FIXED POINT THEOREM FOR WEAKLY C-CONTRACTIVE TYPE MAPPING ON COMPLEX VALUED METRIC SPACES

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Abstract: In this article, we prove that if $f$ is a weakly $C$-contractive type self map on a complete complex valued metric space $(X, d)$ then it has a unique fixed point and also, we extend this result to complete complex valued $b$-metric spaces.

AMS Subject Classification: 47H10, 54H25

Key Words: complex valued metric space, complex valued $b$-metric space, weakly $C$-contractive type map, fixed point

1. Introduction and Preliminaries

According to Azam et. al [1], we have the following notation and definitions. Let $\mathbb{C}$ denote the set of complex numbers and $a, b \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows: $a \preceq b$ if and only if $Re(a) \leq Re(b), Im(a) \leq Im(b)$.

**Definition 1.** Let $X$ be a non-empty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies:

Received: August 12, 2017
Revised: July 12, 2018
Published: July 21, 2018
1. $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$

2. $d(x, y) = d(y, x)$, for all $x, y \in X$

3. $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then $d$ is called a complex valued metric on $X$ and $(X, d)$ is called a complex valued metric space.

**Definition 2.** Let $(X, d)$ be a complex valued metric space and $\{x_n\}_{n \geq 1}$ be a sequence in $X$ and $x \in X$. We say that:

1. The sequence $\{x_n\}_{n \geq 1}$ converges to $x$ if for every $r \in \mathbb{C}$, with $0 \prec r$ there is a positive integer $n_0$ such that for all $n > n_0$, $d(x_n, x) \prec r$. We write $x_n \to x$, as $n \to \infty$.

2. The sequence $\{x_n\}_{n \geq 1}$ is Cauchy sequence if for every $r \in \mathbb{C}$, with $0 \prec r$ there is a positive integer $n_0$ such that for all $n, m > n_0$, $d(x_n, x_m) \prec r$.

3. The metric space $(X, d)$ is a complete complex valued metric space if every Cauchy sequence is convergent.

There are several number of works in contraction type mappings, among these we consider one such mapping namely weakly $C$-contractive map.

**Definition 3.** [4] Let $(X, d)$ be a metric space, the map $f : X \to X$ is called weakly $C$-contractive if

$$d(f(x), f(y)) \leq \frac{1}{2}[d(x, f(y)) + d(y, f(x))] - \phi(d(x, f(y)), d(y, f(x)))$$

(1)

for all $x, y \in X$ and $\phi : [0, \infty)^2 \to [0, \infty)$ is continuous and $\phi(x, y) = 0$ if and only if $x = y = 0$.

In [4], Choudhury proved that in complete metric space, weakly $C$-contractive self map has a unique fixed point.

**2. Main Results**

In this section, we present our main results. Before proceed further we first introduce the generalization of weakly $C$-contractive map.
Definition 4. Let \((X, d)\) be a complex valued metric space, the map \(f : X \rightarrow X\) is called weakly \(C\)-contractive type map if

\[
d(f(x), f(y)) \geq \frac{1}{2}[d(x, f(y)) + d(y, f(x))] - \varphi(M(x, y))
\]

for all \(x, y \in X\) and \(M(x, y) = \max\{Re\ d(x, y), Im\ d(x, y)\}\) with the map \(\varphi : [0, \infty) \rightarrow A = \{a + ib : a, b \in (0, \infty)\} \cup \{0\}\) has the following property:

1. \(\varphi\) is continuous
2. \(x \leq y \Rightarrow \varphi(x) \leq \varphi(y)\)
3. \(\varphi(x) = 0\) if and only if \(x = 0\).

Now we prove our first main result about existence of fixed point of the weakly \(C\)-contractive type map defined on complete complex valued metric space. To prove this result, we need the following lemma in the sequel.

Lemma 1. Let \((X, d)\) be a complex valued metric space and let \(\{x_n\}\) be a sequence in \(X\). Then \(\{x_n\}\) is a Cauchy sequence if and only if to every \(\epsilon > 0\), there is a positive integer \(n_0\) such that \(Re\ d(x_n, x_m) < \epsilon\) and \(Im\ d(x_n, x_m) < \epsilon\), for all \(n, m > n_0\).

Proof. Let \(\{x_n\}\) be a Cauchy sequence and \(\epsilon > 0\) be given. Then take \(0 < r = \epsilon + i\epsilon\), and by the definition of Cauchy sequence there is a positive integer \(n_0\) such that \(d(x_n, x_m) < r\), for all \(n, m > n_0\). Which in turn implies \(Re\ d(x_n, x_m) < \epsilon\) and \(Im\ d(x_n, x_m) < \epsilon\), for all \(n, m > n_0\).

Conversely, let \(0 < r\) be given, then \(Re\ r > 0\) and \(Im\ r > 0\) and by the hypothesis, there are positive integers \(n_1, n_2\) such that \(Re\ d(x_n, x_m) < Re\ r\) and \(Im\ d(x_n, x_m) < Im\ r\), for all \(n, m > n_1\) and \(Re\ d(x_n, x_m) < Im\ r\) and \(Im\ d(x_n, x_m) < Im\ r\), for all \(n, m > n_2\). Let \(n_0 = \max\{n_1, n_2\}\), then \(Re\ d(x_n, x_m) < Re\ r\) and \(Im\ d(x_n, x_m) < Im\ r\), for all \(n, m > n_0\) and hence \(d(x_n, x_m) < r\), for all \(n, m > n_0\), so that \(\{x_n\}\) is a Cauchy sequence.

Theorem 1. Let \((X, d)\) be a complete complex valued metric space and \(f : X \rightarrow X\) be a weakly \(C\)-contractive type map, then \(f\) has a unique fixed point.

Proof. Let \(x_0 \in X\) be an arbitrary point and let \(x_n = f(x_{n-1})\), \(n = 1, 2, 3, \ldots\), Suppose there is some positive integer \(n\) such that \(d(x_n, f(x_n)) = 0\), then \(x_n\) becomes a fixed point of \(f\), which completes the proof. So we assume that \(d(x_n, f(x_n)) \neq 0\) for all \(n \in \mathbb{N}\). Hence we have,

\[
d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1}))
\]
\[
\begin{align*}
\frac{1}{2}[d(x_n, f(x_{n-1})) + d(x_{n-1}, f(x_n))] - \phi(M(x_n, x_{n-1})) \\
\frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] - \phi(M(x_n, x_{n-1}))
\end{align*}
\]

(3)

\[
d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}), n = 1, 2, 3, ...
\]

From this, for \(n = 1, 2, 3, \ldots\), we have, \(Re \ d(x_{n+1}, x_n) \leq Re \ d(x_n, x_{n-1})\) and \(Im \ d(x_{n+1}, x_n) \leq Im \ d(x_n, x_{n-1})\) and which in turn implies \(\{Re \ d(x_{n+1}, x_n)\}\) and \(\{Im \ d(x_{n+1}, x_n)\}\) are decreasing sequences of non-negative real numbers and hence convergent. Let \(\lim_{n \to \infty} Re \ d(x_{n+1}, x_n) = r\) and \(\lim_{n \to \infty} Im \ d(x_{n+1}, x_n) = s\). Since \(Re \ d(x_n, x_{n-1}) \leq M(x_n, x_{n-1})\) and by the property of \(\phi\), we have \(\phi(Re \ d(x_n, x_{n-1})) \leq \phi(M(x_n, x_{n-1}))\) Hence for \(n = 1, 2, 3, \ldots\), the equation (3) becomes

\[
d(x_{n+1}, x_n) \leq \frac{1}{2}[d(x_n, x_{n-1}) + d(x_{n+1}, x_n)] - \phi(Re \ d(x_n, x_{n-1}))
\]

(4)

and therefore,

\[
Re \ d(x_{n+1}, x_n) \leq \frac{1}{2}[Re \ d(x_n, x_{n-1}) + Red(x_{n+1}, x_n)] - Re \ \phi(Re \ d(x_n, x_{n-1}))
\]

(5)

making \(n \to \infty\) on (5) and by the continuity of \(\phi\) we have \(r \leq \frac{1}{2}[r + r] - Re \ \phi(r)\), that is \(Re \ \phi(r) = 0\). Hence, by the property of \(\phi\), \(r = 0\). Thus,

\[
\lim_{n \to \infty} Re \ d(x_{n+1}, x_n) = 0
\]

(6)

Similarly,

\[
\lim_{n \to \infty} Im \ d(x_{n+1}, x_n) = 0.
\]

(7)

Now we have to prove that the sequence \(\{x_n\}\) is a Cauchy sequence. Arguing by contradiction, using lemma (1) we may assume that there is an \(\epsilon > 0\) and the sequences \(\{a(n)\}\) and \(\{b(n)\}\) of positive integers such that \(a(n) > b(n) > n\), \(Re \ d(x_{a(n)}, x_{b(n)}) \geq \epsilon\) and \(Re \ d(x_{a(n)-1}, x_{b(n)}) < \epsilon\), for all positive integers \(n\). Now,

\[
\epsilon \leq Re \ d(x_{a(n)}, x_{b(n)}) = Re \ d(f(x_{a(n)-1}), f(x_{b(n)-1}))
\]

\[
\leq \frac{1}{2}[Re \ d(x_{a(n)-1}, x_{b(n)}) + Re \ d(x_{b(n)-1}, x_{a(n)})] - Re \ \phi(Re \ d(x_{a(n)-1}, x_{b(n)-1}))
\]

(8)
Also

\[ \epsilon \leq \text{Re } d(x_{a(n)}, x_{b(n)}) \]
\[ \leq \text{Re } d(x_{a(n)}, x_{a(n)-1}) + \text{Re } d(x_{a(n)-1}, x_{b(n)}) \]
\[ \leq \text{Re } d(x_{a(n)}, x_{a(n)-1}) + \epsilon \]  \hspace{1cm} (9)

making \( n \to \infty \) on (9) and using (6) we get,

\[ \lim_{n \to \infty} \text{Re } d(x_{a(n)}, x_{b(n)}) = \epsilon \] and \[ \lim_{n \to \infty} \text{Re } d(x_{a(n)-1}, x_{b(n)}) = \epsilon \] \hspace{1cm} (10)

Next

\[ \epsilon \leq \text{Re } d(x_{a(n)}, x_{b(n)}) \]
\[ \leq \text{Re } d(x_{a(n)}, x_{a(n)-1}) + \text{Re } d(x_{a(n)-1}, x_{b(n)-1}) \]
\[ + \text{Re } d(x_{b(n)-1}, x_{b(n)}) \]  \hspace{1cm} (11)

\[ \text{Re } d(x_{a(n)-1}, x_{b(n)-1}) \leq \text{Re } d(x_{a(n)-1}, x_{a(n)}) + \text{Re } d(x_{a(n)}, x_{b(n)}) \]
\[ + \text{Re } d(x_{b(n)}, x_{b(n)-1}) \]  \hspace{1cm} (12)

making \( n \to \infty \) on both inequalities (11) and (12) together with equations (6) and (10), we obtain

\[ \lim_{n \to \infty} \text{Re } d(x_{a(n)-1}, x_{b(n)-1}) = \epsilon \] \hspace{1cm} (13)

Also

\[ \text{Re } d(x_{a(n)-1}, x_{b(n)}) \leq \text{Re } d(x_{a(n)-1}, x_{a(n)}) + \text{Re } d(x_{a(n)}, x_{b(n)-1}) \]
\[ + \text{Re } d(x_{b(n)-1}, x_{b(n)}) \]

\[ \text{Re } d(x_{a(n)}, x_{b(n)-1}) \leq \text{Re } d(x_{a(n)}, x_{a(n)-1}) + \text{Re } d(x_{a(n)-1}, x_{b(n)-1}) \]

making \( n \to \infty \) on the above two inequalities and using (6), (10), (13) we get,

\[ \lim_{n \to \infty} \text{Re } d(x_{a(n)}, x_{b(n)-1}) = \epsilon \] \hspace{1cm} (14)

now allow \( n \to \infty \) in (8) and using (10), (13), (14) with the continuity of \( \phi \) we have \( \epsilon \leq \frac{1}{2} \epsilon \leq \epsilon + \epsilon \) \hspace{1cm} (15)

We have \( \text{Re } \phi(\epsilon) = 0 \) and by the property of \( \phi \), we have \( \epsilon = 0 \). Which is a contradiction to \( \epsilon > 0 \), so that \( \{x_n\} \) is a Cauchy sequence and since \( X \) is complete, there is a \( p \in X \) so that \( x_n \to p \), as \( n \to \infty \).
Next we prove that this $p$ is a fixed point of $f$.

\[
d(p, f(p)) \leq d(p, x_{n+1}) + d(f(x_n), f(p)) \\
\leq d(p, x_{n+1}) + \frac{1}{2}[d(x_n, f(p)) + d(p, f(x_n))] - \phi(M(x_n, p))
\]

\[
Re \, d(p, f(p)) \leq Re(d(p, x_{n+1}) + \frac{1}{2}[Red(x_n, p) + Re \, d(p, f(p)) \\
+ Re \, d(p, x_{n+1})] - Re \, \phi(Re \, d(x_n, p))
\]

which on making $n \to \infty$ by using continuity of $\phi$ and lemma 2 (see [1]), we have

\[
Re \, d(p, f(p)) \leq \frac{1}{2}Re \, d(p, f(p)) - Re \, \phi(0) \leq \frac{1}{2}Re \, d(p, f(p))
\]

which is possible only if $Re \, d(p, f(p)) = 0$. Similarly, we have $Im \, d(p, f(p)) = 0$. Thus, $d(p, f(p)) = 0$ and therefore $f(p) = p$. To prove the uniqueness, suppose for $p$ and $q$ are two fixed points of $f$. Now, $d(p, q) = d(f(p), f(q)) \leq \frac{1}{2}[d(p, f(q)) + d(q, f(p))] - \phi(M(p, q))$ and $0 \leq -\phi(Re \, d(p, q))$ by the property of $\phi$, $Re \, d(p, q) = 0$ and similarly $Im \, d(p, q) = 0$, so that $d(p, q) = 0$ or $p = q$ completes the proof.

In [6], Rao et. al., introduced the complex valued $b$-metric space and studied their consequences. Now, we extended Theorem 1 in complete complex valued $b$-metric spaces as follows.

**Theorem 2.** Let $(X, d)$ be a complete complex valued $b$-metric space with $s > 1$ and the map $f : X \to X$ satisfies

\[
d(f(x), f(y)) \leq \frac{1}{2s^4}[d(x, f(y)) + d(y, f(x))] - \phi(M(x, y)) \tag{15}
\]

for all $x, y \in X$ with $\phi$ and $M(x, y)$ as in definition 8. Then $f$ has a unique fixed point (Note that here the weakly $C$-contractive type map depend on $s$).

**Proof.** Let $x_0 \in X$ be an arbitrary point and let $x_n = f(x_{n-1}), n = 1, 2, 3...$ If some positive integer $n$ such that $d(x_n, f(x_n)) = 0$, then $x_n$ becomes a fixed point of $f$, which completes the proof. So we assume that $d(x_n, f(x_n)) \neq 0$, for all $n$. As in theorem 1, we can easily shows that

\[
\lim_{n \to \infty} Re \, d(x_{n+1}, x_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} Im \, d(x_{n+1}, x_n) = 0. \tag{16}
\]

because,

\[
d(f(x), f(y)) \leq \frac{1}{2s^4}[d(x, f(y)) + d(y, f(x))] - \phi(M(x, y)) \\
\leq \frac{1}{2}[d(x, f(y)) + d(y, f(x))] - \phi(M(x, y))
\]

Now we have to prove that the sequence $\{x_n\}$ is a Cauchy sequence. Arguing by contradiction, using Lemma (1) we may assume that there is an $\epsilon > 0$ and
the sequences \( \{a(n)\} \) and \( \{b(n)\} \) of positive integers such that \( a(n) > b(n) > n \), \( \text{Re} \ d(x_{a(n)}, x_{b(n)}) \geq \epsilon \) and \( \text{Re} \ d(x_{a(n)−1}, x_{b(n)}) < \epsilon \), for all positive integers \( n \). Now,

\[
\epsilon \leq \text{Re} \ d(x_{a(n)}, x_{b(n)}) = \text{Re} \ d(f(x_{a(n)−1}), f(x_{b(n)−1})) \\
\leq \frac{1}{2s^4} [\text{Re} \ d(x_{a(n)−1}, x_{b(n)}) + \text{Re} \ d(x_{a(n)}, x_{b(n)−1})] \\
- \text{Re} \ \phi(\text{Re} \ d(x_{a(n)−1}, x_{b(n)−1})) \quad (17)
\]

also

\[
\epsilon \leq \text{Re} \ d(x_{a(n)}, x_{b(n)}) \\
\leq s[\text{Re} \ d(x_{a(n)}, x_{a(n)−1}) + \text{Re} \ d(x_{a(n)−1}, x_{b(n)})] \\
\leq s \text{Re} \ d(x_{a(n)}, x_{a(n)−1}) + s\epsilon \quad (18)
\]

so from (16), we get

\[
\epsilon \leq \limsup_{n \to \infty} \text{Re} \ d(x_{a(n)}, x_{b(n)}) \leq s\epsilon \quad (19)
\]

and

\[
\frac{\epsilon}{s} \leq \limsup_{n \to \infty} \text{Re} \ d(x_{a(n)−1}, x_{b(n)}) \leq \epsilon \quad (20)
\]

Next

\[
\epsilon \leq \text{Re} \ d(x_{a(n)}, x_{b(n)}) \\
\leq s \text{Re} \ d(x_{a(n)}, x_{a(n)−1}) + s^2 \text{Re} \ d(x_{a(n)−1}, x_{b(n)−1}) \\
+ s^2 \text{Re} \ d(x_{b(n)−1}, x_{b(n)}) \quad (21)
\]

\[
\text{Re} \ d(x_{a(n)−1}, x_{b(n)−1}) \leq s \text{Re} \ d(x_{a(n)−1}, x_{a(n)}) + s^2 \text{Re} \ d(x_{a(n)}, x_{b(n)}) \\
+ s^2 \text{Re} \ d(x_{b(n)}, x_{b(n)−1}) \quad (22)
\]

It follows from (16), (19), (21) and (22) that

\[
\frac{\epsilon}{s^2} \leq \limsup_{n \to \infty} \text{Re} \ d(x_{a(n)−1}, x_{b(n)−1}) \leq s^3\epsilon \quad (23)
\]

Also, we can show that

\[
\frac{\epsilon}{s^2} \leq \liminf_{n \to \infty} \text{Re} \ d(x_{a(n)−1}, x_{b(n)−1}) \leq s^3\epsilon \quad (24)
\]
Further

\[ Re \, d(x_{a(n)-1}, x_{b(n)}) \leq s \, Re \, d(x_{a(n)-1}, x_{a(n)}) + s^2 \, Re \, d(x_{a(n)}, x_{b(n)-1}) \]
\[ + s^2 \, Re \, d(x_{b(n)-1}, x_{b(n)}) \]
\[ Re \, d(x_{a(n)}, x_{b(n)-1}) \leq s \, Re \, d(x_{a(n)}, x_{a(n)-1}) + s \, Re \, d(x_{a(n)-1}, x_{b(n)-1}) \]

From the above two inequalities with (16), (20) and (23), we get

\[ \frac{\epsilon}{s^3} \leq \limsup_{n \to \infty} Re \, d(x_{a(n)}, x_{b(n)-1}) \leq s^4 \epsilon \quad (25) \]

Now using (17), (20), (23) and (25) we have

\[ \epsilon \leq \frac{1}{2s^4}[s^4 \epsilon + s^4 \epsilon] + \limsup_{n \to \infty} [-Re \, \phi(Re \, d(x_{a(n)-1}, x_{b(n)-1}))] \]
\[ 0 \leq - \liminf_{n \to \infty} [Re \, \phi(Re \, d(x_{a(n)-1}, x_{b(n)-1}))] \]

\[ \liminf_{n \to \infty} Re \, \phi(Re \, d(x_{a(n)-1}, x_{b(n)-1})) = 0 \text{ as } Re \, \phi(.) \geq 0. \]

Take

\[ \alpha_n = Re \, d(x_{a(n)-1}, x_{b(n)-1}), \]

then \( \liminf_{n \to \infty} Re \, \phi(\alpha_n) = 0. \) Therefore, there is a subsequence \( \{\alpha_{n_k}\} \) such that

\[ \lim_{k \to \infty} Re \, \phi(\alpha_{n_k}) = 0. \]

Since \( \phi \) is continuous, we have \( Re \, \phi(\lim_{k \to \infty} \alpha_{n_k}) = \lim_{k \to \infty} Re \, \phi(\alpha_{n_k}) = 0. \) Also by the property of \( \phi \), we obtain \( \lim_{k \to \infty} \alpha_{n_k} = 0, \) which in turn implies \( \liminf_{n \to \infty} \alpha_n \leq 0. \) That is \( \liminf Re(d(x_{a(n)-1}, x_{b(n)-1})) \leq 0, \) which is a contradiction to (24). So that \( \{x_n\} \) is a Cauchy sequence and the rest of the proof is followed by Theorem 1.

References