International Journal of Pure and Applied Mathematics
Volume 119 No. 4 2018, 695-704
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v119i4.11

ijpam.eu

THE FORCING EDGE STEINER NUMBER OF A GRAPH

A. Siva Jothi[1 §], J. John[2], S. Robinson Chellathurai[3]

[1]Department of Mathematics
Marthandam College of Engineering and Technology
Kuttakuzhi, 629 177, INDIA

[2]Department of Mathematics
Government College of Engineering
Tirunelveli, 627 001, INDIA

[3]Department of Mathematics
Scott Christian College
Nagercoil, 629 003, INDIA

Abstract: For a connected graph $G = (V, E)$, a set $W \subseteq V(G)$ is called an edge Steiner set of G if every edge of G is contained in a Steiner W-tree of G. The edge Steiner number $s_1(G)$ of G is the minimum cardinality of its edge Steiner sets and any edge Steiner set of cardinality $s_1(G)$ is a minimum edge Steiner set of G. For a minimum edge Steiner set W of G, a subset $T \subseteq W$ is called a forcing subset for W if W is the unique minimum edge Steiner set containing T. A forcing subset for W of minimum cardinality is a minimum forcing subset of W. The forcing edge Steiner number of W, denoted by $fs_1(W)$, is the cardinality of a minimum forcing subset of W. The forcing edge Steiner number of G, denoted by $fs_1(G)$, is $fs_1(G) = \min\{fs_1(W)\}$, where the minimum is taken over all minimum edge Steiner sets W in G. Some general properties satisfied by this concept are studied. The forcing edge Steiner numbers of certain classes of graphs are determined. It is shown for every pair of integers with $0 \leq a \leq b$, $b \geq 2$ and $b - a - 1 > 0$, there exists a connected graph G such that $fs_1(G) = a$ and $s_1(G) = b$.

AMS Subject Classification: 05C12

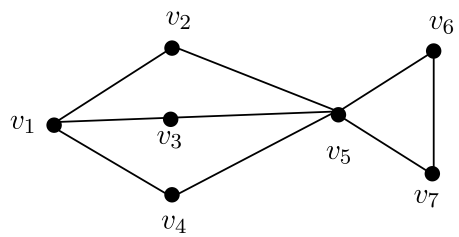
Key Words: Steiner distance, Steiner number, edge Steiner number, forcing Steiner number, forcing edge Steiner number

Received: August 28, 2017
Revised: August 7, 2018
Published: August 10, 2018

[§]Correspondence author

1. Introduction

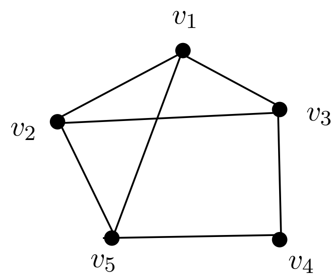
By a graph $G = (V, E)$,we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic. For basic graph theoretic terminology, we refer to Harary [1]. For a non-empty set W of vertices in a connected graph G, the Steiner distance $d(W)$ of W is the minimum size of a connected subgraph of G containing W. Necessarily, each such subgraph is a tree and is called a Steiner tree with respect to W or a Steiner W-tree. It is to be noted that $d(W) = d(u, v)$, when $W = \{u, v\}$. The set of all vertices of G that lie on some Steiner W-tree is denoted by $S(W)$. If $S(W) = V$, then W is called a Steiner set for G. A Steiner set of minimum cardinality is a minimum Steiner set or simply a s-set of G and this cardinality is the Steiner number $s(G)$ of G. The Steiner number of a graph was introduced and studied in [2] and further studied in [3,4,5,6]. When $W = \{u, v\}$, every Steiner W-tree in G is a $u - v$ geodesic. Also $S(W)$ equals the set of vertices lying in $u - v$ geodesic,inclusive of u, v. Hence Steiner sets, Steiner numbers can be consider as extensions of geodesic concepts. For the graph G given in Figure 1.1, $W = \{v_1, v_6, v_7\}$ is a minimum Steiner set of G so that $s(G) = 3$.



G
Figure 1.1

A subset $T \subseteq S$ is called a forcing subset for S if S is the unique minimum Steiner set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing Steiner number of S, denoted by f(S), is the cardinality of a minimum forcing subset of S. The forcing Steiner number of G, denoted by f(G), is $f(G) = \min\{f(S)\}$, where the minimum is taken over all minimum Steiner sets S in G. The forcing Steiner number of a graph was introduced and studied in [2] and further studied in [4,6]. An edge Steiner set of G is a set $W \subseteq V(G)$ such that every edge of G is contained in a

Steiner W-tree. The edge Steiner number $s_1(G)$ is the minimum cardinality of its edge Steiner sets and any edge Steiner set of cardinality $s_1(G)$ is a minimum edge Steiner set or simply a s_1-set of G. For the graph G given in Figure 1.2, $W = \{v_3, v_5\}$ is a minimum Steiner set of G so that $s(G) = 2$ and $W_1 = \{v_1, v_2, v_4\}$ is a minimum edge Steiner set of G so that $s_1(G) = 3$.



G

Figure 1.2

A vertex v is an extreme vertex of a graph G if the subgraph induced by its neighbors is complete. Throughout the following denotes a connected graph with at least two vertices. The following theorems are used in the sequel.

Theorem 1.1. [**5**] *Each extreme vertex of a graph G belongs to every edge Steiner set of G.*

Theorem 1.2. [**5**] *For the complete graph $G = K_p$, $s_1(G) = p$.*

2. The Forcing edge Steiner Number of a Graph

Even though every connected graph contains a minimum edge Steiner set, some connected graphs may contain several minimum edge Steiner sets. For each minimum edge Steiner set W in a connected graph G, there is always some subset T of W that uniquely determines W as the minimum edge Steiner set containing T. Such "forcing subsets" will be considered in this section.

Definition 2.1. *Let G be a connected graph and W a minimum edge Steiner set of G. A subset $T \subseteq W$ is called a forcing subset for W if W is the unique minimum edge Steiner set containing T. A forcing subset for W of minimum cardinality is a minimum forcing subset of W. The forcing edge Steiner number of W, denoted by $fs_1(W)$, is the cardinality of a minimum* *forcing subset of* W. *The forcing edge Steiner number of* G, *denoted by* $fs_1(G)$, *is* $fs_1(G) = \min\{fs_1(W)\}$, *where the minimum is taken over all minimum edge Steiner sets* W *in* G.

Example 2.2. For the graph G given in Figure 1.1, $W_1 = \{v_1, v_6, v_7\}$ is the unique minimum edge Steiner set of G so that $fs_1(G) = 0$. For the graph G given in Figure 2.1, $W_1 = \{v_1, v_2, v_4, v_5\}$ and $W_2 = \{v_1, v_3, v_4, v_6\}$ are the only two minimum edge Steiner sets of G. It is clear that $fs_1(W_1) = fs_1(W_2)$ $= 1$ and so $fs_1(G) = 1$.

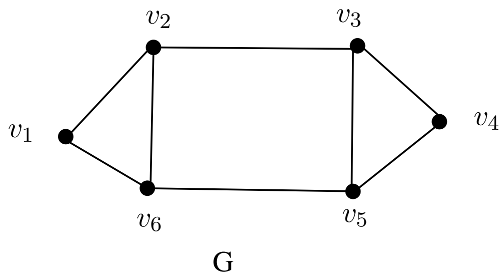


Figure 2.1

The next theorem follows immediately from the definitions of the edge Steiner number and the forcing edge Steiner number of a connected graph G.

Theorem 2.3. *For every connected graph* G, $0 \leq fs_1(G) \leq s_1(G)$
The following theorem characterizes graphs G *for which the bounds in the Theorem 2.3 attained and also graph for which* $fs_1(G) = 1$. *Since the proof of the theorem is straight forward, we omit it.*

Theorem 2.4. *Let* G *be a connected graph. Then*

i) $fs_1(G) = 0$ *if and only if* G *has a unique minimum edge Steiner set.*

ii) $fs_1(G) = 1$ *if and only if* G *has at least two minimum edge Steiner sets, one of which is a unique minimum edge Steiner set containing one of its elements, and*

iii) $fs_1(G) = s_1(G)$ *if and only if no minimum edge Steiner set of* G *is the unique minimum edge Steiner set containing any of its proper subsets.*

Definition 2.5. *A vertex* v *of a graph* G *is said to be an edge Steiner vertex if* v *belongs to every minimum edge Steiner set of* G.

Example 2.6. For the graph G given in Figure 2.2, $W_1 = \{v_1, v_3, v_4\}$ and $W_2 = \{v_1, v_3, v_5\}$ are the only two minimum edge Steiner sets of G so that v_1 and v_3 are the edge Steiner vertices of G.

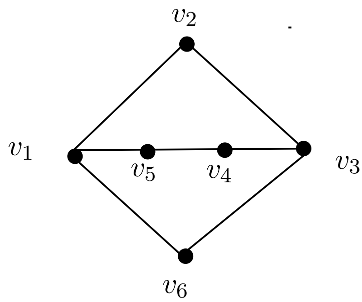


Figure 2.2

Theorem 2.7. *Let G be a connected graph and W the set of all edge Steiner vertices of G. Then* $fs_1(G) \leq s_1(G)$ - $|W|$.

Proof. Let S be any minimum edge Steiner set of G. Then $s_1(G) = |S|$, $W \subseteq S$ and S is the unique minimum edge Steiner set containing $S - W$. Thus $fs_1(G) \leq |S - W| = |S| - |W| = s_1(G) - |W|$. □

Corollary 2.8. *If G is a connected graph with k extreme vertices, then* $fs_1(G) \leq s_1(G) - k$.

Proof. This follows from Theorems 1.1 and 2.7. □

Remark 2.9. The bound in Theorem 2.7 is sharp. For the graph G given in Figure 2.2, $S_1 = \{v_1, v_3, v_4\}$ and $S_2 = \{v_1, v_3, v_5\}$ are the only two s_1-sets so that $s_1(G) = 3$ and $fs_1(G) = 1$. Also,W $= \{v_1, v_3\}$ is the set of all edge Steiner vertices of G and so $fs_1(G) = s_1(G) - \mid W \mid$. Also, the inequality in Theorem 2.7 can be strict. For the graph G given in Figure 2.1, $s_1(G) = 4$ and $fs_1(G)$ $= 1$. Since $W = \{v_1, v_4\}$ is the set of all edge Steiner vertices of G, we have and so $fs_1(G) \leq s_1(G) - \mid W \mid$.

In the following we determine the forcing edge Steiner numbers of certain standard graphs.

Theorem 2.10. *For an even cycle $G = C_p (p \geq 4)$, a set $S \subseteq V$ is a s_1-set if and only if S consists of two antipodal vertices. In particular for an even cycle $G = C_p$, $s_1(G) = 2$.*

Proof. If S consists of two antipodal vertices, then it is clear that S is a s_1-set of C_p. Conversely, let S be any s_1-set of C_p. Then $s(C_p) = \mid S \mid$. Then it follows from the first part of the proof that S consists of two vertices, say $S = \{u, v\}$. If u and v are not antipodal, then any edge that is not on the $u - v$ geodesic does not lie on the Steiner S-tree. Thus S is not a s_1-set, which is a contradiction. □

Theorem 2.11. *For a cycle C_p $(p \geq 4)$, $fs_1(C_p) = \begin{cases} 1 & if p \text{ is even} \\ 2 & if p \text{ is odd.} \end{cases}$*

Proof. For p is even, it follows from Theorem 2.10 that C_p contains $p/2$ s_1-sets and it is clear that each singleton set is the minimum forcing set for exactly one s_1 of C_p. Hence it follows from Theorem 2.4 (i) that $fs_1(C_p) = 1$.

Let p be odd and $p = 2n + 1$. Let the cycle be $C_p : v_1, v_2, ..., v_n, v_{n+1}, v_{n+2}, .., v_{2n+1}, v_1$. If $S = \{u, v\}$ is any set of two vertices of C_p, then no edge on the $u - v$ longest path lies on the Steiner S-tree in C_p and so no two element subset of C_p is a Steiner set of C_p. Now, it is clear that the sets $S_1 = \{v_1, v_{n+1}, v_{n+2}\}$, $S_2 = \{v_2, v_{n+2}, v_{n+3}\}$,..., $S_{n+2} = \{v_{n+2}, v_1, v_2\}$,...,$S_{2n+1} = \{v_{2n+1}, v_n, v_{n+1}\}$ are s_1-sets of C_p.(Note that there are more s-sets of C_p, for example, $S^{'} = \{v_1, v_{n+1}, v_{n+3}\}$ is a s_1-set different from these). It is clear from the s_1-sets S_i $(1 \leq i \leq 2n + 1)$ that each $\{v_i\}(1 \leq i \leq 2n + 1)$ is a subset of more than one s_1-set S_i. Hence it follows from Theorem 2.4 (i) and (ii) that $fs_1(C_p) \geq 2$. Now, since v_{n+1} and v_{n+2} are antipodal to v_1, it is clear that S_1 is the unique s_1-set containing $\{v_{n+1}, v_{n+2}\}$ and so $fs_1(C_p) = 2$. □

Theorem 2.12. *For the complete graph $G = K_p$ $(p \geq 2)$,$fs_1(G)$=0.*

Proof. Since $W = V(G)$ is the unique minimum s_1-set of G, the result follows from Theorem 2.4(i). □

Theorem 2.13. *For the complete bipartite graph $G = K_{m,n}$ (m,n ≥ 2),*
$$fs_1(K_{m,n}) = \begin{cases} 0 & if m \neq n \\ 1 & if m = n. \end{cases}$$

Proof. First assume that $m < n$. Let $U = \{u_1, u_2, ..., u_m\}$ and $W = \{w_1, w_2, ..., w_n\}$ be the bipartition sets of G. Let $S = U$. We prove that S is a s_1-set of G. Any Steiner S-tree T is a star centered at w_j $(1 \leq j \leq n)$ with u_i $(1 \leq i \leq m)$ as end vertices of T. Hence every edge of G lies on a Steiner S-tree of G so that S is an edge Steiner set of G. Let X be any set of vertices such that $\mid X \mid<\mid S \mid$. Then there exists a vertex $u_i \in U$ such that $u_i \notin X$. Since any Steiner X-tree is a star centered at w_j $(1 \leq j \leq n)$, whose end-vertices are elements of X, the edge w_j u_i does not lie on any Steiner X-tree of G. Thus X is not an edge Steiner set of G. Hence S is a s_1-set so that $s_1(K_{m,n}) = \mid S \mid = m$. Now, let S_1 be a set of vertices such that $\mid S_1 \mid = m$. If S_1 is a subset of W, then since $m < n$, there exists a vertex $w_j \in W$ such that $w_j \notin S_1$. Then the edges $w_j u_i$ do not lie on any Steiner S_1-tree of G. If $S_1 U \cup W$ such that S_1 contains at least one vertex from each of U and W, then since $S_1 \neq U$, there exists vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin$S$_1$ and $w_j \notin$S$_1$. Then clearly the edges $u_i w_j$ do not lie on any Steiner S_1-tree of G and so S_1 is not a Steiner set of G. It follows that U is the unique s_1-set of G. Hence it follows from Theorem 2.4(i) that $fs_1(G) = 0$. Now, let $m = n$. Then as in the first part of this theorem, both U and W are s_1-sets of G. Now, let $S^{'}$ be any set of vertices such that $\mid S' \mid = m$ and $S' \neq U, W$. Then there exist vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin S^{'}$ and $w_j \notin S^{'}$. Then as earlier, $S^{'}$ is not an edge Steiner set of G. Hence it follows that U and W are the only s_1-sets of G. Since U is the unique edge Steiner set containing $\{u_i\}$, it follows that $fs_1(G) = 1$. □

Theorem 2.14. *If $S = \{u, v\}$ is a s_1-set of a connected graph G, then u and v are antipodal vertices of G.*

Proof. Let $S = \{u, v\}$ be a s_1-set of G.Then every edge of G lies on a Steiner S-tree of G. Hence every vertex of G lies on a Steiner S-tree of G. Since every Steiner S-tree is a $u - v$ geodesic, every vertex of G lies on a geodesic joining u and v. We claim that $d(u, v) = d(G)$. If $d(u, v) < d(G)$, then let x and y be two vertices of G such that $d(x, y) = d(G)$. Now,it follows that x and y lie on distinct geodesics joining u and v. Hence $d(u, v) = d(u, x) + d(x, v)$ ... (1) and $d(u, v) = d(u, y) + d(y, v)$ ... (2). By the triangle inequality, $d(x, y) < d(x, u) + d(u, y)$... (3). Since $d(u, v) < d(x, y)$, (3) becomes $d(u, v) < d(x, u) + d(u, y)$ ... (4). Using (4) in (1), $d(x, v) < d(u, y)$ ... (5). Also, by triangle inequality, we have $d(x, y) \leq d(x, v) + d(v, y)$ ... (6). Now, using (5) and (2),(6) becomes $d(x, y) < d(u, y) + d(v, y) =$ d(u,v). Thus, $d(G) < d(u, v)$, which is a contradiction. Hence $d(u, v) = d(G)$ and so u and v are antipodal vertices of G. □

Theorem 2.15. *If G is a connected graph with $s_1(G) = 2$, then $fs_1(G) \leq 1$.*

Proof. Let $S = \{u, v\}$ be any s_1-set of G. Then by Theorem 2.14, u and v are antipodal vertices of G. Suppose that $fs_1(G) = 2$. It follows from Theorem 2.4(iii) that S is not the unique s_1-set containing u and so there exists $x \neq u$ such that $S' = \{u, x\}$ is also a s_1-set of G. By Theorem 2.14, u and x are two antipodal vertices of G and v is an internal vertex of some $u - x$ geodesic in G. Therefore, $d(u, v) < d(u, x)$, which is a contradiction. □

Theorem 2.16. *If G is a connected geodetic graph with $s_1(G) = 2$, then $fs_1(G) = 0$.*

Proof. Let $s_1(G) = 2$. Let $W = \{u, v\}$ be a minimum edge Steiner set of G. Then it is clear that every edge of G lies on a $u - v$ geodesic. Since G is a geodetic graph it follows that $G = P_n$. Hence it follows from Theorem 2.4(i) that $fs_1(G) = 0$. □

Theorem 2.17. *Let G be a connected graph with s(G)= $s_1(G)$. Then, $fs_1(G) \leq fs(G)$.*

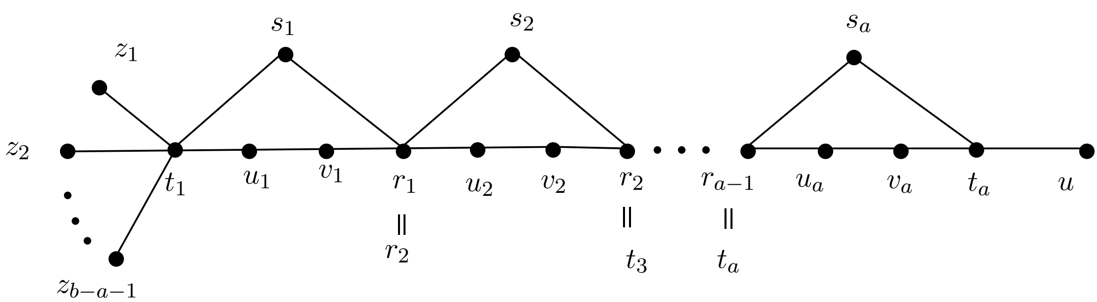
Proof. Let T be any forcing subset of any s-set. We show that T is a forcing subset of a s_1-set of G. Otherwise, T is a subset of more than one s_1-set. Since every edge Steiner set of G is a set Steiner set of G and since $s(G) = s_1(G)$, it follows that T is a subset of more than one s-set of G, which is a contradiction. Thus every forcing subset of any s-set of G is also a forcing subset of a s_1-set of G. Hence $fs_1(G) \leq fs(G)$. □

Theorem 2.18. *For every pair a,b of integers with $0 \leq a < b$, b2 and $b - a - 1 > 0$, there exists a connected graph G such that $fs_1(G) =$ a and $s_1(G)$ = b.*

Proof. If $a = 0$, let $G = K_b$. Then by Theorem 2.11, $fs_1(G) = 0$ and by Theorem 1.2, $s_1(G) =$ b. Now, we assume that $a1$. Let $F_i : s_i, t_i, u_i, v_i, r_i, s_i$ $(1 \leq i \leq a)$ be a copy of the cycle C_5. Let G be the graph obtained from F_i $(1 \leq i \leq a)$ by first identifying the vertices r_{i-1} of F_{i-1} and t_i of F_i $(2 \leq i \leq a)$ and then adding the $b - a$ new vertices z_1,z_2, ..., z_{b-a-1},u and joining the $b - a$ edges $t_1 z_i$ $(1 \leq i \leq b - a - 1)$ and $r_a u$. The graph G is given in Figure 2.3. Let $Z = \{z_1, z_2, .., z_{b-a-1}, u\}$ be the set of end-vertices of G. By Theorem 1.1, every s_1-set contains Z. Let $H_i = \{u_i, v_i\}$ $(1 \leq i \leq a)$.

First we show that $s_1(G) = b$. Since the edges u_iv_i do not lie on the unique Steiner Z-tree of G, it is clear that Z is not an edge Steiner set of G. Hence it follows from Theorem 1.1 that every s_1-set of G must contain exactly one vertex from each H_i $(1 \le i \le a)$ and so $s_1(G)b - a + a = b$. On the other hand, since the set $W_1 = Z \cup \{v_1, v_2, ..., v_a\}$ is an edge Steiner set of G, it follows that $s_1(G) \le \mid W_1 \mid = b$. Thus, $s_1(G) =$ b.

Next, we show that $fs_1(G) = a$. By Theorem 1.1, every edge Steiner set of G contains Z and so it follows from Theorem 2.7 that $fs_1(G)$ $\le$s$_1(G)$ - $|Z| = b - (b - a) = a$. Now, since $s_1(G) = b$ and every $s_1(G)$-set of G contains Z, it is easily seen that every $s_1(G)$-set S is of the form $Z \cup \{c_1, c_2, ..., c_a\}$, where $c_i \in H_i$ $(1 \le i \le a)$. Let T be any proper subset of S with $\mid T \mid < a$. Then there is a vertex c_j $(1 \le j \le a)$ such that c$_j \notin T$. Let d_j be a vertex of H_j distinct from c_j. Then $S_1 = (S - \{c_j\}) \cup \{d_j\}$ is a s_1-set properly containing T. Thus S is not the unique s_1-set containing T and so T is not a forcing subset of S. This is true for all s_1-sets of G and so $fs_1(G) = a$. □



G

Figure 2.3

References

[1] Buckley, F. Harary, *Distance in Graphs*, Addition- Wesley, Redwood City, CA, 1990.

[2] G. Chartrand and P. Zhang, *The Steiner number of a graph*, Discrete Mathematics 242 (2002) 41 - 54.

[3] G. Chartrand, P. Zhang, *The Forcing Geodetic Number of a Graph*, Discussiones Mathematicae Vol. 19, (1999) 45 - 58.

[4] A. P. Santhakumaran and J. John, *The forcing Steiner number of a graph.* Discussiones Mathematicae Graph Theory 31 (1) (2011) 171-181.

[5] A. P. Santhakumaran and J. John, *The edge Steiner number of a graph*, Journal of Discrete Mathematical Science and Cryptography Vol. 10, No. 5, (2007) 677 - 696.

[6] A.P. Santhakumaran and J. John, *On the forcing Geodetic and the forcing Steiner Numbers of a graph*, Discussiones Mathematicae Graph Theory 31 (4) (2011), 611-624.