrightarrow g_f,$$

where $g_f : I \to X$, $g_f(t) = F_G f(t)$ for $t\in I$.
As F_G is continuous and f is measurable, then g_f is measurable. Since $\|F_G f\|_p \le 1$ then $g_f \in L^P(I,B_G)$. Now we show A is a farthest point map. Since

$$\|f(t) - F_G f(t)\| \ge \|f(t)-z\|, \quad for\ z\in B_G,$$

therefore we get $\|f-F_G f\|_p \ge \|f-g\|_p$ for $g\in L^P(I,B_G)$ i.e., $Af \in \mathbf{F}_{L^P(I,B_G)}(f)$. Since f is arbitrary then $L^P(I,B_G)$ and in consequence $L^p(I)\otimes_\epsilon B_G$ are remotal in $L^p(I)\otimes_{\alpha_p} X$. □

Theorem 4.6. *Let G be a subspace of X and (I,μ) be the unit interval with the Lebesgue measure. For $1\le p<\infty$. If there exist a continuous farthest point map from $L^p(I)\otimes_{\alpha_p} X$ onto $L^p(I)\otimes_{\alpha_p} B_G$. Then G is ball remotal.*

Proof. Let $\phi : L^p(I) \otimes_{\alpha_p} X \longrightarrow L^p(I) \otimes_{\alpha_p} B_G$, be continuous farthest point map, For $z \in X$, define $f_z : I \to X$, $f_z(t) = z$. By assumption for $z \in B_G$, $\|f_x - \phi f_x\|_p \geq \|f_x - f_z\|_p$ hence

$$\int_I \|f_x(t) - \phi f_x(t)\|^p d\mu(t) \geq \|x - z\|^p.$$

Since (I, μ) is probability measure, there is an $E \subseteq I$, $\mu(E) = 1$ such that for $t \in E$

$$\|f_x(t) - \phi f_x(t)\| \geq \|x - z\|.$$

Hence $\|x - \phi f_x(t)\| \geq \|x - z\|$ for $z \in B_G$ and $t \in E$. This implies that G is ball remotal. □

Corollary 4.7. *Let G be a finite dimensional subspace of a Banach space X and T be a compact space. Then $C(T, G)$ is ball remotal in $C(T, X)$.*

Proof. Since B_G is an compact set then farthest point map from X to B_G is continues. Thus ball remotal $C(T, G)$ is a consequence of Theorem 4.1. □

Corollary 4.8. *Let G be a ball strongly remotal subset of X.*

a) $C(I) \otimes_\epsilon G$ is ball remotal in $C(I) \otimes_\epsilon X$.

b) $L^p(I) \otimes_{\alpha_p} G$ is ball remotal in $L^p(I) \otimes_{\alpha_p} X$.

Proof. (a)This follows by Theorems 3.6 and 4.1.
(b) This follows by Theorems 3.6 and 4.6. □

Theorem 4.9. *Let X be a reflexive Banach space and G be a closed finite dimensional subspace of Banach space Y. Then $L(X, G)$ is ball remotal in $L(X, Y)$.*

Proof. Let $T \in L(X, Y)$, $\mathbf{S}_{B_{L(X,G)}}(T) = r$ and $T_n \in B_{L(X,G)}$ such that $\lim \|T - T_n\| = r$. Since $L(X, G)$ is a reflexive space[see Theorem 4.9 [13]] then $B_{L(X,G)}$ is weakly compact. So there exists a subsequence T_{n_k} and $A \in B_{L(X,G)}$ such that $T_{n_k} \to^w A$ and this implies that $T_{n_k} - T \to^w A - T$ i,e. for $\psi \in Y^*$ and $x \in X$ we have

$$\lim_{n_k} \psi((T_{n_k} - T)(x)) = \psi((A - T)(x)) \leq \|\psi\| \|(A - T)(x)\|.$$

Thus there exist $n_{k_0} \in \mathbb{N}$ such that for $n_k \geq n_{k_0}$ we have

$$\psi((T_{n_k} - T)(x)) \leq \|\psi\| \|(A - T)(x)\|.$$

Now for each $\psi \in Y^*$ with $\|\psi\| \leq 1$ we get

$$\sup_{\|\psi\| \leq 1} \psi((T_{n_k} - T)(x)) \leq \|(A - T)(x)\|.$$

Therefore

$$\|(T_{n_k} - T)(x)\| \leq \|(A - T)(x)\|.$$

By taking sup on X we have $\|T_{n_k} - T\| \leq \|A - T\|$. Then $r = \lim \|T_{n_k} - T\| \leq \|A - T\|$. Hence $\|A - T\| = r$, i.e $A \in F_{B_{L(X,G)}}(T)$. This implies that $L(X, G)$ is ball remotal in $L(X, Y)$ □

Corollary 4.10. *Let X be a reflexive space, X^* has the approximation property and G be a finite dimensional closed subspace of Y^*. Then $X^* \otimes_\epsilon G$ is ball remotal in $X^* \otimes_\epsilon Y^*$.*

Proof. By Lemma 2.6, we have

$$X^* \otimes_\epsilon G \simeq \overline{FR}(X, G) = K(X, G) \subseteq K(X, Y^*) \simeq X^* \otimes_\epsilon Y^*$$

and by Theorem 4.9, $L(X, G)$ is ball remotal in $L(X, Y^*)$. This implies $L(X, G)$ is ball remotal in $K(X, Y^*)$. By the isometrically, we give $X^* \otimes_\epsilon G$ is ball remotal in $X^* \otimes_\epsilon Y^*$. □

Corollary 4.11. *If p, q be real numbers satisfying $1 \leq p \leq q < \infty$ with conjugate indices p', q' and G be a finite dimensional subspace of l_p. Then $l_{q'} \otimes_\epsilon G$ is ball remotal in $l_{q'} \otimes_\epsilon l_p$.*

Proof. To see this fact take $X = l_q$, $Y^* = l_p$ and use Corollary 4.10. □

Corollary 4.12. *Let S and T be two σ-finite measure spaces and G be a finite dimensional closed subspace of space $L_\infty(T)$. Then $L_\infty(S) \otimes_\pi G$ is ball remotal in $L_\infty(S \times T)$.*

Proof. Take $X = L_1(S)$ and $Y = L_1(T)$. Then $X^* = L_\infty(S)$ and $Y^* = L_\infty(T)$. By Lemmas 2.4 and 2.6 we have

$$\begin{aligned} L_\infty(S) \otimes_\epsilon G = \overline{FR}(X, G) &\subseteq L(X, Y^*) \\ &= [L_1(S) \otimes_\pi L_1(T)]^* \\ &= L_1(S \times T)^* = L_\infty(S \times T). \end{aligned}$$

Now we proof by application Theorem 4.9 □

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