THE FORCING EDGE STEINER NUMBER OF A GRAPH

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Abstract: For a connected graph $G = (V,E)$, a set $W \subseteq V(G)$ is called an edge Steiner set of $G$ if every edge of $G$ is contained in a Steiner $W$-tree of $G$. The edge Steiner number $s_1(G)$ of $G$ is the minimum cardinality of its edge Steiner sets and any edge Steiner set of cardinality $s_1(G)$ is a minimum edge Steiner set of $G$. For a minimum edge Steiner set $W$ of $G$, a subset $T \subseteq W$ is called a forcing subset for $W$ if $W$ is the unique minimum edge Steiner set containing $T$. A forcing subset for $W$ of minimum cardinality is a minimum forcing subset of $W$. The forcing edge Steiner number of $W$, denoted by $fs_1(W)$, is the cardinality of a minimum forcing subset of $W$. The forcing edge Steiner number of $G$, denoted by $fs_1(G)$, is $fs_1(G) = \min\{fs_1(W)\}$, where the minimum is taken over all minimum edge Steiner sets $W$ in $G$. Some general properties satisfied by this concept are studied. The forcing edge Steiner numbers of certain classes of graphs are determined. It is shown for every pair of integers with $0 \leq a \leq b$, $b \geq 2$ and $b - a - 1 > 0$, there exists a connected graph $G$ such that $fs_1(G) = a$ and $s_1(G) = b$.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. The distance $d(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u - v$ path in $G$. An $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic. For basic graph theoretic terminology, we refer to Harary [1]. For a non-empty set $W$ of vertices in a connected graph $G$, the Steiner distance $d(W)$ of $W$ is the minimum size of a connected subgraph of $G$ containing $W$. Necessarily, each such subgraph is a tree and is called a Steiner tree with respect to $W$ or a Steiner $W$-tree. It is to be noted that $d(W) = d(u, v)$, when $W = \{u, v\}$. The set of all vertices of $G$ that lie on some Steiner $W$-tree is denoted by $S(W)$. If $S(W) = V$, then $W$ is called a Steiner set for $G$. A Steiner set of minimum cardinality is a minimum Steiner set or simply a $s$-set of $G$ and this cardinality is the Steiner number $s(G)$ of $G$. The Steiner number of a graph was introduced and studied in [2] and further studied in [3,4,5,6]. When $W = \{u, v\}$, every Steiner $W$-tree in $G$ is a $u - v$ geodesic. Also $S(W)$ equals the set of vertices lying in $u - v$ geodesic inclusive of $u,v$. Hence Steiner sets, Steiner numbers can be consider as extensions of geodesic concepts. For the graph $G$ given in Figure 1.1, $W = \{v_1, v_6, v_7\}$ is a minimum Steiner set of $G$ so that $s(G) = 3$. 

![Figure 1.1](image-url)

A subset $T \subseteq S$ is called a forcing subset for $S$ if $S$ is the unique minimum Steiner set containing $T$. A forcing subset for $S$ of minimum cardinality is a minimum forcing subset of $S$. The forcing Steiner number of $S$, denoted by $f(S)$, is the cardinality of a minimum forcing subset of $S$. The forcing Steiner number of $G$, denoted by $f(G)$, is $f(G) = \min\{f(S)\}$, where the minimum is taken over all minimum Steiner sets $S$ in $G$. The forcing Steiner number of a graph was introduced and studied in [2] and further studied in [4,6]. An edge Steiner set of $G$ is a set $W \subseteq V(G)$ such that every edge of $G$ is contained in a
Steiner $W$-tree. The edge Steiner number $s_1(G)$ is the minimum cardinality of its edge Steiner sets and any edge Steiner set of cardinality $s_1(G)$ is a minimum edge Steiner set or simply a $s_1$-set of $G$. For the graph $G$ given in Figure 1.2, $W = \{v_3, v_5\}$ is a minimum Steiner set of $G$ so that $s(G) = 2$ and $W_1 = \{v_1, v_2, v_4\}$ is a minimum edge Steiner set of $G$ so that $s_1(G) = 3$.

A vertex $v$ is an extreme vertex of a graph $G$ if the subgraph induced by its neighbors is complete. Throughout the following denotes a connected graph with at least two vertices. The following theorems are used in the sequel.

**Theorem 1.1.** [5] Each extreme vertex of a graph $G$ belongs to every edge Steiner set of $G$.

**Theorem 1.2.** [5] For the complete graph $G = K_p$, $s_1(G) = p$.

### 2. The Forcing edge Steiner Number of a Graph

Even though every connected graph contains a minimum edge Steiner set, some connected graphs may contain several minimum edge Steiner sets. For each minimum edge Steiner set $W$ in a connected graph $G$, there is always some subset $T$ of $W$ that uniquely determines $W$ as the minimum edge Steiner set containing $T$. Such “forcing subsets” will be considered in this section.

**Definition 2.1.** Let $G$ be a connected graph and $W$ a minimum edge Steiner set of $G$. A subset $T \subseteq W$ is called a forcing subset for $W$ if $W$ is the unique minimum edge Steiner set containing $T$. A forcing subset for $W$ of minimum cardinality is a minimum forcing subset of $W$. The forcing edge Steiner number of $W$, denoted by $fs_1(W)$, is the cardinality of a minimum
forcing subset of \( W \). The forcing edge Steiner number of \( G \), denoted by \( f_s_1(G) \), is \( f_s_1(G) = \min \{ f_s_1(W) \} \), where the minimum is taken over all minimum edge Steiner sets \( W \) in \( G \).

**Example 2.2.** For the graph \( G \) given in Figure 1.1, \( W_1 = \{ v_1, v_6, v_7 \} \) is the unique minimum edge Steiner set of \( G \) so that \( f_s_1(G) = 0 \). For the graph \( G \) given in Figure 2.1, \( W_1 = \{ v_1, v_2, v_4, v_5 \} \) and \( W_2 = \{ v_1, v_3, v_4, v_6 \} \) are the only two minimum edge Steiner sets of \( G \). It is clear that \( f_s_1(W_1) = f_s_1(W_2) = 1 \) and so \( f_s_1(G) = 1 \).

The next theorem follows immediately from the definitions of the edge Steiner number and the forcing edge Steiner number of a connected graph \( G \).

**Theorem 2.3.** For every connected graph \( G \), \( 0 \leq f_s_1(G) \leq s_1(G) \)

The following theorem characterizes graphs \( G \) for which the bounds in the Theorem 2.3 attained and also graph for which \( f_s_1(G) = 1 \). Since the proof of the theorem is straightforward, we omit it.

**Theorem 2.4.** Let \( G \) be a connected graph. Then

i) \( f_s_1(G) = 0 \) if and only if \( G \) has a unique minimum edge Steiner set.

\[\text{i)}\ f_s_1(G) = 0 \text{ if and only if } G \text{ has a unique minimum edge Steiner set.}\]

ii) \( f_s_1(G) = 1 \) if and only if \( G \) has at least two minimum edge Steiner sets, one of which is a unique minimum edge Steiner set containing one of its elements, and

\[\text{ii)}\ f_s_1(G) = 1 \text{ if and only if } G \text{ has at least two minimum edge Steiner sets, one of which is a unique minimum edge Steiner set containing one of its elements.}\]

iii) \( f_s_1(G) = s_1(G) \) if and only if no minimum edge Steiner set of \( G \) is the unique minimum edge Steiner set containing any of its proper subsets.

**Definition 2.5.** A vertex \( v \) of a graph \( G \) is said to be an edge Steiner vertex if \( v \) belongs to every minimum edge Steiner set of \( G \).
Example 2.6. For the graph $G$ given in Figure 2.2, $W_1 = \{v_1, v_3, v_4\}$ and $W_2 = \{v_1, v_3, v_5\}$ are the only two minimum edge Steiner sets of $G$ so that $v_1$ and $v_3$ are the edge Steiner vertices of $G$.

![Figure 2.2](image_url)

Theorem 2.7. Let $G$ be a connected graph and $W$ the set of all edge Steiner vertices of $G$. Then $f_{s_1}(G) \leq s_1(G) - |W|$.

Proof. Let $S$ be any minimum edge Steiner set of $G$. Then $s_1(G) = |S|$, $W \subseteq S$ and $S$ is the unique minimum edge Steiner set containing $S - W$. Thus $f_{s_1}(G) \leq |S - W| = |S| - |W| = s_1(G) - |W|$. \qed

Corollary 2.8. If $G$ is a connected graph with $k$ extreme vertices, then $f_{s_1}(G) \leq s_1(G) - k$.

Proof. This follows from Theorems 1.1 and 2.7. \qed

Remark 2.9. The bound in Theorem 2.7 is sharp. For the graph $G$ given in Figure 2.2, $S_1 = \{v_1, v_3, v_4\}$ and $S_2 = \{v_1, v_3, v_5\}$ are the only two $s_1$-sets so that $s_1(G) = 3$ and $f_{s_1}(G) = 1$. Also, $W = \{v_1, v_3\}$ is the set of all edge Steiner vertices of $G$ and so $f_{s_1}(G) = s_1(G) - |W|$. Also, the inequality in Theorem 2.7 can be strict. For the graph $G$ given in Figure 2.1, $s_1(G) = 4$ and $f_{s_1}(G) = 1$. Since $W = \{v_1, v_4\}$ is the set of all edge Steiner vertices of $G$, we have and so $f_{s_1}(G) \leq s_1(G) - |W|$.

In the following we determine the forcing edge Steiner numbers of certain standard graphs.
Theorem 2.10. For an even cycle $G = C_p (p \geq 4)$, a set $S \subseteq V$ is a $s_1$-set if and only if $S$ consists of two antipodal vertices. In particular for an even cycle $G = C_p$, $s_1(G) = 2$.

Proof. If $S$ consists of two antipodal vertices, then it is clear that $S$ is a $s_1$-set of $C_p$. Conversely, let $S$ be any $s_1$-set of $C_p$. Then $s(C_p) = |S|$. Then it follows from the first part of the proof that $S$ consists of two vertices, say $S = \{u, v\}$. If $u$ and $v$ are not antipodal, then any edge that is not on the $u - v$ geodesic does not lie on the Steiner $S$-tree. Thus $S$ is not a $s_1$-set, which is a contradiction.

Theorem 2.11. For a cycle $C_p \ (p \geq 4)$, $f_s(C_p) = \begin{cases} 1 & \text{if } p \text{ is even} \\ 2 & \text{if } p \text{ is odd}. \end{cases}$

Proof. For $p$ is even, it follows from Theorem 2.10 that $C_p$ contains $p/2$ $s_1$-sets and it is clear that each singleton set is the minimum forcing set for exactly one $s_1$ of $C_p$. Hence it follows from Theorem 2.4 (i) that $f_s(C_p) = 1$.

Let $p$ be odd and $p = 2n + 1$. Let the cycle be $C_p : v_1, v_2, \ldots, v_n, v_{n+1}, v_{n+2}, \ldots, v_{2n+1}, v_1$. If $S = \{u, v\}$ is any set of two vertices of $C_p$, then no edge on the $u - v$ longest path lies on the Steiner S-tree in $C_p$ and so no two element subset of $C_p$ is a Steiner set of $C_p$. Now, it is clear that the sets $S_1 = \{v_1, v_{n+1}, v_{n+2}\}, S_2 = \{v_2, v_{n+2}, v_{n+3}\}, \ldots, S_{n+2} = \{v_{n+2}, v_1, v_2\}, \ldots, S_{2n+1} = \{v_{2n+1}, v_n, v_{n+1}\}$ are $s_1$-sets of $C_p$ (Note that there are more s-sets of $C_p$, for example, $S' = \{v_1, v_{n+1}, v_{n+3}\}$ is a $s_1$-set different from these). It is clear from the $s_1$-sets $S_i \ (1 \leq i \leq 2n + 1)$ that each $\{v_i\}(1 \leq i \leq 2n + 1)$ is a subset of more than one $s_1$-set $S_i$. Hence it follows from Theorem 2.4 (i) and (ii) that $f_s(C_p) \geq 2$. Now, since $v_{n+1}$ and $v_{n+2}$ are antipodal to $v_1$, it is clear that $S_1$ is the unique $s_1$-set containing $\{v_{n+1}, v_{n+2}\}$ and so $f_s(C_p) = 2$.

Theorem 2.12. For the complete graph $G = K_p \ (p \geq 2), f_s(G) = 0$.

Proof. Since $W = V(G)$ is the unique minimum $s_1$-set of $G$, the result follows from Theorem 2.4(i).

Theorem 2.13. For the complete bipartite graph $G = K_{m,n} \ (m,n \geq 2)$,

$$f_s(K_{m,n}) = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$
Proof. First assume that $m < n$. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be the bipartition sets of $G$. Let $S = U$. We prove that $S$ is a $s_1$-set of $G$. Any Steiner $S$-tree $T$ is a star centered at $w_j$ ($1 \leq j \leq n$) with $u_i$ ($1 \leq i \leq m$) as end vertices of $T$. Hence every edge of $G$ lies on a Steiner $S$-tree of $G$ so that $S$ is an edge Steiner set of $G$. Let $X$ be any set of vertices such that $|X| < |S|$. Then there exists a vertex $u_i \in U$ such that $u_i \notin X$. Since any Steiner $X$-tree is a star centered at $w_j$ ($1 \leq j \leq n$), whose end-vertices are elements of $X$, the edge $w_j u_i$ does not lie on any Steiner $X$-tree of $G$. Thus $X$ is not an edge Steiner set of $G$. Hence $S$ is a $s_1$-set so that $s_1(K_{m,n}) = |S| = m$. Now, let $S_1$ be a set of vertices such that $|S_1| = m$. If $S_1$ is a subset of $W$, then since $m < n$, there exists a vertex $w_j \in W$ such that $w_j \notin S_1$. Then the edges $w_j u_i$ do not lie on any Steiner $S_1$-tree of $G$. If $S_1 U \cup W$ such that $S_1$ contains at least one vertex from each of $U$ and $W$, then since $S_1 \neq U$, there exists vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin S_1$ and $w_j \notin S_1$. Then clearly the edges $u_i w_j$ do not lie on any Steiner $S_1$-tree of $G$ and so $S_1$ is not a Steiner set of $G$. It follows that $U$ is the unique $s_1$-set of $G$. Hence it follows from Theorem 2.4(i) that $fs_1(G) = 0$. Now, let $m = n$. Then as in the first part of this theorem, both $U$ and $W$ are $s_1$-sets of $G$. Now, let $S'$ be any set of vertices such that $|S'| = m$ and $S' \neq U, W$. Then there exist vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin S'$ and $w_j \notin S'$. Then as earlier, $S'$ is not an edge Steiner set of $G$. Hence it follows that $U$ and $W$ are the only $s_1$-sets of $G$. Since $U$ is the unique edge Steiner set containing $\{u_i\}$, it follows that $fs_1(G) = 1$. \hfill \Box

Theorem 2.14. If $S = \{u, v\}$ is a $s_1$-set of a connected graph $G$, then $u$ and $v$ are antipodal vertices of $G$.

Proof. Let $S = \{u, v\}$ be a $s_1$-set of $G$. Then every edge of $G$ lies on a Steiner $S$-tree of $G$. Hence every vertex of $G$ lies on a Steiner $S$-tree of $G$. Since every Steiner $S$-tree is a $u-v$ geodesic, every vertex of $G$ lies on a geodesic joining $u$ and $v$. We claim that $d(u, v) = d(G)$. If $d(u, v) < d(G)$, then let $x$ and $y$ be two vertices of $G$ such that $d(x, y) = d(G)$. Now, it follows that $x$ and $y$ lie on distinct geodesics joining $u$ and $v$. Hence $d(u, v) = d(u, x) + d(x, v) \ldots$ (1) and $d(u, v) = d(u, y) + d(y, v) \ldots$ (2). By the triangle inequality, $d(x, y) < d(x, u) + d(u, y) \ldots$ (3). Since $d(u, v) < d(x, y)$, (3) becomes $d(u, v) < d(x, u) + d(u, y) \ldots$ (4). Using (4) in (1), $d(x, v) < d(u, y) \ldots$ (5). Also, by triangle inequality, we have $d(x, y) \leq d(x, v) + d(v, y) \ldots$ (6). Now, using (5) and (2),(6) becomes $d(x, y) < d(u, y) + d(v, y) = d(u,v)$. Thus, $d(G) < d(u, v)$, which is a contradiction. Hence $d(u, v) = d(G)$ and so $u$ and $v$ are antipodal vertices of $G$. \hfill \Box
Theorem 2.15. If $G$ is a connected graph with $s_1(G) = 2$, then $f_{s_1}(G) \leq 1$.

Proof. Let $S = \{u, v\}$ be any $s_1$-set of $G$. Then by Theorem 2.14, $u$ and $v$ are antipodal vertices of $G$. Suppose that $f_{s_1}(G) = 2$. It follows from Theorem 2.4(iii) that $S$ is not the unique $s_1$-set containing $u$ and so there exists $x \neq u$ such that $S' = \{u, x\}$ is also a $s_1$-set of $G$. By Theorem 2.14, $u$ and $x$ are two antipodal vertices of $G$ and $v$ is an internal vertex of some $u - x$ geodesic in $G$. Therefore, $d(u, v) < d(u, x)$, which is a contradiction. □

Theorem 2.16. If $G$ is a connected geodetic graph with $s_1(G) = 2$, then $f_{s_1}(G) = 0$.

Proof. Let $s_1(G) = 2$. Let $W = \{u, v\}$ be a minimum edge Steiner set of $G$. Then it is clear that every edge of $G$ lies on a $u - v$ geodesic. Since $G$ is a geodetic graph it follows that $G = P_n$. Hence it follows from Theorem 2.4(i) that $f_{s_1}(G) = 0$. □

Theorem 2.17. Let $G$ be a connected graph with $s(G) = s_1(G)$. Then, $f_{s_1}(G) \leq f_s(G)$.

Proof. Let $T$ be any forcing subset of any $s$-set. We show that $T$ is a forcing subset of a $s_1$-set of $G$. Otherwise, $T$ is a subset of more than one $s_1$-set. Since every edge Steiner set of $G$ is a set Steiner set of $G$ and since $s(G) = s_1(G)$, it follows that $T$ is a subset of more than one $s$-set of $G$, which is a contradiction. Thus every forcing subset of any $s$-set of $G$ is also a forcing subset of a $s_1$-set of $G$. Hence $f_{s_1}(G) \leq f_s(G)$. □

Theorem 2.18. For every pair $a, b$ of integers with $0 \leq a < b$, $b^2$ and $b - a - 1 > 0$, there exists a connected graph $G$ such that $f_{s_1}(G) = a$ and $s_1(G) = b$.

Proof. If $a = 0$, let $G = K_b$. Then by Theorem 2.11, $f_{s_1}(G) = 0$ and by Theorem 1.2, $s_1(G) = b$. Now, we assume that $a1$. Let $F_i : s_i, t_i, u_i, v_i, r_i, s_i (1 \leq i \leq a)$ be a copy of the cycle $C_5$. Let $G$ be the graph obtained from $F_i$ (1 \leq i \leq a) by first identifying the vertices $r_{i-1}$ of $F_{i-1}$ and $t_i$ of $F_i (2 \leq i \leq a)$ and then adding the $b - a$ new vertices $z_1, z_2, ..., z_{b-a-1}, u$ and joining the $b - a$ edges $t_1z_i (1 \leq i \leq b - a - 1)$ and $r_au$. The graph $G$ is given in Figure 2.3. Let $Z = \{z_1, z_2, ..., z_{b-a-1}, u\}$ be the set of end-vertices of $G$. By Theorem 1.1, every $s_1$-set contains $Z$. Let $H_i = \{u_i, v_i\} (1 \leq i \leq a)$. 
First we show that $s_1(G) = b$. Since the edges $u_iv_i$ do not lie on the unique Steiner $Z$-tree of $G$, it is clear that $Z$ is not an edge Steiner set of $G$. Hence it follows from Theorem 1.1 that every $s_1$-set of $G$ must contain exactly one vertex from each $H_i$ ($1 \leq i \leq a$) and so $s_1(G) = b - a + a = b$. On the other hand, since the set $W_1 = Z \cup \{v_1, v_2, \ldots, v_a\}$ is an edge Steiner set of $G$, it follows that $s_1(G) \leq |W_1| = b$. Thus, $s_1(G) = b$.

Next, we show that $fs_1(G) = a$. By Theorem 1.1, every edge Steiner set of $G$ contains $Z$ and so it follows from Theorem 2.7 that $fs_1(G) \leq s_1(G) - |Z| = b - (b - a) = a$. Now, since $s_1(G) = b$ and every $s_1(G)$-set of $G$ contains $Z$, it is easily seen that every $s_1(G)$-set $S$ is of the form $Z \cup \{c_1, c_2, \ldots, c_a\}$, where $c_i \in H_i$ ($1 \leq i \leq a$). Let $T$ be any proper subset of $S$ with $|T| < a$. Then there is a vertex $c_j$ ($1 \leq j \leq a$) such that $c_j \notin T$. Let $d_j$ be a vertex of $H_j$ distinct from $c_j$. Then $S_1 = (S - \{c_j\}) \cup \{d_j\}$ is a $s_1$-set properly containing $T$. Thus $S$ is not the unique $s_1$-set containing $T$ and so $T$ is not a forcing subset of $S$. This is true for all $s_1$-sets of $G$ and so $fs_1(G) = a$. 

\[\square\]

Figure 2.3

References


