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BOUNDEDNESS OF MAXIMAL FUNCTIONS WITH MIXED HOMOGENEITY

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Abstract: In this article, we study a certain class of maximal functions with mixed homogeneity when the kernels in $L^q(\mathbf{S}^{n-1})$ for $1 < q \leq 2$. We obtain appropriate L^p estimates for such maximal operators. These estimates will allow us to use extrapolation arguments to establish the L^p boundedness of our maximal functions when their kernels belong to $L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1}) \cup B_q^{(0,-1/\gamma)}(\mathbf{S}^{n-1})$ with $q > 1$ and $1 \leq \gamma \leq 2$. Our results essentially improve and extend some known results on maximal functions as well as singular integrals.

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1. Preliminaries and Statement of Results

Let $\mathbf{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue surface measure $d\sigma_n = d\sigma(\cdot)$. Also, let p' denote the exponent conjugate to p; that is $1/p + 1/p' = 1$.

Let $\alpha_i \geq 1$ $(i = 1, 2, \cdots, n)$ be fixed real numbers. Define $F : \mathbf{R}^n \times \mathbf{R}^+ \to \mathbf{R}$ by $F(x,\rho) = \sum_{i=1}^{n} \frac{x_i^2}{\rho^{2\alpha_i}}$. Then for fixed $x \in \mathbf{R}^n$, it is easy to see that the function $F(x,\rho)$ is strictly decreasing function in $\rho > 0$. The unique solution of the

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equation $F(x,\rho)=1$ is denoted by $\rho(x)$. It was proved in [12] that $\rho(x)$ is a metric on $\mathbf{R}^n$, and $(\mathbf{R}^n,\rho)$ is the mixed homogeneity space related to $\{\alpha_i\}_{i=1}^n$.

For $\lambda>0$, let A_λ be the diagonal $n\times n$ matrix

$$A_\lambda = diag\{\lambda^{\alpha_1},\lambda^{\alpha_2},\cdots,\lambda^{\alpha_n}\} = \begin{bmatrix} \lambda^{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{\alpha_n} \end{bmatrix}.$$

The change of variables related to the space $(\mathbf{R}^n,\rho)$ is given by the transformation

$$\begin{aligned} x_1 &= \rho^{\alpha_1}\cos\vartheta_1\cdots\cos\vartheta_{n-2}\cos\vartheta_{n-1}, \\ x_2 &= \rho^{\alpha_2}\cos\vartheta_1\cdots\cos\vartheta_{n-2}\sin\vartheta_{n-1}, \\ &\vdots \\ x_{n-1} &= \rho^{\alpha_{n-1}}\cos\vartheta_1\sin\vartheta_2, \\ x_n &= \rho^{\alpha_n}\sin\vartheta_1; \end{aligned}$$

where $x=(x_1,x_2,\cdots,x_n)\in\mathbf{R}^n$. Thus, $dx=\rho^{\alpha-1}J(x')d\rho d\sigma(x')$, where $\rho^{\alpha-1}J(x')$ is the Jacobian of the above transforms,

$$\alpha=\sum_{k=1}^n \alpha_k \quad and \quad J(x')=\sum_{k=1}^n \alpha_k (x_k')^2.$$

It was shown in [12] that $J(x')$ is a $\mathcal{C}^\infty(\mathbf{S}^{n-1})$ function in the variable $x'\in\mathbf{S}^{n-1}$, and that a real constant $C\geq 1$ exists so that $1\leq J(x')\leq C$.

Let Ω be a real valued and measurable function on $\mathbf{R}^n$ with $\Omega\in L^1(\mathbf{S}^{n-1})$ that satisfies the conditions

$$\Omega(A_\lambda z) \quad = \quad \Omega(z), \quad \forall\lambda>0, \tag{1}$$

$$\int_{\mathbf{S}^{n-1}} \Omega(x')J(x')d\sigma(x')=0, \tag{2}$$

and let $P:\mathbf{R}^n\to\mathbf{R}$ be a real-valued polynomial. Define the maximal function $\mathcal{M}_{\Omega,P}$ by

$$\mathcal{M}_{\Omega,P}(f)(x)=\sup_{h\in L^\gamma(\mathbf{R}^+,\frac{d\rho}{\rho})}\left|\int_{\mathbf{R}^n} e^{iP(y)}f(x-y)\frac{\Omega(y')h(\rho(y))}{\rho(y)^\alpha}dy\right|, \tag{3}$$

where $L^\gamma(\mathbf{R}^+,\frac{d\rho}{\rho})$ be the set of all measurable functions $h:\mathbf{R}^+\to\mathbf{R}$ that satisfy

$$\|h\|_{L^\gamma(\mathbf{R}^+,\frac{d\rho}{\rho})}=\left(\int_0^\infty |h(r)|^\gamma \frac{d\rho}{\rho}\right)^{1/\gamma}\leq 1 \quad with\ \gamma\geq 1.$$

When $P(y) = 0$ and $\alpha_1 = \cdots = \alpha_n = 1$, then $\rho(x) = |x|$, $\alpha = n$ and $(\mathbf{R}^n, \rho) = (\mathbf{R}^n, |\cdot|)$. In this case, $\mathcal{M}_{\Omega,P}$ reduces to the classical maximal operator denoted by μ_Ω. The operator μ_Ω was introduced by Chen and Lim in [11] in which the authors proved that if $\Omega \in \mathcal{C}(\mathbf{S}^{n-1})$, then μ_Ω is bounded on $L^p(\mathbf{R}^n)$ for all $p > \frac{2n}{2n-1}$. Subsequently, the study of the L^p boundedness of μ_Ω under various conditions on the kernels has received a large amount of attention of many authors. For more information about the importance of such operators and their developments, the readers are refereed to [1], [7], [8], [10], [13], [14], [15], [16], [17], [19], among numerous references.

We shall state our main results as follows:

Theorem 1. *Suppose that $\Omega \in L^q(\mathbf{S}^{n-1})$, $1 < q \leq 2$ and $\|\Omega\|_{L^1(\mathbf{S}^{n-1})} \leq 1$. Let $P : \mathbf{R}^n \to \mathbf{R}$ be a real-valued polynomial and $h \in L^\gamma(\mathbf{R}^+, \frac{d\rho}{\rho})$ for some $1 \leq \gamma \leq 2$. Suppose that $\mathcal{M}_{\Omega,P}$ be given by (3). Then there exists a constant $C_p > 0$ such that*

$$\|\mathcal{M}_{\Omega,P}(f)\|_{L^p(\mathbf{R}^n)} \leq C_p \left(1 + \beta_\Omega\right)^{1/\gamma'} \|f\|_{L^p(\mathbf{R}^n)} \tag{4}$$

for all $\gamma' \leq p < \infty$ with $1 < \gamma \leq 2$; and

$$\|\mathcal{M}_{\Omega,P}(f)\|_{L^\infty(\mathbf{R}^n)} \leq C_p \|f\|_{L^\infty(\mathbf{R}^n)}, \tag{5}$$

where $\beta_\Omega = \log(e + \|\Omega\|_{L^q(\mathbf{S}^{n-1})})$.

The power of our theorem lies in using its conclusion and the extrapolation arguments as those in (see [3], [4] and [18]) to get the following result:

Theorem 2. *Suppose that Ω satisfies (1)-(2), $h \in L^\gamma(\mathbf{R}^+, \frac{d\rho}{\rho})$ for some $1 \leq \gamma \leq 2$ and $\mathcal{M}_{\Omega,P}$ is given by (3).*
(a) If $\Omega \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$, then

$$\|\mathcal{M}_{\Omega,P}(f)\|_{L^p(\mathbf{R}^n)} \leq C_p \|f\|_{L^p(\mathbf{R}^n)} \left(1 + \|\Omega\|_{L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})}\right)$$

for all $\gamma' \leq p < \infty$ with $1 < \gamma \leq 2$;
(b) If $\Omega \in B_q^{(0,-1/\gamma)}(\mathbf{S}^{n-1})$ for some $q > 1$, then

$$\|\mathcal{M}_{\Omega,P}(f)\|_{L^p(\mathbf{R}^n)} \leq C_p \|f\|_{L^p(\mathbf{R}^n)} \left(1 + \|\Omega\|_{B_q^{(0,-1/\gamma)}(\mathbf{S}^{n-1})}\right)$$

for all $\gamma' \leq p < \infty$ with $1 < \gamma \leq 2$.

By Theorem 2, we get the following:

Corollary 3. *Assume that $P : \mathbf{R}^n \to \mathbf{R}$ is a real-valued polynomial and $h \in L^\gamma(\mathbf{R}^+, \frac{dr}{r})$ for some $1 < \gamma \leq 2$. Let $\Omega \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1}) \cup B_q^{(0,-1/\gamma)}(\mathbf{S}^{n-1})$ and satisfies (1)-(2). Then the singular integral operator $T_{\Omega,P}$ given by*

$$T_{\Omega,P}(f) = p.v. \int_{\mathbf{R}^n} e^{iP(y)} f(x-y) \frac{\Omega(y')h(\rho(y))}{\rho(y)^\alpha}(y)dy$$

is bounded on $L^p(\mathbf{R}^n)$ for all $p \geq \gamma'$ with L^p bounds that may depend on the degree of the polynomial P but they are independent of the coefficients of the polynomial P.

Remark

(1) Al-Salman in [6] improved the results in [11]. In fact, he established the $L^p(\mathbf{R}^n)$ boundedness of μ_Ω for all $p \geq 2$ under the condition

$$\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1}).$$

Moreover, he pointed out that the condition $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ is optimal in the sense that the operator μ_Ω may lose the L^2 boundedness if Ω is assumed to be in the space $L(\log L)^\varepsilon(\mathbf{S}^{n-1})$ for some $\varepsilon < 1/2$.

(2) The author of [5] was able to show that the $L^p(\mathbf{R}^n)$ $(p \geq 2)$ boundedness of μ_Ω holds if Ω belongs to the block space $B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$, $q > 1$. Also, he established the optimality of the condition $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ in the sense that $-1/2$ in $B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ cannot be replaced by any smaller number.

(3) When $\alpha_1 = \cdots = \alpha_n = 1$, the operator $\mathcal{M}_{\Omega,P}$ was studied in [4] only for the case $\gamma = 2$. Precisly, he proved that $\mathcal{M}_{\Omega,P}$ is bounded on $L^p(\mathbf{R}^n)$ for all $p \geq 2$ provide that $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1}) \cup B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$.

(4) Very recently, under the conditions

$$\Omega \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1}) \cup B_q^{(0,-1/\gamma)}(\mathbf{S}^{n-1}),$$

$h \in L^\gamma(\mathbf{R}^+, \frac{d\rho}{\rho})$ with $1 \leq \gamma \leq 2$, Ali in [9] found that $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbf{R}^{n+1})$ for all $(p \geq \gamma')$, where the revolution was about $y = \phi(y)$ and ϕ is any polynomial.

Here and henceforth, the letter C denotes a bounded positive constant that may vary at each occurrence but independent of the essential variables.

2. Preliminary Lemmas

In this section, we present and establish two lemmas used in the sequel. The first lemma is obtained by using [[9], Lemma 2.2] with a special case $\phi(\rho) = \rho$.

Lemma 4. *Let $\Omega \in L^1(\mathbf{S}^{n-1})$ satisfy the conditions* (1)-(2)*. Define the maximal function $\mathcal{M}_\Omega^\rho f$ by*

$$\mathcal{M}_\Omega^\rho f(x) = \sup_{\mathbf{j} \in \mathbf{Z}} \int\limits_{2^j \le \rho(y) \le 2^{j+1}} \frac{|f(x-y)|\,|\Omega(y)|}{\rho(y)^n} dy.$$

Then, for $1 < p \le \infty$, we have

$$\left\|\mathcal{M}_\Omega^\rho f\right\|_{L^p(\mathbf{R}^n)} \le C_p \left\|f\right\|_{L^p(\mathbf{R}^n)} \left\|\Omega\right\|_{L^1(\mathbf{S}^{n-1})}.$$

The next lemma can be obtained by applying the arguments (with only minor modifications) used in [4] as well as [9].

Lemma 5. *Let $1 < q \le 2$, and let $\Omega \in L^q(\mathbf{S}^{n-1})$ that satisfies the conditions* (1)-(2) *with $\|\Omega\|_{L^1(\mathbf{S}^{n-1})} \le 1$. Assume that*

$$P(x) = \sum_{|\gamma| \le m} a_\gamma x^\gamma$$

is a real-valued polynomial of degree m with $m > 1$ such that $|x|^m$ is not one of its terms. For $k \in \mathbf{N}$, define $\mathcal{J}_{k,\Omega} : \mathbf{R}^n \to \mathbf{R}$ by

$$\mathcal{J}_{k,\Omega}(\zeta) = \int_1^{2^{2\beta_\Omega}} \left| \int\limits_{\mathbf{S}^{n-1}} \Omega(u) J(u) \mathcal{G}_{k,\Omega}(\rho, P(u), A_\rho u \cdot \zeta) d\sigma(u) \right|^2 \frac{d\rho}{\rho}, \qquad (6)$$

where

$$\mathcal{G}_{k,\Omega}(\rho, P(u), s) = e^{-i\left[P\left(2^{-(k+1)\beta_\Omega} A_\rho u\right) + 2^{-(k+1)\beta_\Omega} s\right]}. \qquad (7)$$

Then, there exists a constat $C > 0$ such that

$$\sup_{\zeta \in \mathbf{R}^n} \mathcal{J}_{k,\Omega}(\zeta) \le C 2^{\frac{(k+1)}{4\tau q'}} \beta_\Omega,$$

where $\tau = \max\{\alpha_1, \cdots, \alpha_n\}$.

Proof. On one hand, by using the same approaches as in [[4], ineq. (2.8)], we acheive that

$$P\left(2^{-(k+1)\beta_\Omega}A_\rho u\right)+2^{-(k+1)\beta_\Omega}A_\rho u\cdot\zeta-P\left(2^{-(k+1)\beta_\Omega}A_\rho v\right)-2^{-(k+1)\beta_\Omega}A_\rho v\cdot\zeta$$

$$=2^{-m(k+1)\beta_\Omega}\left(A_{\rho^m}\sum_{|\gamma|=m}a_\gamma\left(u^\gamma-v^\gamma\right)\right)+2^{-(k+1)\beta_\Omega}A_\rho(u-v)\cdot\zeta+H_{\rho,u,v,\zeta},$$

where $\frac{d^s}{d\rho^s}H_{\rho,u,v,\zeta}=0$ and $s=m\tau$. So, by Van der Court lemma, we obtain

$$\begin{aligned}&\left|\int_1^{2^{2\beta_\Omega}}\mathcal{G}_{k,\Omega}(\rho,P(u),\zeta\cdot u)\overline{\mathcal{G}_{k,\Omega}(\rho,P(v),\zeta\cdot v)}\frac{d\rho}{\rho}\right|\\ \leq\ & C_s\left|2^{-m(k+1)\beta_\Omega}\left\{P(u)-P(v)\right\}\right|^{-1/s},\end{aligned}\tag{8}$$

which when combined with the trivial estimate

$$\left|\int_1^{2^{2\beta_\Omega}}\mathcal{G}_{k,\Omega}(\rho,P(u),\zeta\cdot u)\overline{\mathcal{G}_{k,\Omega}(\rho,P(v),\zeta\cdot v)}\frac{d\rho}{\rho}\right|\leq C\beta_\Omega$$

leads to

$$\begin{aligned}&\left|\int_1^{2^{2\beta_\Omega}}\mathcal{G}_{k,\Omega}(\rho,P(u),\zeta\cdot u)\overline{\mathcal{G}_{k,\Omega}(\rho,P(v),\zeta\cdot v)}\frac{d\rho}{\rho}\right|\\ &\qquad\leq C_s\left|2^{-m(k+1)\beta_\Omega}\left\{P(u)-P(v)\right\}\right|^{\frac{-1}{4sq'\beta_\Omega}}\beta_\Omega^{1-\frac{1}{4q'\beta_\Omega}}.\end{aligned}$$

Now, by Hölder's inequality, we obtain

$$\begin{aligned}&\left(\mathcal{J}_{k,\Omega}(\zeta)\right)^{q'}\leq C\left\|\Omega\right\|^{2q'}_{L^q(\mathbf{S}^{n-1})}\\ &\int_{\mathbf{S}^{n-1}}\int_{\mathbf{S}^{n-1}}\left|\int_1^{2^{2\beta_\Omega}}\mathcal{G}_{k,\Omega}(\rho,P(u),\zeta\cdot u)\ \overline{\mathcal{G}_{k,\Omega}(\rho,P(v),\zeta\cdot v)}\frac{d\rho}{\rho}\right|^{q'}d\sigma(u)d\sigma(v).\end{aligned}$$

Hence, as $1<q\leq 2$, we deduce

$$\mathcal{J}_{k,\Omega}(\zeta)\ \leq\ C_s2^{\frac{(k+1)}{4\tau q'}}\left\|\Omega\right\|^2_{L^1(\mathbf{S}^{n-1})}\beta_\Omega^{1-\frac{1}{4q'\beta_\Omega}}\leq C_m2^{\frac{(k+1)}{4\tau q'}}\beta_\Omega$$

□

3. Proof of Main Result

In the proof of Theorem 1, we will use some ideas from [2], [4] and [9]. By duality,

$$\mathcal{M}_{\Omega,P}(f)(x) = \left(\int_0^\infty \left| \int_{\mathbf{S}^{n-1}} e^{iP(A_\rho u)} f(x - A_\rho u)\Omega(u)J(u)d\sigma(u) \right|^{\gamma'} \frac{d\rho}{\rho} \right)^{1/\gamma'},$$

and so we have

$$\|\mathcal{M}_{\Omega,P}(f)\|_{L^p(\mathbf{R}^n)} = \|N(f)\|_{L^p(L^{\gamma'}(\mathbf{R}^+,\frac{d\rho}{\rho}),\mathbf{R}^n)}, \tag{9}$$

where $N : L^p(\mathbf{R}^n) \to L^p(L^{\gamma'}(\mathbf{R}^+, \frac{d\rho}{\rho}), \mathbf{R}^n)$ is defined by

$$N(f)(x,\rho) = \int_{\mathbf{S}^{n-1}} e^{iP(A_\rho u)} f(x - A_\rho u)\Omega(u)J(u)d\sigma(u).$$

Now, assume that Theorem 1 is true for the cases $\gamma = 1$ and $\gamma = 2$; which means that

$$\begin{aligned} \|\mathcal{M}_{\Omega,P}(f)\|_{L^\infty(\mathbf{R}^{n+1})} &= \|N(f)\|_{L^\infty(L^\infty(\mathbf{R}^+,\frac{d\rho}{\rho}),\mathbf{R}^{n+1})} \\ &\le C_p \|f\|_{L^\infty(\mathbf{R}^{n+1})} \end{aligned} \tag{10}$$

and

$$\begin{aligned} \|\mathcal{M}_{\Omega,P}(f)\|_{L^p(\mathbf{R}^n)} &= \|N(f)\|_{L^p(L^2(\mathbf{R}^+,\frac{d\rho}{\rho}),\mathbf{R}^n)} \\ &\le C_p(1+\beta_\Omega)^{1/2} \|f\|_{L^p(\mathbf{R}^n)} \end{aligned} \tag{11}$$

for all $2 \le p < \infty$. Then by the interpolation theorem to (10)-(11), we directly deduce

$$\|\mathcal{M}_{\Omega,P}(f)\|_{L^p(\mathbf{R}^n)} \le C_p(1+\beta_\Omega)^{1/\gamma'} \|f\|_{L^p(\mathbf{R}^n)} \tag{12}$$

for $\gamma' \le p < \infty$ and for $1 < \gamma < 2$.

Thus, we finish the proof of Theorem 1 for $\gamma \in [1, 2]$ once we prove it only for the cases $\gamma = 1$ and $\gamma = 2$.

Case 1 (if $\gamma = 1$). Assume that $h \in L^1(\mathbf{R}^+, \frac{d\rho}{\rho})$ and $f \in L^\infty(\mathbf{R}^n)$. Then for all $x \in \mathbf{R}^n$, we have

$$\left| \int_0^\infty h(\rho) \int_{\mathbf{S}^{n-1}} e^{iP(A_\rho u)} f(x - A_\rho u)\Omega(u)J(u)d\sigma(u) \frac{d\rho}{\rho} \right| \le C \|f\|_{L^\infty(\mathbf{R}^n)} \|h\|_{L^1(\mathbf{R}^+,\frac{d\rho}{\rho})}.$$

Hence, Taking the supremum on both sides over all h with $\|h\|_{L^1(\mathbf{R}^+,\frac{d\rho}{\rho})} \leq 1$ gives

$$\mathcal{M}_{\Omega,P}f(x) \leq C\,\|f\|_{L^\infty(\mathbf{R}^n)}$$

for almost every where $x \in \mathbf{R}^n$, which leads to

$$\|\mathcal{M}_{\Omega,P}f\|_{L^\infty(\mathbf{R}^n)} \leq C\,\|f\|_{L^\infty(\mathbf{R}^n)}\,.$$

Case 2 (**if** $\gamma = \mathbf{2}$). We prove Theorem 1 for the case $\gamma = 2$ by applying the approaches used in [4]. We use the induction on the degree of the polynomial P. If the degree of P is 0, then by [[9], Theorem 1.1] with $\phi(t) = t$ plus the duality, we get

$$\|\mathcal{M}_{\Omega,P}(f)\|_{L^p(\mathbf{R}^n)} \quad \leq \quad C_p\ (1+\beta_\Omega)^{1/2}\,\|f\|_{L^p(\mathbf{R}^n)} \tag{13}$$

for $p \geq 2$. Now, if $\deg(P) = 1$, then $P(y) = \kappa \cdot y$ for some $\kappa \in \mathbf{R}^n$. Define the function $\psi(y) = e^{-i\kappa\cdot y}f(y)$. So we have

$$\mathcal{M}_{\Omega,P}(f)(x) \quad = \quad \sup_{h\in U}\left|\int\limits_{\mathbf{R}^n} e^{i\kappa\cdot x}\psi(x-y)|\frac{\Omega(y')h(\rho(y))}{\rho(y)^\alpha}dy\right| \leq \mathcal{M}_{\Omega,P}(\psi)(x),$$

which gives by (13) that the inequality

$$\|\mathcal{M}_{\Omega,P}(f)\|_{L^p(\mathbf{R}^n)} \quad \leq \quad C_p\ (1+\beta_\Omega)^{1/2}\,\|f\|_{L^p(\mathbf{R}^n)} \tag{14}$$

holds for all $p \geq 2$. Next, assume that (11) is satisfied for any polynomial of degree less than or equal to m with $m \geq 2$. We need to show that (11) is still true if $deg(P) = m+1$. Let

$$P(x) = \sum_{|\alpha|\leq m+1} a_\gamma x^\gamma$$

be a polynomial of degree $m+1$. Without loss of generality, we may assume that $\sum\limits_{|\gamma|=m+1} |a_\gamma| = 1$, and also we may assume that P does not contain $|x|^{m+1}$ as one of its terms. Let $\{\varphi_k\}_{k\in\mathbf{Z}}$ be a collection of $\mathcal{C}^\infty(0,\infty)$ functions satisfying the following conditions:

$$\text{supp}\ \varphi_k \quad \subseteq \quad \mathcal{I}_{k,\beta_\Omega} = \left[2^{-(k+1)\beta_\Omega}, 2^{-(k-1)\beta_\Omega}\right];\ \ \varphi_k(u) = \varphi_k(\rho(u)))$$

$$0 \le \varphi_k \le 1; \quad \sum_{k \in \mathbf{Z}} \varphi_k(\rho) = 1; \quad and \quad \left| \frac{d^k \varphi_k(\rho)}{d\rho^k} \right| \le \frac{C_k}{\rho^k}.$$

Set

$$\Gamma_\infty(\rho) = \sum_{k=-\infty}^{0} \varphi_k(\rho) \quad and \quad \Gamma_0(\rho) = \sum_{k=1}^{\infty} \varphi_k(\rho).$$

Thanks to Minkowski's inequality, we have

$$\mathcal{M}_{\Omega,P}(f)(x) \le \mathcal{M}_{\Omega,P,\infty}(f)(x) + \mathcal{M}_{\Omega,P,0}(f)(x), \tag{15}$$

where

$$\mathcal{M}_{\Omega,P,\infty}(f)(x) = \left(\int_{2^{-\beta_\Omega}}^{\infty} \left| \Gamma_\infty(\rho) \int_{\mathbf{S}^{n-1}} e^{iP(A_\rho u)} f(x - A_\rho u)\Omega(u)J(u)d\sigma(u) \right|^2 \frac{d\rho}{\rho} \right)^{1/2},$$

and

$$\mathcal{M}_{\Omega,P,0}(f)(x) = \left(\int_0^1 \left| \Gamma_0(\rho) \int_{\mathbf{S}^{n-1}} e^{iP(A_\rho u)} f(x - A_\rho u)\Omega(u)J(u)d\sigma(u) \right|^2 \frac{d\rho}{\rho} \right)^{1/2}.$$

Let us first compute L^p-norm of $\mathcal{M}_{\Omega,P,\infty}$. Define

$$\mathcal{M}_{\Omega,P,\infty,k}(f)(x) = \left(\int_{2^{-(k+1)\beta_\Omega}}^{2^{-(k-1)\beta_\Omega}} \left| \int_{\mathbf{S}^{n-1}} e^{iP(A_\rho u)} f(x - A_\rho u)\Omega(u)J(u)d\sigma(u) \right|^2 \frac{d\rho}{\rho} \right)^{1/2}.$$

Hence, by generalized Minkowski's inequality, it is easy to show that

$$\mathcal{M}_{\Omega,P,\infty}(f)(x) \le \sum_{k=-\infty}^{0} \mathcal{M}_{\Omega,P,\infty,k}(f)(x). \tag{16}$$

If $p = 2$, then by a simple change of variables, Plancherel's theorem, Fubini's theorem, and Lemma 5, we get that

$$\|\mathcal{M}_{\Omega,P,\infty,k}(f)\|_{L^2(\mathbf{R}^n)} = \left(\int_{\mathbf{R}^n} \left| \widehat{f}(\zeta) \right|^2 \mathcal{J}_{k,\Omega}(\zeta) d\zeta \right)^{1/2}$$

$$\leq \quad C2^{\frac{(k+1)}{8q'}} \left(\beta_\Omega + 1\right)^{1/2} \|f\|_{L^2(\mathbf{R}^n)} . \tag{17}$$

However, if $p > 2$, then by the duality, there exists $\Psi \in L^{(p/2)'}(\mathbf{R}^n)$ with $\|\Psi\|_{L^{(p/2)'}(\mathbf{R}^n)} = 1$ such that

$$\|\mathcal{M}_{\Omega,P,\infty,k}(f)\|^2_{L^p(\mathbf{R}^n)}$$
$$= \int\limits_{\mathbf{R}^n} \int\limits_1^{2^{2\beta_\Omega}} \left| \int\limits_{\mathbf{S}^{n-1}} \mathcal{G}_{k,\Omega}(\rho, P(u), 0) f(x - 2^{-(k+1)\beta_\Omega} A_\rho u) J(u) d\sigma(u) \right|^2 \frac{d\rho}{\rho} |\Psi(x)|\, dx$$

So, by Hölder's inequality and Lemma 4, we conclude that

$$\begin{aligned}
\|\mathcal{M}_{\Omega,P,\infty,k}(f)\|^2_{L^p(\mathbf{R}^n)} \leq & C \int\limits_{\mathbf{R}^n} |f(z)|^2 \\
& \int\limits_1^{2^{2\beta_\Omega}} \int\limits_{\mathbf{S}^{n-1}} |\Omega(u)| \left|\Psi(z + 2^{-(k+1)\beta_\Omega} A_\rho u)\right| d\sigma(u) \frac{d\rho}{\rho} dz \\
\leq & C\beta_\Omega \left\| |f|^2 \right\|_{L^{(p/2)}(\mathbf{R}^n)} \left\| \mathcal{M}^\rho_\Omega \widetilde{\Psi}(z) \right\|_{L^{(p/2)'}(\mathbf{R}^n)} \\
\leq & C_p \beta_\Omega \|f\|^2_{L^p(\mathbf{R}^n)} \|\Psi\|_{L^{(p/2)'}(\mathbf{R}^n)} \|\Omega\|_{L^1(\mathbf{S}^{n-1})} ,
\end{aligned}$$

where $\widetilde{\Psi}(z) = \Psi(-z)$. Thus,

$$\|\mathcal{M}_{\Omega,P,\infty,k}(f)\|_{L^p(\mathbf{R}^n)} \leq C_p \left(1 + \beta_\Omega\right)^{1/2} \|f\|_{L^p(\mathbf{R}^n)} ,$$

which when Combined with (17) gives that there is $0 < \varepsilon < 1$ so that

$$\|\mathcal{M}_{\Omega,P,\infty,k}(f)\|_{L^p(\mathbf{R}^n)} \leq C_p 2^{\varepsilon(k+1)/8} \left(1 + \beta_\Omega\right)^{1/2} \|f\|_{L^p(\mathbf{R}^n)} \tag{18}$$

for all $p \geq 2$. Therefore, by (16) and (18), we obtain

$$\|\mathcal{M}_{\Omega,P,\infty}(f)\|_{L^p(\mathbf{R}^n)} \leq C_p \left(1 + \beta_\Omega\right)^{1/2} \|f\|_{L^p(\mathbf{R}^n)} . \tag{19}$$

Now, let us estimate the L^p-norm of $\mathcal{M}_{\Omega,P,0}$. Let $Q(x) = \sum\limits_{|\gamma| \leq m} a_\gamma x^\gamma$. Define $\mathcal{M}^{(1)}_{\Omega,Q,0}$ and $\mathcal{M}^{(2)}_{\Omega,P,Q,0}$ by

$$\mathcal{M}^{(1)}_{\Omega,Q,0}(f)(x) = \left(\int\limits_0^1 \left| \int\limits_{\mathbf{S}^{n-1}} e^{iQ(A_\rho u)} f(x - A_\rho u) \Omega(u) J(u) d\sigma(u) \right|^2 \frac{d\rho}{\rho} \right)^{1/2} ,$$

$$\mathcal{M}^{(2)}_{\Omega,P,Q,0}(f)(x) =$$
$$\left(\int_0^1 \left| \int_{\mathbf{S}^{n-1}} \left(e^{iP(A_\rho u)} - e^{iQ(A_\rho u)}\right) f(x - A_\rho u)\Omega(u)J(u)d\sigma(u)\right|^2 \frac{d\rho}{\rho}\right)^{1/2}.$$

Thus, by Minkowski's inequality, we deduce

$$\mathcal{M}_{\Omega,P,0}(f)(x) \le \mathcal{M}^{(1)}_{\Omega,Q,0}(f)(x) + \mathcal{M}^{(2)}_{\Omega,P,Q,0}(f)(x). \tag{20}$$

On one hand, since $\deg(Q) \le m$, then by our assumption,

$$\left\|\mathcal{M}^{(1)}_{\Omega,Q,0}(f)\right\|_{L^p(\mathbf{R}^n)} \le C_p \ (1+\beta_\Omega)^{1/2} \|f\|_{L^p(\mathbf{R}^n)} \tag{21}$$

for all $p \ge 2$. On the other hand, since we have

$$\left|e^{iP(A_\rho u)} - e^{iQ(A_\rho u)}\right| \ \le \ \rho^{\tau(m+1)} \left| \sum_{|\gamma|=m+1} a_\gamma(u)^\gamma \right| \le \rho^{\tau(m+1)},$$

then by Cauchy-Schwartz inequality, we reach that

$$\begin{aligned}
\mathcal{M}^{(2)}_{\Omega,P,Q,0}(f)(x) \le& C\left(\int_0^1 \int_{\mathbf{S}^{n-1}} \rho^{2\tau(m+1)} |\Omega(u)| \, |f(x-A_\rho u)|^2 \, d\sigma(u) \frac{d\rho}{\rho}\right)^{1/2} \\
\le& \left(\sum_{j=1}^{\infty} 2^{-j(2\tau(m+1))} \int_{2^{-j}}^{2^{-j+1}} \int_{\mathbf{S}^{n-1}} |\Omega(u)| \, |f(x-A_\rho u)|^2 \, d\sigma(u) \frac{d\rho}{\rho}\right)^{1/2} \\
\le& C\left(\mathcal{M}^\rho_\Omega\left(|f|^2\right)\right)^{1/2}.
\end{aligned}$$

Hence, by Lemma 4, we get that

$$\begin{aligned}
\left\|\mathcal{M}^{(2)}_{\Omega,P,Q,0}(f)\right\|_{L^p(\mathbf{R}^n)} \ &\le \ C_p \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \left\||f|^2\right\|^{1/2}_{L^{p/2}(\mathbf{R}^n)} \\
&\le \ C_p \|f\|_{L^p(\mathbf{R}^n)} \le C_p \ (1+\beta_\Omega)^{1/2} \|f\|_{L^p(\mathbf{R}^n)} \qquad (22)
\end{aligned}$$

for all $p \ge 2$. Therefore, by (20)-(22), we obtain

$$\|\mathcal{M}_{\Omega,P,0}(f)\|_{L^p(\mathbf{R}^n)} \le C_p \ (1+\beta_\Omega)^{1/2} \|f\|_{L^p(\mathbf{R}^n)}. \tag{23}$$

Consequently, by (15), (19) and (23), we finish the proof of Theorem 1.

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