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SOME ARITHMETIC FUNCTIONS AND THEIR MEANS

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Abstract: We give four formulas for $d(n^2)$ in terms of arithmetical functions ω, μ, d_3 and d:

$$d(n^2) = \sum_{k|n} d(k)\mu^2(\frac{n}{k}) = \sum_{k^2|n} \mu(k)d_3(\frac{n}{k^2}) = \sum_{k^2|n}(\sum_{m|n} d(m))\mu(k) = \sum_{k|n} 2^{\omega(k)}.$$

We estimate the partial sum

$$\sum_{n\le x} d(n^2) = (\frac{1}{\zeta(2)} + \circ(1))x log^2 x$$

using Tauberian theorem. Similarly, we estimate the partial sums for $\frac{1}{d(n)}$, $\frac{logn}{d(n)}$, and $\frac{\sigma(n)}{d(n)}$.

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1. Introduction

We consider some arithmetical functions whose mean is not finite; in such cases the order of growth of the partial sums is determined. Our examples are (i) $d(n^2)$ (ii) $\frac{1}{d(n)}$ (iii) $\frac{logn}{d(n)}$ (iv) $\frac{\sigma(n)}{d(n)}$. Our main tools are Abel's summation formula and a generalized Tauberian theorem, due to H. Delange.

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I: $d(n^2)$

Lemma 1. $d(n^2) = \sum_{k|n} 2^{\omega(k)}$ *for all positive integers* k *(here* d *is the number of divisors and* ω *is the number of distinct prime divisors).*

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_l^{\alpha_l}$ so that $n^2 = p_1^{2\alpha_1} p_2^{2\alpha_2} \dots p_l^{2\alpha_l}$.If k divides n, then $k = p_1^{p_1} \dots p_l^{p_l}$ with $\beta_i \leq \alpha_i$ for all i. k contributes divisors of n^2 of the form $k(\prod_i p_i^{r_i})$, with $r_i \leq \alpha_i$ for all i. These are $2^{\omega(k)}$ in number. All divisors of n^2 arise this way. Hence $d(n^2) = \sum_{k|n} 2^{\omega(k)}$. □

Lemma 2. *(i)* $\sum_{k=1}^{\infty} \frac{2^{\omega(k)}}{k^s} = \frac{\zeta^2(s)}{\zeta(2s)}$, *and*
(ii) $\frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\sum_{k|n} 2^{\omega(k)}}{n^s} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s}$

Proof. (i) is standard ([2], p156, [1],p247) and (ii) follows from the formula for product of two Dirichlet series. □

Proposition 1. $\sum_{n \leq x} d(n^2) = (\frac{1}{\zeta(2)} + \circ(1)) x log^2 x$

Proof. We apply the Tauberian theorem (see [4], p. 42, Cor. 4.1.3):

Tauberian Theorem. *Suppose* $b_n \geq 0$, $F(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s} = \frac{H(s)}{(s-\beta)^{\alpha+1}}$ *with* H *analytic for* $Re\ s \geq \beta$, $\alpha > -1$ *real and* $H(\beta) = c\Gamma(\beta + 1)$.
Then $\sum_{n \leq x} b_n = (c + \circ(1)) x^{\beta} log^{\alpha} x$.
We choose

$$F(s) = \zeta(s)(\frac{\zeta^2(s)}{\zeta(2s)}) = \sum_{n=1}^{\infty} \frac{(\sum_{k|n} 2^{\omega(k)})}{n^s} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s} = [\frac{1}{(s-1)^{2+1}}] H(s)$$

for $H(s) = \frac{1}{\zeta(2s)}.g(s)$ with g analytic, $g(1) = 1$, $\beta = 1$, $\alpha = 2$, $H(1) = \frac{1}{\zeta(2)} = c$
So

$$\sum_{n \leq x} d(n^2) = (\frac{1}{\zeta(2)} + \circ(1)) x^1 log^2 x.$$ □

Lemma 3. *Let $d_3(n)$ = no of factorizations of n into 3 distinct positive integers. Then*

$$\sum_{n=1}^{\infty} \frac{d_3(n)}{n^s} = \zeta^3(s)$$

and so

$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\left(\sum_{k^2|n} \mu(k) d_3(\frac{n}{k^2})\right)}{n^s} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s}.$$

Proof. This is as in the case of Lemma 2. □

Lemma 4. *(i)*$(\frac{\zeta(s)}{\zeta(2s)})\zeta^2(s) = \sum_{n=1}^{\infty} \frac{(\sum_{k|n} d(k)\mu^2(\frac{n}{k}))}{n^s} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s}$,

(ii)$(\frac{\zeta(s)}{\zeta(2s)})(\zeta(s))(\zeta(s)) = \frac{(\sum_{k^2|n} (\sum_{m|n} d(m))\mu(k))}{n^s} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s}$.

Proof. Same proof as in Lemma2 or 3. These give alternate expressions for $d(n^2)$. □

II: $\frac{1}{d(n)}$

We have (see [2], p. 137):

$$\sum_{n=1}^{\infty} \frac{(\frac{1}{d(n)})}{n^s} = \zeta^{\frac{1}{2}}(s) g(s),$$

with g analytic for Re $s > \frac{1}{2}$.

Note that this gives a singularity of the form $\frac{1}{(s-1)^{\frac{1}{2}}}.H(s)$, we apply the Tauberian Theorem above with $\beta = 1$, $\alpha = -\frac{1}{2}$ and obtain

$$\sum_{n \le x} \frac{1}{d(n)} = (c + \circ(1)) \frac{x}{\sqrt{logx}}.$$

This estimate answers a question of Ramanujan. There is a different proof in [8].

Corollary 1. $\sum_{n \le x} \frac{n}{d(n)} = (k + \circ(1)) \frac{x^2}{\sqrt{logx}}$.

Proof. We "shift" the variable to $(s-1)$ above and write

$$\sum_{n=1}^{\infty} \frac{(\frac{n}{d(n)})}{n^{s+1}} = \zeta^{\frac{1}{2}}(s)g(s) \quad (Res > 1).$$

Then $\beta = 2$, $\alpha = -\frac{1}{2}$ to give the estimate. □

III: $\frac{\sigma(n)}{logn}$

$\sigma(n)$ is the sum of divisors of n. It is known that (see [1], p. 60, Theorem 3.4)

$$\sum_{n\le x} \sigma(n) = \frac{1}{2}\zeta(2)x^2 + \bigcirc(xlogx).$$

Abel's Summation Formula:

$$\sum_{y<n\le x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt,$$

where $\sum_{n\le x} a(n) = A(x)$, $f(x)$ is a differentiable function (see [1], p. 77, Theorem 4.2).

Let us choose $a(n) = \sigma(n)$, $f(x) = \frac{1}{logx}$ so that $f'(x) = -\frac{1}{xlog^2x}$. Then

$$\sum_{2<n\le x} \frac{\sigma(n)}{logn} = [\frac{\zeta(2)}{2}x^2 + \bigcirc(xlogx)].\frac{1}{logx} - \frac{A(2)}{log2} + \int_2^x \frac{(\frac{\zeta(2)}{2}x^2 + \bigcirc(xlogx))}{xlog^2x}dx$$

$$= \frac{\zeta(2)}{2}\frac{x^2}{logx} + \bigcirc(x) - \frac{3}{log2} + \int_2^x \frac{\zeta(2)}{2}\frac{t}{log^2t}dt + \bigcirc(\int_2^x \frac{1}{logt}dt)$$

$$= \frac{\zeta(2)}{2}\frac{x^2}{logx} + \bigcirc(x) + \bigcirc(\int_2^{x^2} \frac{1}{log^2u}du) - \frac{3}{2} + \bigcirc(\int_2^x \frac{1}{logt}dt)$$

$$= \frac{\zeta(2)}{2}\frac{x^2}{logx} - \frac{3}{2} + \bigcirc(x) + \bigcirc(\frac{x^2}{(logx^2)^2}) + \bigcirc(li(x))$$

$$= \frac{\zeta(2)}{2}\frac{x^2}{logx} - \frac{3}{2} + \bigcirc(x) + \bigcirc((\frac{x}{logx})^2) + \bigcirc(li(x))$$

$$= \frac{\zeta(2)}{2}\frac{x^2}{logx} - \frac{3}{2} + \bigcirc((\frac{x}{logx})^2).$$

IV: $\frac{logn}{d(n)}$

To estimate $\sum\limits_{2\le n\le x} \frac{(logn)}{d(n)}$ we use Abel summation with $a(n) = \frac{1}{d(n)}$, $f(x) = logx$, $f'(x) = \frac{1}{x}$, $A(x) = (c + \circ(1))\frac{x}{\sqrt{logx}}$:

$$\sum_{2\le n\le x} \frac{(logn)}{d(n)} = [(c+\circ(1))\frac{x}{\sqrt{logx}}]logx - k + \int_2^x \frac{1}{x}[(c+\circ(1))]\frac{x}{\sqrt{logx}}dx$$
$$= (c+\circ(1))x\sqrt{logx} - k + \bigcirc(\int_2^x \frac{1}{\sqrt{logt}}dt)$$
$$= cx\sqrt{logx} + \bigcirc(\frac{\sqrt{x}}{\sqrt{logx}}) + \circ(1)x\sqrt{logx} - k$$
$$= cx\sqrt{logx} + \bigcirc(\sqrt{\frac{x}{logx}})$$

V: $\frac{d(n)}{logn}$

We use Abel Summation to estimate $\sum\limits_{2<n\le x} \frac{d(n)}{logn}$.

Note that $a(n) = d(n)$, $A(x) = \sum\limits_{n\le x} d(n) = xlogx + (2\gamma - 1)x + \bigcirc(\sqrt{x})$, $f(t) = \frac{1}{logt}$, $f'(t) = -\frac{1}{tlog^2t}$. So

$$\sum_{2\le n\le x} \frac{d(n)}{logn} = A(x)f(x) - A(2)f(2) - \int_2^x A(t)f'(t)dt$$
$$= \frac{(xlogx)}{logx} - (1+2)(\frac{1}{log2}) + \int_2^x \frac{tlogt}{tlogt^2}dt + \frac{(2\gamma-1)x}{logx} + \bigcirc(\frac{\sqrt{x}}{logx})$$
$$= x - \frac{3}{log2} + lix - li2 + \frac{(2\gamma-1)x}{logx} + \bigcirc(\frac{\sqrt{x}}{logx})$$
$$= x + lix + \frac{(2\gamma-1)x}{logx} + \bigcirc(\frac{\sqrt{x}}{logx}).$$

Remark 1. The leading term in $\sum\limits_{y<n\le x} \frac{logn}{d(n)}$ is $cx\sqrt{logx}$ which is larger

than that of $\sum_{y<n\leq x} \frac{d(n)}{logn}$, namely x. Hardy-Ramanujan's Theorem asserts that $d(n) \approx 2^{loglogn} \simeq (logn)^{0.6}$ for almost all n ([3], p55, [8]). The estimate for the mean of $\sum_{2<n\leq x} \frac{d(n)}{logn}$ gives 1 as the average value of the ratio.

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