A STAGE-STRUCTURED MODEL WITH MEDIA EFFECT FOR INNOVATION DIFFUSION OF NEW PRODUCT

Rishi Tuli\textsuperscript{1,}\textsuperscript{§}, Joydip Dhar\textsuperscript{2}, Harbax S. Bhatti\textsuperscript{3}

\textsuperscript{1}I.K.G. Punjab Technical University
Punjab, INDIA

\textsuperscript{2}ABV-Indian IIITM
Gwalior-474015, M.P., INDIA

\textsuperscript{3}B.B.S.B. Engineering College
Fatehgarh Sahib, Punjab, INDIA

Abstract: The current paper aims to study the dynamics of the mature non-adopter population who has to become an adopter population by the direct interaction with adopter population. Media plays a significant role in influencing the mature non-adopter population so they can meet the adopter class. It is shown that steady-state solutions are positive and bounded. Asymptotic stability analysis is carried out for all possible equilibrium points. At the interior equilibrium, the sensitivity analysis of the variables concerning the model parameters has been examined. Ultimately numerical simulation has been carried out to support our analytical findings with the different set of parameters.

AMS Subject Classification: 34C23, 34D20, 92D30

Key Words: boundedness, positivity, delay, sensitivity analysis

1. Introduction

Particularly market development is predicted and determined by implementing the Lotka-Volterra model, which recognizes the competing interaction regard-
Lotka-Volterra models have previously performed to the modeling market dynamics and market competition, fundamentally in a monopolistic business [3, 4]. It is broadly accepted that various internal and external determinants which incorporate, income groups, purchasing ability, social structuring, marketing enterprises like advertisements and door to door association with the end user or verbal information of people between the community, etc. It has a crucial role in the adoption or dispersion of innovation of latest technology with the society[5, 6, 7, 8, 9, 10, 11]. One can determine a real analysis of these innovation diffusion models among their utilization[9]. Researchers have performed several analyses with different aspects of innovation diffusion employing mathematical models [9].

In this research, the whole population has distributed into distinct age groups and every age group into two subgroups, especially, people using this know-how and the other which does not. The model under this study has a nonlinear diffusion model, and all the parameters are innovation coefficients correlated with specific age group and associated to advertisement moreover the imitation coefficients concerned to the impact of using technology group on non-user group [12, 13]. The stability of steady-state and the improved Lotka-Volterra Model to affect the growth of purchases of DTs and NBs made in Taiwan has been examined in [14]. Many researchers analyzed the innovation diffusion model, by including cost, promotion, and delivery, the innovation diffusion model, including different types of buyers, and the replacement model within the latest technology and the traditional one implemented systems dynamics, probabilistic system, an optimization system, and simulation on the computer[15]. The initial concern of the current work is the matter of the growth of a markets structure and by application, select in ways of the evolutionary approach to population dynamics and population biology[16]. The number of diseases, which often concerns the kids, e.g., Measles, Rubella, etc.. Rubella is the disease that transpires extensive, generally affecting kids up to the age of 10 years old [17]. The author introduces an epidemic model as the infection spread from an interaction of infected person to premature person. Thus, in that way, in the innovation diffusion model, the non-adopter person can become the user of innovation by the direct interaction with the adopter person [18].

Keeping in the totality of above discussion, in the present analysis, a stage-structured model with media effect on the mature non-adopter population has been proposed. This paper is planned as follows: In Section 1, describe the introduction of the research study. In Section 2, representation of the proposed model. In Section 3, positivity and boundedness of the proposed model have
studied. In Section 4, asymptotic stability for all possible equilibrium position has analyzed. In Section 5, the sensitivity analysis of the essential variables at interior equilibrium position concerning model parameters has achieved. At last, in part Section 6, numerically simulation has been carried out to support our analytical findings with a different set of parameters.

2. Representation of Proposed Model

The assumption of the proposed model is discussed below:

(i) The total population at time $t$ is classified into three classes premature non-adopter population $N_1(t)$, mature non-adopter population $N_2(t)$ and adopter population $A(t)$.

(ii) $\tau$ is the maturity period for premature to mature non-adopter population.

(iii) Here, we suppose that only mature non-adopter population can become user of the innovation.

(iv) Mature non-adopter population can become the user of the innovation with media effect or direct interactions with the user of that particular innovation.

(v) Assume, $\alpha_1$ is the birth rate of premature non-adopter, Next $\delta_1, \delta_2, \delta_3$ are the death rates of premature non-adopter $N_1(t)$, mature non-adopter $N_2(t)$ and adopter population $A(t)$. Where, $\gamma_1$ is the interaction rate of mature adopter population $N_2(t)$ and adopter population $A(t)$, Finally, $\gamma_2$ is the frustration rate of adopter population $A(t)$ and joins back the mature non-adopter class $N_2(t)$, $m_1$ is the media effect.

The proposed system is of the form:

$$\frac{dN_1}{dt} = \alpha_1(N_2 + A) - \alpha_1 e^{-\delta_1 \tau}(N_2 + A)(t - \tau) - \delta_1 N_1, \quad (1)$$

$$\frac{dN_2}{dt} = \alpha_1 e^{-\delta_1 \tau}(N_2 + A)(t - \tau) - \gamma_1 N_2 A + \gamma_2 A - m_1 N_2 - \delta_2 N_2, \quad (2)$$

$$\frac{dA}{dt} = \gamma_1 N_2 A - \gamma_2 A + m_1 N_2 - \delta_3 A, \quad (3)$$

with initial values:

$N_1(0) > 0, N_2(0) > 0$ and $A(0) > 0$ for all $t \geq 0$. 
3. Positivity and Boundedness of Proposed Model

For the positivity and boundedness of the proposed system (1)-(3), we state and evaluate the mentioned below lemmas:

**Lemma (3.1):** The solution for the proposed system (1)-(3), with initial values are positive, for all \( t \geq 0 \).

**Proof.** Here \( t \in [0, \tau] \), for solving the equation (1), can be written as

\[
\frac{dN_1}{dt} \geq -\alpha_1 e^{-\delta_1 \tau} (N_2 + A) - \delta_1 N_1,
\]

which exhibit that

\[
N_1(t) \geq e^{-K} [N_1(0) - \int_0^t (\alpha_1 e^{-\delta_1 \tau} (N_2 + A)) e^K dv] > 0,
\]

where \( k = \int_0^t \delta_1 du \).

For \( t \in [0, \tau] \), the expression (2), can be recast as

\[
\frac{dN_2}{dt} \geq -\delta_2 N_2 - \gamma_1 AN_2 - m_1 N_2,
\]

which exhibit that

\[
N_2(t) \geq N_2(0) \exp \left\{ - \int_0^t (\delta_2 + \gamma_1 A + m_1) dv \right\} > 0.
\]

For \( t \in [0, \tau] \), the expression (3) can be recast as

\[
\frac{dA}{dt} \geq -\gamma_2 A - \delta_3 A,
\]

which exhibit that

\[
A(t) \geq A(0) \exp \left\{ - \int_0^t (\gamma_2 + \delta_3) dv \right\} > 0.
\]

From the above discussion, we prove that \( N_1(t) > 0, N_2(t) > 0 \) and \( A(t) > 0 \) for all \( t \geq 0 \).

**Lemma (3.2):** The solution for proposed model equations (1)-(3), with initial conditions is bounded uniformly in \( \zeta \), where

\[
\zeta = \{(N_1, N_2, A) : 0 \leq N_1(t) \leq ce^{-\delta_1 t}, 0 \leq N_2(t) + A(t) \leq c_1 e^{-K_1 t}\}.
\]
Proof. Here just outline concerning the proof is presented. For more detail see [19]. Here we supposed that all the values of a variable supposed in the model equation (1)-(3), concerns to the region $\zeta$ and are positive. From first equation of the model (1)-(3), we have:

$$\frac{dN_1}{dt} + \delta_1 N_1 = 0.$$ 

It implies that $0 \leq N_1(t) \leq ce^{-\delta_1 t}$. Now, From second and third equation of the model (1)-(3), we have

\[
\frac{d(N_2 + A)}{dt} = \alpha_1 e^{-\delta_1 \tau} (N_2 + A) - \delta_2 N_2 - \delta_3 A, \\
\frac{d(N_2 + A)}{dt} \leq \alpha_1 e^{-\delta_1 \tau} (N_2 + A) - \delta(N_2 + A); \text{ Where } \delta = \min(\delta_1, \delta_2).
\]

On solving, we obtain $0 \leq (N_2 + A) \leq c_1 e^{-K_1 t}$ where $K_1 = \delta - \alpha_1 e^{-\delta_1 \tau}$. Hence, the solution of the proposed system (1)-(3), is bounded. $\square$

4. Dynamical Nature of the System

(i) The equilibrium $E_0(0, 0, 0)$ always exists.

(ii) The interior equilibrium $E^*(N_1^*, N_2^*, A^*)$ exists, if $H_1$ and $H_2$ holds, where $N_1^*, N_2^*, A^*$ is given by

\[
N_1^* = \frac{\alpha_1 (1 - e^{-\delta_1 \tau}) (\delta_3 - \delta_2) (\alpha_1 (m_1 + \gamma_2 + \delta_3) e^{-\delta_1 \tau} - (\gamma_2 \delta_2 + m_1 \delta_3 + \delta_2 \delta_3))}{\gamma_1 \delta_1 (\alpha_1 e^{-\delta_1 \tau} - \delta_2) (\delta_3 - e^{-\delta_1 \tau})},
\]

\[
N_2^* = \frac{\alpha_1 (1 - e^{-\delta_1 \tau}) (\delta_3 - \delta_2) (\alpha_1 (m_1 + \gamma_2 + \delta_3) e^{-\delta_1 \tau} - (\gamma_2 \delta_2 + m_1 \delta_3 + \delta_2 \delta_3))}{\gamma_1 (\alpha_1 e^{-\delta_1 \tau} - \delta_2)},
\]

\[
A^* = \frac{\alpha_1 (1 - e^{-\delta_1 \tau}) (\delta_3 - \delta_2) (\alpha_1 (m_1 + \gamma_2 + \delta_3) e^{-\delta_1 \tau} - (\gamma_2 \delta_2 + m_1 \delta_3 + \delta_2 \delta_3))}{\gamma_1 (\delta_3 - \alpha_1 e^{-\delta_1 \tau})}.
\]

(4)

For the existence of interior equilibrium $H_1$ and $H_2$ holds, where $H_1 : \delta_3 > \delta_2$, and $H_2: \frac{1}{\delta_1} \ln \frac{\gamma_1}{\delta_2} < \tau < \Max[\frac{1}{\delta_1} \ln \frac{\alpha_1 (m_1 + \gamma_2 + \delta_3)}{\gamma_2 \delta_2 + m_1 \delta_3 + \delta_2 \delta_3}, \frac{1}{\delta_1} \ln \frac{\alpha_1}{\delta_2}]$ holds.

As the transcendental polynomials equation concerning the 2nd degree of the pattern

$$\lambda^2 + p\lambda + r + (s\lambda + q)e^{-\lambda \tau} = 0.$$ 

(5)

was studied by [20, 18] and examined the mentioned below results:
[B1] \( p + s > 0; \)

[B2] \( q + r > 0; \)

[B3] either \( s^2 - p^2 + 2r < 0 \) and \( r^2 - q^2 > 0 \) or \( (s^2 - p^2 + 2r)^2 < 4(r^2 - q^2); \)

[B4] either \( r^2 - q^2 < 0 \) or \( s^2 - p^2 + 2r > 0 \) and \( (s^2 - p^2 + 2r)^2 = 4(r^2 - q^2); \)

[B5] either \( r^2 - q^2 > 0 \), \( s^2 - p^2 + 2r > 0 \) and \( (s^2 - p^2 + 2r)^2 > 4(r^2 - q^2). \)

**Lemma (4.1)**

(i) If [B1]-[B3] exists, entire roots of (5) posses negative real parts for every \( \tau \geq 0. \)

(ii) If [B1], [B2] and [B4] exists and \( \tau = \tau_j^+ \), then equation (5) posses a pair of purely imaginary roots \( \pm iw_+ \). When \( \tau = \tau_j^+ \) then entire roots of (5) except \( \pm iw_+ \) posses negative real parts.

(iii) If [B1], [B2] and [B5] exists and \( \tau = \tau_j^+ (\tau = \tau_j^- \) respectively) then equation (5) has a pair of purely imaginary roots \( \pm iw_+ (\pm iw_- \) respectively). Moreover \( \tau = \tau_j^+ (\tau_j^- \) respectively), then entire roots of (5) except \( \pm iw_+ (\pm iw_- \) respectively) posses negative real parts.

**Theorem 1.** For the system (1)-(4), we have If \( \tau > \frac{1}{\delta_1} \text{Max}(\text{In} \alpha_1 p, \text{In} \alpha_2 p) \), \( \text{In} \alpha_1 p, \frac{1}{2} \text{In} \alpha_2 p \) exists, then equilibrium \( E_0(0, 0, 0) \) is locally asymptotically stable for all \( \tau. \)

**Proof.** The variational matrix corresponding to model equation (1)-(3), is given as

\[
J = \begin{pmatrix} -\delta_1 & \alpha_1 - e^{-\delta_1 \tau} \alpha_1 e^{-\lambda \tau} & \alpha_1 - e^{-\delta_1 \tau} \alpha_1 e^{-\lambda \tau} \\ 0 & e^{-\delta_1 \tau} \alpha_1 e^{-\lambda \tau} - \delta_2 - A \gamma_1 - m_1 & e^{-\delta_1 \tau} \alpha_1 e^{-\lambda \tau} + \gamma_2 - N_2 \gamma_1 \\ 0 & A \gamma_1 + m_1 & N_2 \gamma_1 - \gamma_2 - \delta_3 \end{pmatrix}.
\]

The corresponding characteristic expression regarding jacobian matrix at the equilibrium position \( E_0(0, 0, 0) \) can be written as \( \lambda = -\delta_1 \) and

\[
\lambda^2 + (m_1 + \gamma_2 + \delta_2 + \delta_3) \lambda + (\gamma_2 \delta_2 + m_1 \delta_3 + \delta_2 \delta_3) - [(\alpha_1 \lambda + m_1 \alpha_1 + \alpha_1 \gamma_2 + \alpha_1 \delta_3) e^{-\delta_1 \tau} e^{-\lambda \tau} = 0.
\]

Which is the similar form given by [20].

\[
\lambda^2 + p \lambda + r + (s \lambda + q) e^{-\lambda \tau} = 0.
\]
On comparing equation (6)-(7),

\[ p = m_1 + \gamma_2 + \delta_2 + \delta_3, \]
\[ r = \gamma_2 \delta_2 + m_1 \delta_3 + \delta_2 \delta_3, \]
\[ s = -\alpha_1 e^{-\delta_1 \tau}, \]
\[ q = -(m_1 \alpha_1 + \alpha_1 \gamma_2 + \alpha_1 \delta_3)e^{-\delta_1 \tau}. \]

We have,

\[ B_1 = p + s = m_1 + \gamma_2 + \delta_2 + \delta_3 - \alpha_1 e^{-\delta_1 \tau}, \]
\[ B_2 = r + q = \gamma_2 \delta_2 + m_1 \delta_3 + \delta_2 \delta_3 - (m_1 \alpha_1 + \alpha_1 \gamma_2 + \alpha_1 \delta_3)e^{-\delta_1 \tau}. \]

Using Lemma (4.1), \([B_1], [B_2]\) and \([B_3]\) holds, if \(\tau > \frac{1}{\delta_1} \text{Max}(\ln \frac{\alpha_1 p}{m_1}, \ln \frac{\alpha_1 r}{m_2 \alpha_1}, \frac{1}{2} \ln \frac{\alpha_2^2}{H_3})\)

entire roots of (1)-(3) posses negative real components for every \(\tau\) and therefore the system is locally asymptotically stable. Where \(H_3 = (m_1 + \delta_2)^2 + (\gamma_2 + \delta_3)^2 + 2m_1 \gamma_2.\)

\[ \Box \]

**Theorem 2.** If Minimum \(\frac{1}{\delta_1} \left(\ln \frac{\alpha_1 p_1}{m_1}, \ln \frac{H_3}{H_5}, \frac{1}{2} \ln \frac{H_6}{H_7}\right) \)< \(\tau \)< \(\frac{1}{\delta_1} \ln \frac{H_8}{H_9}\), the interior equilibrium point \(E^*(N_1^*, N_2^*, A^*)\) is locally asymptotically stable for all \(\tau\).

**Proof.** The corresponding characteristic equation regarding jacobian matrix at the interior equilibrium position \(E^*(N_1^*, N_2^*, A^*)\) can be written as:

\[ (\delta_1 + \lambda) F_1(\lambda) = 0. \tag{8} \]

On solving equation (8) \(\lambda = -\delta_1\) and

\[ F_1(\lambda) = \lambda^2 + (m_1 + \frac{m_1 N_3^*}{A^*} + A^* \gamma_1 + \delta_2) \lambda + \left(\frac{m_2 N_2^*}{A^*} + 2m_1 N_2^* \gamma_1\right) + A^* N_2^* \gamma_1^2 - m_1 \gamma_2 - A^* \gamma_1 \gamma_2 - [\alpha_1 \lambda + m_1 \alpha_1 + \frac{m_1 \alpha_1 N_2^*}{A^*} + \alpha_1 \gamma_1 A^*] e^{-\delta_1 \tau} e^{-\lambda \tau} = 0. \tag{9} \]

From above \(F_1(\lambda)\) which is the similar form given by [20].

\[ \lambda^2 + p_1 \lambda + r_1 + (s_1 \lambda + q_1)e^{-\lambda \tau} = 0. \tag{10} \]

On comparing equation (9) and (10)

where \(p_1 = m_1 + \frac{m_1 N_3^*}{A^*} + A^* \gamma_1 + \delta_2, \]
\[ r_1 = \frac{m_2 N_2^*}{A^*} + 2m_1 N_2^* \gamma_1 + AN_2^* \gamma_1^2 - m_1 \gamma_2 - A^* \gamma_1 \gamma_2, \]
Using the Lemma (4.1),

\[ B_1 = p_1 + s_1 = m_1 + \frac{m_1 N_2^*}{A^*} + A^* \gamma_1 + \delta_2 - \alpha_1 e^{-\delta_1 \tau}, \]

\[ B_2 = r_1 + q_1 = \frac{m_1^2 N_2^*}{A^*} + 2 m_1 N_2^* \gamma_1 + A N_2^* \gamma_1^2 - m_1 \gamma_2 - A^* \gamma_1 \gamma_2\]

\[ - (m_1 \alpha_1 + \frac{m_1 \alpha_1 N_2^*}{A^*} + \alpha_1 \gamma_1 A^*) e^{-\delta_1 \tau}. \]

From Lemma (4.1), \([B1], [B2]\) and \([B3]\) exists, If minimum

\[ \frac{1}{\delta_1} (\ln \frac{\alpha_1}{p_1}, \ln \frac{H_4}{H_5}, \frac{1}{2} \ln \frac{H_6}{H_7}) < \tau < \frac{1}{\delta_1} \ln \frac{H_8}{H_9} \]

holds, then entire roots of equations (1)-(3) having negative real components for every \(\tau\) and hence the projected system is locally asymptotically stable.

Here: \(H_4 = \alpha_1 (m_1 + \frac{m N_2^*}{A^*} + \gamma_1 A^*) (m_1 \gamma_2) (A^* \gamma_1 \gamma_2),\)

\(H_5 = \frac{m_1 N_2^*}{A^*} (m_1 + \delta_2 + N_2^* (2m_1 + A^* \gamma_1),\)

\(H_6 = (\alpha_1 \gamma_1) (2m_1 N_2^* \gamma_1) (2A^* N_2^* \gamma_1^2),\)

\(H_7 = (m_1 + \delta_2 + A^* \gamma_1)^2 + \frac{(m_1 N_2^* \gamma_1)}{A^*} + 2 \gamma_2 (m_1 + A^* \gamma_1),\)

\(H_8 = \frac{m_1 N_2^*}{A^*} (m_1 + \delta_2 + \alpha_1) + (N_2^* \gamma_1 + \alpha_1) (m_1 + A^* \gamma_1) + N_2^* \gamma_1 m_1,\)

\(H_9 = m_1 \gamma_2 + A^* \gamma_1 \gamma_2.\)

5. Sensitivity Analysis

In this section, sensitivity examination of essential variables for the proposed model equations (1)-(3), concerning the context of the model parameters at the interior equilibrium position, has achieved. The respective sensitive parameters of the essential variables at interior equilibrium are shown in the Table 2 using parameter values given in the Table 1. It is observe that \(\alpha_1\) has a positive impact and \(\delta_1, \tau\) have negative impact on the \(N_1^*\). Hence we recognized significant variations in \(N_1^*\) by a little variation in these parameter. Further, \(\gamma_2\) and \(\alpha_1\) are positive influence parameter to \(N_2^*\) and \(\tau, \delta_1\) have the negative impact on \(N_2^*\). Other parameters are less sensible as compared to these parameters in \(N_2^*\). Finally, \(\alpha_1, \delta_1, \tau, \delta_3\) are the most sensitive parameter to \(A^*\) and all the other parameters are less sensitive to \(A^*\).
Parameter
\[ \begin{array}{|l|c|}
\hline
\alpha_1 & 0.25 \\
\delta_1 & 0.13 \\
\delta_2 & 0.02 \\
\delta_3 & 0.08 \\
\gamma_1 & 0.1 \\
\gamma_2 & 0.35 \\
\tau & 10 \\
m_1 & 0.1 \\
\hline
\end{array} \]

Table 1: Parametric values used for sensitivity examination

Table 2: The sensitivity indices \( \gamma_{x_i}^{y_j} = \frac{\partial x_i}{\partial y_j} \times y_j / x_i \) of the essential variables for the intended model equation (1-4) to the parameters \( y_j \) for the parameter values given in Table t2.

6. Numerical Simulations

To justify the analytic findings of the model (1)-(3), Numerical simulations are carried out by using MATLAB. It gives a touch of completeness to the analytic findings. It is observed that the equilibrium \( E_0(0,0,0) \) is stable for the parametric values \( \alpha_1 = 0.1; \delta_1 = 0.03; \delta_2 = 0.02; \delta_3 = 0.24; \gamma_1 = 0.5; \gamma_2 = 0.4; \tau = 80; m_1 = 0.1. \) (see Figure 1).

The interior equilibrium point \( E^*(28.67, 4.05, 16.44) \) is stable for parameter values \( \alpha_1 = 0.25; \delta_1 = 0.13; \delta_2 = 0.02; \delta_3 = 0.08; \gamma_1 = 0.1; \gamma_2 = 0.35; \tau = 10; m_1 = 0.1. \) (see Figure 2).

The impact of media effect \( m_1 \) on the fraction of mature non-adopter pop-
The equilibrium point $E_0(0, 0, 0)$ is stable for the parametric values $\alpha_1 = 0.1; \delta_1 = 0.03; \delta_2 = 0.02; \delta_3 = 0.24; \gamma_1 = 0.5; \gamma_2 = 0.4; \tau = 80; m_1 = 0.1$.

7. Conclusions

In this paper, a three compartment age-structured population model is developed to study the behavior of the mature non-adopter population with the effect of media towards the adoption of a new product. The existence and stability nature of the model is calculated for all feasible equilibrium positions. We have also obtained the local stability conditions for the adopter-free along with interior equilibrium position. Sensitivity analysis at interior equilibrium is also studied. At last, to defend the analytical investigation, we perform the numerical experimentation with the different set of parameters. It is also observed that as the impact of media effect increases on the mature non-adopter population($N_2$), which consequences the mature non-adopter population($N_2$) converted into the adopter population($A$).
Figure 2: The interior equilibrium point $E^*(28.67, 4.05, 16.44)$ is stable for parameter values $\alpha_1 = 0.25; \delta_1 = 0.13; \delta_2 = 0.02; \delta_3 = 0.08; \gamma_1 = 0.1; \gamma_2 = 0.35; \tau = 10; m_1 = 0.1$.

Figure 3: The effect of media coefficient $m_1$ on mature non-adopter population ($N_2$) for the parametric values $\alpha_1 = 0.1; \delta_1 = 0.03; \delta_2 = 0.02; \delta_3 = 0.24; \gamma_1 = 0.5; \gamma_2 = 0.4; \tau = 80; m_1 = 0.1$. 
Figure 4: The effect of media coefficient $m_1$ on mature non-adopter population ($N_2$) for the parametric values $\alpha_1 = 0.25; \delta_1 = 0.13; \delta_2 = 0.02; \delta_3 = 0.08; \gamma_1 = 0.1; \gamma_2 = 0.35; \tau = 10; m_1 = 0.1$.

Acknowledgements

I express my warm thanks to I.K.G. Punjab Technical University, Punjab for providing me the facilities for the research being required.

References


