

FIXED POINT THEOREM FOR WEAKLY C-CONTRACTIVE TYPE MAPPING ON COMPLEX VALUED METRIC SPACES

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Abstract: In this article, we prove that if f is a weakly C -contractive type self map on a complete complex valued metric space (X, d) then it has a unique fixed point and also, we extend this result to complete complex valued b -metric spaces.

AMS Subject Classification: 47H10, 54H25

Key Words: complex valued metric space, complex valued b -metric space, weakly C -contractive type map, fixed point

1. Introduction and Preliminaries

According to Azam et. al [1], we have the following notation and definitions. Let $\mathbb{C}$ denote the set of complex numbers and $a, b \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows: $a \preceq b$ if and only if $Re(a) \leq Re(b)$, $Im(a) \leq Im(b)$.

Definition 1. Let X be a non-empty set. Suppose that the mapping $d : X \times X \rightarrow \mathbb{C}$ satisfies:

Received: August 12, 2017

Revised: July 12, 2018

Published: July 21, 2018

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url: www.acadpubl.eu

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1. $0 \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$, for all $x, y \in X$
3. $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Definition 2. Let (X, d) be a complex valued metric space and $\{x_n\}_{n \geq 1}$ be a sequence in X and $x \in X$. We say that:

1. The sequence $\{x_n\}_{n \geq 1}$ converges to x if for every $r \in \mathbb{C}$, with $0 \prec r$ there is a positive integer n_0 such that for all $n > n_0$, $d(x_n, x) \prec r$. We write $x_n \rightarrow x$, as $n \rightarrow \infty$.
2. The sequence $\{x_n\}_{n \geq 1}$ is Cauchy sequence if for every $r \in \mathbb{C}$, with $0 \prec r$ there is a positive integer n_0 such that for all $n, m > n_0$, $d(x_n, x_m) \prec r$.
3. The metric space (X, d) is a complete complex valued metric space if every Cauchy sequence is convergent.

There are several number of works in contraction type mappings, among these we consider one such mapping namely weakly C -contractive map.

Definition 3. [4] Let (X, d) be a metric space, the map $f : X \rightarrow X$ is called weakly C -contractive if

$$d(f(x), f(y)) \leq \frac{1}{2}[d(x, f(y)) + d(y, f(x))] - \phi(d(x, f(y)), d(y, f(x))) \quad (1)$$

for all $x, y \in X$ and $\phi : [0, \infty)^2 \rightarrow [0, \infty)$ is continuous and $\phi(x, y) = 0$ if and only if $x = y = 0$.

In [4], Choudhury proved that in complete metric space, weakly C -contractive self map has a unique fixed point.

2. Main Results

In this section, we present our main results. Before proceed further we first introduce the generalization of weakly C -contractive map.

Definition 4. Let (X, d) be a complex valued metric space, the map $f : X \rightarrow X$ is called weakly C -contractive type map if

$$d(f(x), f(y)) \preceq \frac{1}{2}[d(x, f(y)) + d(y, f(x))] - \phi(M(x, y)) \quad (2)$$

for all $x, y \in X$ and $M(x, y) = \max\{Re\ d(x, y), Im\ d(x, y)\}$ with the map $\phi : [0, \infty) \rightarrow A = \{a + ib : a, b \in (0, \infty)\} \cup \{0\}$ has the following property:

1. ϕ is continuous
2. $x \leq y \implies \phi(x) \preceq \phi(y)$
3. $\phi(x) = 0$ if and only if $x = 0$.

Now we prove our first main result about existence of fixed point of the weakly C -contractive type map defined on complete complex valued metric space. To prove this result, we need the following lemma in the sequel.

Lemma 1. Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is a Cauchy sequence if and only if to every $\epsilon > 0$, there is a positive integer n_0 such that $Re\ d(x_n, x_m) < \epsilon$ and $Im\ d(x_n, x_m) < \epsilon$, for all $n, m > n_0$.

Proof. Let $\{x_n\}$ be a Cauchy sequence and $\epsilon > 0$ be given. Then take $0 \prec r = \epsilon + i\epsilon$, and by the definition of Cauchy sequence there is a positive integer n_0 such that $d(x_n, x_m) \prec r$, for all $n, m > n_0$. Which in turn implies $Re\ d(x_n, x_m) < \epsilon$ and $Im\ d(x_n, x_m) < \epsilon$, for all $n, m > n_0$.

Conversely, let $0 \prec r$ be given, then $Re\ r > 0$ and $Im\ r > 0$ and by the hypothesis, there are positive integers n_1, n_2 such that $Re\ d(x_n, x_m) < Re\ r$ and $Im\ d(x_n, x_m) < Im\ r$, for all $n, m > n_1$ and $Re\ d(x_n, x_m) < Im\ r$ and $Im\ d(x_n, x_m) < Re\ r$, for all $n, m > n_2$. Let $n_0 = \max\{n_1, n_2\}$, then $Re\ d(x_n, x_m) < Re\ r$ and $Im\ d(x_n, x_m) < Im\ r$, for all $n, m > n_0$ and hence $d(x_n, x_m) \prec r$, for all $n, m > n_0$, so that $\{x_n\}$ is a Cauchy sequence.

Theorem 1. Let (X, d) be a complete complex valued metric space and $f : X \rightarrow X$ be a weakly C -contractive type map, then f has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point and let $x_n = f(x_{n-1})$, $n = 1, 2, 3, \dots$. Suppose there is some positive integer n such that $d(x_n, f(x_n)) = 0$, then x_n becomes a fixed point of f , which completes the proof. So we assume that $d(x_n, f(x_n)) \neq 0$ for all $n \in \mathbb{N}$. Hence we have,

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1}))$$

$$\begin{aligned}
&\preceq \frac{1}{2}[d(x_n, f(x_{n-1})) + d(x_{n-1}, f(x_n))] - \phi(M(x_n, x_{n-1})) \\
&\preceq \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] - \phi(M(x_n, x_{n-1})) \quad (3) \\
d(x_{n+1}, x_n) &\preceq d(x_n, x_{n-1}), n = 1, 2, 3, \dots
\end{aligned}$$

From this, for $n = 1, 2, 3, \dots$, we have, $Re\ d(x_{n+1}, x_n) \leq Re\ d(x_n, x_{n-1})$ and $Im\ d(x_{n+1}, x_n) \leq Im\ d(x_n, x_{n-1})$ and which inturn implies $\{Re\ d(x_{n+1}, x_n)\}$ and $\{Im\ d(x_{n+1}, x_n)\}$ are decreasing sequences of non-negative real numbers and hence convergent. Let $\lim_{n \rightarrow \infty} Re\ d(x_{n+1}, x_n) = r$ and $\lim_{n \rightarrow \infty} Im\ d(x_{n+1}, x_n) = s$. Since $Re\ d(x_n, x_{n-1}) \leq M(x_n, x_{n-1})$ and by the property of ϕ , we have $\phi(Re\ d(x_n, x_{n-1})) \preceq \phi(M(x_n, x_{n-1}))$ Hence for $n = 1, 2, 3, \dots$, the equation(3) becomes

$$d(x_{n+1}, x_n) \preceq \frac{1}{2}[d(x_n, x_{n-1}) + d(x_{n+1}, x_n)] - \phi(Re\ d(x_n, x_{n-1})) \quad (4)$$

and therefore,

$$\begin{aligned}
Re\ d(x_{n+1}, x_n) &\leq \frac{1}{2}[Re\ d(x_n, x_{n-1}) + Red(x_{n+1}, x_n)] \\
&\quad - Re\ \phi(Re\ d(x_n, x_{n-1})) \quad (5)
\end{aligned}$$

making $n \rightarrow \infty$ on (5) and by the continuity of ϕ we have $r \leq \frac{1}{2}[r+r] - Re\ \phi(r)$, that is $Re\ \phi(r) = 0$. Hence, by the property of ϕ , $r = 0$. Thus,

$$\lim_{n \rightarrow \infty} Re\ d(x_{n+1}, x_n) = 0 \quad (6)$$

Similarly,

$$\lim_{n \rightarrow \infty} Im\ d(x_{n+1}, x_n) = 0. \quad (7)$$

Now we have to prove that the sequence $\{x_n\}$ is a Cauchy sequence. Arguing by contradiction, using lemma(1) we may assume that there is an $\epsilon > 0$ and the sequences $\{a(n)\}$ and $\{b(n)\}$ of positive integers such that $a(n) > b(n) > n$, $Re\ d(x_{a(n)}, x_{b(n)}) \geq \epsilon$ and $Re\ d(x_{a(n)-1}, x_{b(n)}) < \epsilon$, for all positive integers n . Now,

$$\begin{aligned}
\epsilon &\leq Re\ d(x_{a(n)}, x_{b(n)}) = Re\ d(f(x_{a(n)-1}), f(x_{b(n)-1})) \\
&\leq \frac{1}{2}[Re\ d(x_{a(n)-1}, x_{b(n)}) + Re\ d(x_{b(n)-1}, x_{a(n)})] \\
&\quad - Re\ \phi(Re\ d(x_{a(n)-1}, x_{b(n)-1})) \quad (8)
\end{aligned}$$

Also

$$\begin{aligned}\epsilon &\leq Re\,d(x_{a(n)}, x_{b(n)}) \\ &\leq Re\,d(x_{a(n)}, x_{a(n)-1}) + Re\,d(x_{a(n)-1}, x_{b(n)}) \\ &\leq Re\,d(x_{a(n)}, x_{a(n)-1}) + \epsilon\end{aligned}\quad (9)$$

making $n \rightarrow \infty$ on (9) and using (6) we get,

$$\lim_{n \rightarrow \infty} Re\,d(x_{a(n)}, x_{b(n)}) = \epsilon \text{ and } \lim_{n \rightarrow \infty} Re\,d(x_{a(n)-1}, x_{b(n)}) = \epsilon \quad (10)$$

Next

$$\begin{aligned}\epsilon &\leq Re\,d(x_{a(n)}, x_{b(n)}) \\ &\leq Re\,d(x_{a(n)}, x_{a(n)-1}) + Re\,d(x_{a(n)-1}, x_{b(n)-1}) \\ &\quad + Re\,d(x_{b(n)-1}, x_{b(n)})\end{aligned}\quad (11)$$

$$\begin{aligned}Re\,d(x_{a(n)-1}, x_{b(n)-1}) &\leq Re\,d(x_{a(n)-1}, x_{a(n)}) + Re\,d(x_{a(n)}, x_{b(n)}) \\ &\quad + Re\,d(x_{b(n)}, x_{b(n)-1})\end{aligned}\quad (12)$$

making $n \rightarrow \infty$ on both inequalities (11) and (12) together with equations (6) and (10), we obtain

$$\lim_{n \rightarrow \infty} Re\,d(x_{a(n)-1}, x_{b(n)-1}) = \epsilon \quad (13)$$

Also

$$\begin{aligned}Re\,d(x_{a(n)-1}, x_{b(n)}) &\leq Re\,d(x_{a(n)-1}, x_{a(n)}) + Re\,d(x_{a(n)}, x_{b(n)-1}) \\ &\quad + Re\,d(x_{b(n)-1}, x_{b(n)})\end{aligned}$$

$$Re\,d(x_{a(n)}, x_{b(n)-1}) \leq Re\,d(x_{a(n)}, x_{a(n)-1}) + Re\,d(x_{a(n)-1}, x_{b(n)-1})$$

making $n \rightarrow \infty$ on the above two inequalities and using (6), (10), (13) we get,

$$\lim_{n \rightarrow \infty} Re\,d(x_{a(n)}, x_{b(n)-1}) = \epsilon \quad (14)$$

now allow $n \rightarrow \infty$ in (8) and using (10), (13), (14) with the continuity of ϕ we have $\epsilon \leq \frac{1}{2}[\epsilon + \epsilon] - Re\,\phi(\epsilon)$, that is $Re\,\phi(\epsilon) = 0$ and by the property of ϕ , we have $\epsilon = 0$. Which is a contradiction to $\epsilon > 0$, so that $\{x_n\}$ is a Cauchy sequence and since X is complete, there is a $p \in X$ so that $x_n \rightarrow p$, as $n \rightarrow \infty$.

Next we prove that this p is a fixed point of f .

$$\begin{aligned} d(p, f(p)) &\preceq d(p, x_{n+1}) + d(f(x_n), f(p)) \\ &\preceq d(p, x_{n+1}) + \frac{1}{2}[d(x_n, f(p)) + d(p, f(x_n))] - \phi(M(x_n, p)) \\ \operatorname{Re} d(p, f(p)) &\leq \operatorname{Re} d(p, x_{n+1}) + \frac{1}{2}[\operatorname{Re} d(x_n, p) + \operatorname{Re} d(p, f(p)) \\ &\quad + \operatorname{Re} d(p, x_{n+1})] - \operatorname{Re} \phi(\operatorname{Re} d(x_n, p)) \end{aligned}$$

which on making $n \rightarrow \infty$ by using continuity of ϕ and lemma 2 (see[1]), we have $\operatorname{Re} d(p, f(p)) \leq \frac{1}{2}\operatorname{Re} d(p, f(p)) - \operatorname{Re} \phi(0) \leq \frac{1}{2}\operatorname{Re} d(p, f(p))$ which is possible only if $\operatorname{Re} d(p, f(p)) = 0$. similarly, we have $\operatorname{Im} d(p, f(p)) = 0$. Thus, $d(p, f(p)) = 0$ and therefore $f(p) = p$. To prove the uniqueness, suppose for p and q are two fixed points of f . Now, $d(p, q) = d(f(p), f(q)) \preceq \frac{1}{2}[d(p, f(q)) + d(q, f(p))] - \phi(M(p, q))$ and $0 \preceq -\phi(\operatorname{Re} d(p, q))$ by the property of ϕ , $\operatorname{Re} d(p, q) = 0$ and similarly $\operatorname{Im} d(p, q) = 0$, so that $d(p, q) = 0$ or $p = q$ completes the proof.

In[6], Rao et. al., introduced the complex valued b -metric space and studied their consequences. Now, we extended Theorem 1 in complete complex valued b -metric spaces as follows.

Theorem 2. Let (X, d) be a complete complex valued b -metric space with $s > 1$ and the map $f : X \rightarrow X$ satisfies

$$d(f(x), f(y)) \preceq \frac{1}{2s^4}[d(x, f(y)) + d(y, f(x))] - \phi(M(x, y)) \quad (15)$$

for all $x, y \in X$ with ϕ and $M(x, y)$ as in definition 8. Then f has a unique fixed point (Note that here the weakly C -contractive type map depend on s).

Proof. Let $x_0 \in X$ be an arbitrary point and let $x_n = f(x_{n-1})$, $n = 1, 2, 3, \dots$. If some positive integer n such that $d(x_n, f(x_n)) = 0$, then x_n becomes a fixed point of f , which completes the proof. So we assume that $d(x_n, f(x_n)) \neq 0$, for all n . As in theorem 1, we can easily shows that

$$\lim_{n \rightarrow \infty} \operatorname{Re} d(x_{n+1}, x_n) = 0 \text{ and } \lim_{n \rightarrow \infty} \operatorname{Im} d(x_{n+1}, x_n) = 0. \quad (16)$$

because,

$$\begin{aligned} d(f(x), f(y)) &\preceq \frac{1}{2s^4}[d(x, f(y)) + d(y, f(x))] - \phi(M(x, y)) \\ &\preceq \frac{1}{2}[d(x, f(y)) + d(y, f(x))] - \phi(M(x, y)) \end{aligned}$$

Now we have to prove that the sequence $\{x_n\}$ is a Cauchy sequence. Arguing by contradiction, using Lemma(1) we may assume that there is an $\epsilon > 0$ and

the sequences $\{a(n)\}$ and $\{b(n)\}$ of positive integers such that $a(n) > b(n) > n$, $Re\ d(x_{a(n)}, x_{b(n)}) \geq \epsilon$ and $Re\ d(x_{a(n)-1}, x_{b(n)}) < \epsilon$, for all positive integers n . Now,

$$\begin{aligned}\epsilon &\leq Re\ d(x_{a(n)}, x_{b(n)}) = Re\ d(f(x_{a(n)-1}), f(x_{b(n)-1})) \\ &\leq \frac{1}{2s^4} [Re\ d(x_{a(n)-1}, x_{b(n)}) + Re\ d(x_{a(n)}, x_{b(n)-1})] \\ &\quad - Re\ \phi(Re\ d(x_{a(n)-1}, x_{b(n)-1}))\end{aligned}\quad (17)$$

also

$$\begin{aligned}\epsilon &\leq Re\ d(x_{a(n)}, x_{b(n)}) \\ &\leq s[Re\ d(x_{a(n)}, x_{a(n)-1}) + Re\ d(x_{a(n)-1}, x_{b(n)})] \\ &\leq sRe\ d(x_{a(n)}, x_{a(n)-1}) + s\epsilon\end{aligned}\quad (18)$$

so from (16), we get

$$\epsilon \leq \limsup_{n \rightarrow \infty} Re\ d(x_{a(n)}, x_{b(n)}) \leq s\epsilon \quad (19)$$

and

$$\frac{\epsilon}{s} \leq \limsup_{n \rightarrow \infty} Re\ d(x_{a(n)-1}, x_{b(n)}) \leq \epsilon \quad (20)$$

Next

$$\begin{aligned}\epsilon &\leq Re\ d(x_{a(n)}, x_{b(n)}) \\ &\leq s\ Re\ d(x_{a(n)}, x_{a(n)-1}) + s^2\ Re\ d(x_{a(n)-1}, x_{b(n)-1}) \\ &\quad + s^2\ Re\ d(x_{b(n)-1}, x_{b(n)})\end{aligned}\quad (21)$$

$$\begin{aligned}Re\ d(x_{a(n)-1}, x_{b(n)-1}) &\leq s\ Re\ d(x_{a(n)-1}, x_{a(n)}) + s^2\ Re\ d(x_{a(n)}, x_{b(n)}) \\ &\quad + s^2\ Re\ d(x_{b(n)}, x_{b(n)-1})\end{aligned}\quad (22)$$

It follows from (16), (19), (21) and (22) that

$$\frac{\epsilon}{s^2} \leq \limsup_{n \rightarrow \infty} Re\ d(x_{a(n)-1}, x_{b(n)-1}) \leq s^3\epsilon \quad (23)$$

Also, we can show that

$$\frac{\epsilon}{s^2} \leq \liminf_{n \rightarrow \infty} Re\ d(x_{a(n)-1}, x_{b(n)-1}) \leq s^3\epsilon \quad (24)$$

Further

$$\begin{aligned} \operatorname{Re} d(x_{a(n)-1}, x_{b(n)}) &\leq s \operatorname{Re} d(x_{a(n)-1}, x_{a(n)}) + s^2 \operatorname{Re} d(x_{a(n)}, x_{b(n)-1}) \\ &\quad + s^2 \operatorname{Re} d(x_{b(n)-1}, x_{b(n)}) \end{aligned}$$

$$\operatorname{Re} d(x_{a(n)}, x_{b(n)-1}) \leq s \operatorname{Re} d(x_{a(n)}, x_{a(n)-1}) + s \operatorname{Re} d(x_{a(n)-1}, x_{b(n)-1})$$

from the above two inequalities with (16), (20) and (23), we get

$$\frac{\epsilon}{s^3} \leq \limsup_{n \rightarrow \infty} \operatorname{Re} d(x_{a(n)}, x_{b(n)-1}) \leq s^4 \epsilon \quad (25)$$

now using (17), (20), (23) and (25) we have

$$\begin{aligned} \epsilon &\leq \frac{1}{2s^4} [s^4 \epsilon + s^4 \epsilon] + \limsup_{n \rightarrow \infty} [-\operatorname{Re} \phi(\operatorname{Re} d(x_{a(n)-1}, x_{b(n)-1}))] \\ 0 &\leq -\liminf_{n \rightarrow \infty} [\operatorname{Re} \phi(\operatorname{Re} d(x_{a(n)-1}, x_{b(n)-1}))] \end{aligned}$$

$\liminf_{n \rightarrow \infty} \operatorname{Re} \phi(\operatorname{Re} d(x_{a(n)-1}, x_{b(n)-1})) = 0$ as $\operatorname{Re} \phi(\cdot) \geq 0$. Take

$$\alpha_n = \operatorname{Re} d(x_{a(n)-1}, x_{b(n)-1}),$$

then $\liminf_{n \rightarrow \infty} \operatorname{Re} \phi(\alpha_n) = 0$. Therefore, there is a subsequence $\{\alpha_{n_k}\}$ such that $\lim_{k \rightarrow \infty} \operatorname{Re} \phi(\alpha_{n_k}) = 0$. Since ϕ is continuous, we have $\operatorname{Re} \phi(\lim_{k \rightarrow \infty} \alpha_{n_k}) = \lim_{k \rightarrow \infty} \operatorname{Re} \phi(\alpha_{n_k}) = 0$. Also by the property of ϕ , we obtain $\lim_{k \rightarrow \infty} \alpha_{n_k} = 0$, which in turn implies $\liminf \alpha_n \leq 0$. That is $\liminf_{n \rightarrow \infty} \operatorname{Re} d(x_{a(n)-1}, x_{b(n)-1}) \leq 0$, which is a contradiction to (24). So that $\{x_n\}$ is a Cauchy sequence and the rest of the proof is followed by Theorem 1.

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