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DOMINATOR AND STRONG DOMINATOR CHROMATIC NUMBER OF PRODUCT GRAPHS

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Abstract: A dominator coloring of a graph G is a proper coloring of G in which every vertex dominates every vertex of at least one color class. The minimum number of colors required for a dominator coloring of G is called the dominator chromatic number of G and is denoted by $\chi_d(G)$. Let $G = (V, E)$ be a simple graph. A proper color partition of $V(G)$ is called a strong dominator coloring, if the vertex v strongly dominates u then $deg(v) \geq deg(u)$. In this paper, we obtain the strong dominator chromatic number for the tensor product of $K_m \otimes K_n$, and $P_m \otimes K_{1,n}$ respectively. Also upper bound and lower bound for the the dominator chromatic number of modular product graphs are discussed.

AMS Subject Classification: 05C15
Key Words: coloring, domination, dominator coloring, strong dominator coloring, tensor product

1. Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow the book by D. B. West [11] for basic terminology and definitions. The concept of a dominator coloring in a graph was introduced and studied by Gera et al.,[2], studied further by Gera [4, 3] and recently by Chellai and Maffray[1].

There has been a great deal of interest in relating graph coloring and the

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dominating set. The literature on the subject of domination parameters in graphs up to the year 1997 has been surveyed and detailed in the two books[5, 6]. Gera established the following upper and lower bounds on the dominator chromatic number of an arbitrary graph in terms of its domination number and chromatic number,$max\{\gamma(G), \chi(G)\} \leq \chi_d(G) \leq \gamma(G) + \chi(G)$. The dominator coloring of Central graph of path, cycle and central, middle,total graph of star graph families are studied in [7, 8].

Let $G = (V(G), E(G))$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The degree of a vertex v is $\deg_G(v) = |N_G(v)|$. The minimum and maximum degree of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$ respectively.

Definition 1. A proper vertex coloring of a graph G is an assignment of colors to the vertices of G, one color to each vertex, so that adjacent vertices are assigned distinct colors. If k colors are used, then the coloring is referred to as k coloring. In a given coloring of G, a color class of the coloring is a set consisting of all those vertices assigned the same color.

Definition 2. A dominating set, D of a graph G is a subset of the vertices in G such that for each vertex $v, N_G[v] \cap D \neq \emptyset$ The domination number $\gamma(G)$ of G is the cardinality of a minimum dominating set.

2. Definition and Properties

Definition 3. [9] Let $G = (V, E)$ be a simple graph. A proper color partition of $V(G)$ is called a strong dominator color partition if every vertex strongly dominates some color class. The minimum cardinality of a strong dominator coloring partition of G is called the strong dominator coloring number of G and is denoted by $\chi_{sd}(G)$

Definition 4. The tensor product $G\otimes H$ of G and H has vertex set $V(G\otimes H) = V(G) \times V(H)$,
edge set $E(G\otimes H) = \{(a, b)(c, d)|ac \in E(G) \text{and} bd \in E(H)\}$.

Theorem 5. *For $n, m \geq 4$, the strong dominator chromatic number of the tensor product of $K_m\otimes K_n$ is,*

$$\chi_{sd}(K_m\otimes K_n) = \begin{cases} m + 2 \text{ if } n \geq m \\ n + 2 \text{ if } m > n \end{cases}$$

Proof. Let $G = K_m\otimes K_n$ has the vertex set

$$V(G) = \{(u_i, v_j)/1 \leq i \leq m, 1 \leq j \leq n\}$$

and its strong dominator chromatic number be $\chi_{sd}(G)$.
Case 1. $n \geq m$
Now we define a proper strong dominator coloring function $\varphi : V(K_m \otimes K_n) \to \{c_1, c_2, c_3, \ldots, c_m, c_{m+1}, c_{m+2}\}$

$$\varphi(u_i, v_j) = \begin{cases} c_{i-1} \text{ for } (u_i, v_j)/2 \leq i \leq m, 2 \leq j \leq n \\ c_m \text{ for } (u_1, v_j)/2 \leq j \leq n \\ c_{m+1} \text{ for } (u_i, v_1)/2 \leq i \leq m \\ c_{m+2} \text{ for } (u_1, v_1) \end{cases}$$

By the definition of strong dominator coloring, the vertices $(u_i, v_j)/2 \leq i \leq m, 2 \leq j \leq n$ strongly dominate the color class c_{m+2} and (u_1, v_1) strongly dominates itself. For $2 \leq j \leq n$, the vertices (u_1, v_j) strongly dominate the color class c_{m+1} and $2 \leq i \leq m$, the vertices (u_i, v_1) strongly dominate the color class c_m.

On the other hand, if we assign any one color $c_m (or c_{m+1})$ to (u_1, v_1) then the vertices $(u_i, v_j)/2 \leq i \leq m, 2 \leq j \leq n$ does not strongly dominate any color class. Therefore $\chi_{sd}(K_m \otimes K_n) \not< m + 2$. Hence $\chi_{sd}(K_m \otimes K_n) = m + 2$.
Case 2 $m > n$
Now we define a proper strong dominator coloring function $\varphi : V(K_m \otimes K_n) \to \{c_1, c_2, c_3, \ldots, c_n, c_{n+1}, c_{n+2}\}$

$$\varphi(u_i, v_j) = \begin{cases} c_{j-1} \text{ for } (u_i, v_j)/2 \leq i \leq m, 2 \leq j \leq n \\ c_n \text{ for } (u_1, v_j)/2 \leq j \leq n \\ c_{n+1} \text{ for } (u_i, v_1)/2 \leq i \leq m \\ c_{n+2} \text{ for } (u_1, v_1) \end{cases}$$

By the definition of strong dominator coloring, the vertices $(u_i, v_j)/2 \leq i \leq m, 2 \leq j \leq n$ strongly dominate the color class c_{n+2}. For $2 \leq j \leq n$, the vertices (u_1, v_j) strongly dominate the color class c_{n+1}. and For $2 \leq i \leq m$, the vertices (u_i, v_1) strongly dominate the color class c_n. Hence a verification shows that $\chi_{sd}(K_m \otimes K_n) = n + 2$. □

Theorem 6. *For $m, n \geq 4$, the strong dominator chromatic number of the tensor product of $P_m \otimes K_{1,n}$ is,*

$$\chi_{sd}(P_m \otimes K_{1,n}) = mn + 1.$$

Proof. Let $G = P_m \otimes K_{1,n}$, and its strong dominator chromatic number be $\chi_{sd}(G)$. Let $V(P_m) = \{u_i/1 \leq i \leq m\}$ and $V(K_{1,n}) = \{v, v_j/1 \leq j \leq n\}$, the vertex set V of G is defined by

$$V(G) = \{(u_i, v_j), /1 \leq i \leq m, 1 \leq j \leq n\}$$

. The following procedure gives the strong dominator coloring of $P_m \otimes K_{1,n}$ with $mn + 1$ colors.
Case 1. $n \geq m$
For $1 \leq i \leq m$, assign the color c_i to (u_i, v_1)
For $1 \leq i \leq m$, assign the color c_{m+i} to (u_i, v_2)
For $1 \leq i \leq m$, assign the color c_{2m+i} to (u_i, v_3)
For $1 \leq i \leq m$, assign the color c_{3m+i} to (u_i, v_4)
$\ldots\ldots\ldots\ldots,$
For $1 \leq i \leq m$, assign the color $c_{(n-1)m+i}$ to (u_m, v_{n+1})
The vertex $(u_i, v)/1 \leq i \leq m$ is colored by the color c_{mn+1}.
Case 2. $m > n$
For $1 \leq j \leq n$, assign the color c_i to (u_1, v_j)
For $1 \leq j \leq n$, assign the color c_{n+i} to (u_2, v_j)
For $1 \leq j \leq n$, assign the color c_{2n+i} to (u_3, v_j)
For $1 \leq j \leq n$, assign the color c_{3n+i} to (u_4, v_j)
$\ldots\ldots\ldots\ldots, \ldots$
For $1 \leq j \leq n$, assign the color $c_{(m-1)n+j}$ to (u_m, v_j)
The vertex $(u_i, v)/1 \leq i \leq m$ is colored by the color c_{mn+1}. By definition of strong dominator coloring, the above two cases, the vertex $(u_i, v_j)/1 \leq i \leq m, 2 \leq j \leq n$ dominates their own color class and remaining vertices $(u_i, v)/1 \leq i \leq m)$ strongly dominate any one color class of $(u_i, v_j)/1 \leq i \leq m, 2 \leq j \leq n$. Hence a verification shows that $\chi_{sd}(P_m \otimes K_{1,n}) = mn + 1$. □

Definition 7. A dominator coloring of a graph G is a proper coloring of graph such that every vertex of V dominates all vertices of at least one color class (possibly its own class). i.e., it is coloring of the vertices of a graph such that every vertex is either alone in its color class or adjacent to all vertices of at least one other class and is denoted by $\chi_d(G)$.

Definition 8. [10] The modular product is defined by If G and H are two graphs, then we let $G \diamond H$ represent the graph with vertex set $V(G) \times V(H)$ in which two vertices (x, u) and (y, v) are adjacent if
(a) $xy \in E(G)$ and $uv \in E(H)$, or
(b) $xy \notin E(G) and uv \notin E(H)$.

Theorem 9. *For any two graphs, G and H without isolated vertex, the $\chi_d(G \diamond H)$ is,*

$$\max\{\chi_d(G), \chi_d(H)\} \leq \chi_d(G \diamond H) \leq \min\{\chi_d(G) + n(H), \chi_d(H) + n(G)\}$$

Proof. Assume $V(G \diamond H) = \{u_i, v_j/1 \leq i \leq m, 1 \leq j \leq n\}$ It is not difficult to see that $\chi_d(G \diamond H) \geq \chi_d(G)$. Since $\chi_d(G \diamond H) \geq \chi_d(H)$. Hence we obtain $\chi_d(G \diamond H) \geq \max\{\chi_d(G), \chi_d(H)\}$next we prove the other one. Let min $\{\chi_d(G) + n(H), \chi_d(H) + n(G)\} = \chi_d(G) + n$ or $\chi_d(H) + m$. For the given dominator chromatic number of graph $(G \diamond H)$, the coloring $C = \{V_1, V_2, V_3, \ldots V_k\}$ in G. Next we define a coloring C' on $V(G \diamond H)$ such that $C'(u_i, v_j) = c(u_i, v_j)$ for each vertex (u_i, v_j). Let $(u_i, v_j) \in V(G \diamond H)$. Since C is a dominator coloring in G thus the modular product of $(G \diamond H)$ dominate at least any one color class in $V_i \times \{j\}$ or dominate itself and vice verse. Which implies that each vertex in $(G \diamond H)$ is dominate every vertex of some color class. From the observation we obtain that C' is also a dominator coloring in $(G \diamond H)$ Therefore $\chi_d(G \diamond H) \leq \min\{\chi_d(G) + n(H), \chi_d(H) + n(G)\}$. This completes the proof of the theorem. $\square$

Theorem 10. *For any positive integer $m \geq 3$ and $n \geq 4$, the dominator chromatic number of modular product of $K_m \diamond K_{1,n}$ is,*

$$\chi_d(K_m \diamond K_{1,n}) = 4.$$

Proof. Let

$$V(K_n) = \{u_1, u_2, u_3, \ldots, u_m\}$$
$$V(K_{1,n}) = \{v_1, v_2, v_3, \ldots, v_m, v_{n+1}\}$$

Assume $G = (K_m \diamond K_{1,n})$ has the vertex set
$V(G) = \{(u_i, v_j)/1 \leq i \leq m, 1 \leq j \leq n+1\}$ and its dominator chromatic number be $\chi_d(G)$. Define a proper dominator coloring $\varphi : V(K_m \diamond K_{1,n}) \to \{c_1, c_2, c_3, c_4\}$,

$$\varphi(u_i, v_j) = \begin{cases} c_1 \text{for } (u_1, v_1) \\ c_2 \text{ for } (u_i, v_1)/2 \leq i \leq m \\ c_3 \text{ for } (u_1, v_j)/2 \leq j \leq n+1 \\ c_4 \text{ otherwise} \end{cases}$$

By the definition of dominator coloring, the vertex $(u_1, v_j)/2 \leq j \leq n+1$ dominate the color class c_2 and $(u_i, v_1)/2 \leq i \leq m$ dominate the color class c_3.

The vertex u_1, v_1 dominates itself. Next all the remaining vertices dominate the color class c_1.

On the other hand, if we assign any one color $c_2 (or c_3)$ to (u_1, v_1) the the vertices $(u_i, v_1)/2 \leq i \leq m$ (or $(u_i, v_1)/2 \leq i \leq m$) does not dominate any color class. Therefore $\chi_d(K_m \diamond K_{1,n}) \nless 4$. Hence a verification shows that $\chi_d(K_m \diamond K_{1,n}) = 4$. □

Theorem 11. *For any positive integer $m, n \geq 4$, the dominator chromatic number of modular product of $\overline{K}_m \diamond \overline{K}_{1,n}$ is,*

$$\chi_d(\overline{K}_m \diamond \overline{K}_{1,n}) = 4.$$

Proof. The proof follows from Theorem 3.2 □

Theorem 12. *For any positive integer $m, n \geq 3$, the dominator chromatic number of modular product of $K_m \diamond C_n$ is*

$$\chi_{sd}(K_m \times C_n) = \begin{cases} m + 2 \text{ if } n \geq m \\ n + 2 \text{ if } m > n \end{cases}$$

Proof. Let $G = (K_m \diamond C_n)$ and the vertex set V of G is defined by

$$V = \{(u_i, v_j)/1 \leq i \leq m, 1 \leq j \leq n\}$$

and its dominator chromatic number be $\chi_d(G)$. Define a proper dominator coloring function $\varphi = V(K_m \diamond C_n) \to \{c_1, c_2, c_3, c_4, \ldots, c_{m+2}\}$,

Case 1. $n \geq m$

$$\varphi(u_i, v_j) = \begin{cases} c_i \text{ for } (u_i, v_1)/1 \leq i \leq m \\ c_{m+1} \text{ for } (u_i, v_j)/1 \leq i \leq m, \text{ and } j \text{ is odd} \\ c_{m+2} \text{ for } (u_i, v_j)/j \text{ is even and } 1 \leq i \leq m \end{cases}$$

It is easy to see that above assignment φ is proper coloring in addition to dominator coloring of graph $K_m \diamond C_n$. The set $D = \{u_i, v_1 | /1 \leq i \leq m\}$ is colorful dominating set and every vertex in $V - D$ dominate any one color class in D. Thus $\chi_{sd}(K_m \times c_n) \leq m + 2$.

To prove $\chi_{sd}(K_m \diamond c_n) \leq m + 2$ let us assume that $\chi_{sd}(K_m \times c_n) < m + 2$. Suppose that $\chi_{sd}(K_m \diamond c_n) = m + 1$. If we assign $m + 1$ color to the vertices $(u_i, v_j)/2 \leq i \leq m, 2 \leq j \leq n$ then the subset of atleast ant two vertices receives the same color and also the definition of dominator coloring is not satisfied. Thus $\chi_{sd}(K_m \diamond c_n) \nless m + 2$. Hence $\chi_{sd}(K_m \diamond c_n) = m + 2$.

Case 2. $n < m$
Define proper dominator coloring $\varphi : V(K_m \diamond C_n) \to \{c_1, c_2, c_3, c_4, \ldots, c_{n+2}\}$ by

$$\varphi(u_i, v_j) = \begin{cases} c_j \text{ for } (u_1, v_j) 1 \leq j \leq n \\ c_{n+1} \text{ for } (u_i, v_j)/j \text{ is even and } 1 \leq i \leq m \\ c_{n+2} \text{ for } (u_i, v_j)/j \text{ is odd and } 1 \leq i \leq m \end{cases}$$

It is clear that the above assignment φ is proper coloring in addition to dominator coloring of graph $K_m \diamond C_n$.The set $D = \{u_1, v_j | 1 \leq i \leq m\}$ is a dominating set and every vertex in $V - D$ dominate any one color class in D. Hence a verification shows that $\chi_{sd}(K_m \diamond C_n) = m + 2$. □

Theorem 13. *For any positive integer* $m, n \geq 3$,

$$\chi_d(K_m \diamond P_n) = m + 2.$$

Proof. Assume $G = (K_m \diamond P_n)$ has the vertex set
$V = \{(u_i, v_j), 1 \leq i \leq m, 1 \leq j \leq n\}$ and its dominator chromatic number be $\chi_d(G)$. Define a proper dominator coloring function φ on V such that for every vertex

$$\varphi(u_i, v_j) = \begin{cases} c_j \text{ for } (u_1, v_j)/1 \leq j \leq n \\ c_{m+1} \text{ for } (u_i, v_j)/j \text{ is odd } \text{ and } 2 \leq i \leq m \\ c_{m+2} \text{ for } (u_i, v_j)/j \text{ is even } \text{ and } 2 \leq i \leq m \end{cases}$$

Then the function φ is a dominator coloring in G with $m + 2$ colors.Thus $\chi_d(G) \leq m + 2$.

To prove $\chi_d(G) \leq m + 2$. Let us assume that $\chi_d(G) < m + 2$. Suppose that $\chi_d(G) = m + 1$, if we assign $m + 1$ color to the vertices (u_i, v_j) then the subset of any two vertices receives the same color also the dominator coloring is not satisfied. From the observation, a verification shows that dominator coloring with $m + 1$ color is not possible. Hence $\chi_d(G) = m + 2$. □

Theorem 14. *For any positive integer* $n, m \geq 3$

$$\chi_d(P_m \diamond P_n) = m + n - 1$$

Proof. Let $V(P_m) = \{u_1, u_2, u_3, \dot{;} u_n\}$ and $V(P_n) = \{v_1, v_2, v_3, \dot{;} v_n\}$ Let $G = (P_m \diamond P_n)$, the vertex set V of G is defined by

$$V(P_m \diamond P_n) = \{(u_i, v_j)/1 \leq i \leq m, 1 \leq j \leq n\}$$

Define a proper dominator coloring function

$$\varphi : V(P_m \diamond P_n) \to \{c_1, c_2, c_3, c_4, \ldots, c_{m+n-2}, c_{m+n-1}\}.$$

Case 1. $n \geq m$

$$\varphi(u_i, v_j) = \begin{cases} c_j \text{ for } (u_1, v_j)/1 \leq j \leq n \\ c_{m+i-1} \text{ for } (u_i, v_j)/1 \leq j \leq n, 2 \leq i \leq m \end{cases}$$

Then the function φ is a dominator coloring in G with $m+n-1$ colors. Thus $\chi_d(G) \leq m+n-1$.

on the other hand if we assign m+n colors to $v_i u_j$ then vertices do not satisfies the definition of dominator coloring. From the above observation an easy check shows that dominator coloring with $m+n$ color is not possible. Hence $\chi_d(G) = m+n-1$

Case 2. $m > n$

$$\varphi(u_i, v_j) = \begin{cases} c_i \text{ for } (u_i, v_1)/1 \leq i \leq m \\ c_{n+j-1} \text{ for } (u_i, v_j)/2 \leq j \leq n, 1 \leq i \leq m \end{cases}$$

Then the function f is a dominator coloring in G with $m+n-1$ colors.Thus $\chi_d(G) \leq m+n-1$. Hence An easy verification shows that $\chi_d(G) \leq m+n-1$. □

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