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A DIFFERENTIAL GAME ON MILITARY INTERVENTION AND COMPETITION BETWEEN DRUG CARTELS

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Abstract: We explore the causes of the surge in violence in Mexico. In particular, we expose a dynamic competition game theory model where two drug organizations use warfare to compete for a turf in a certain territory. The inclusion of a military intervention in the model increases the violence in the zone given that the potential gains from holding territory are constant.

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1. Introduction

The surge of violence Mexico has seen in recent years can be attributed to a failure in the federal government strategy. In particular, it has been observed that the drug-related homicides have spiked since the declaration of the "war against drugs" from the year 2006 (see [14] and [12]). It is also noticeable that over 70% of such reported crimes concentrate in only 80 municipalities. This suggests that certain territories have a strategic importance for drug cartels. For example, cities in the border with the United States can be difficult to substitute as a mean to transport drugs to this important market. This means

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that drug cartels have strong incentives to maintain control of such cities.

This also means that a policy of prohibition has little effect deterring criminal's activity. It has been observed [19] that prohibition leads to an increase of violence, since the supplying firms have no legal means to protect their source of profit. A legal firm can resource to marketing strategies to try to increase the demand of their product, and take legal actions when there is a dispute over unfair competition, trademark issues or other. Since a drug cartel does not have these options, violence becomes a natural option. Moreover, the introduction of a military intervention can be disruptive of the market, leading to an increase of the need of military action (see [7] and [13]).

This work presents a model of dynamic competition between drug cartels for control over a given territory. The organizations are modeled as profit maximizing firms whose set of actions is constrained to be warfare. Additionally, an exogenous variable is introduced that can represent military intervention in the turf. Since this problem can be stated as a non-cooperative game, a Nash equilibrium solution is in place.

We find the Nash equilibria for the cartels in terms of their optimal warfare policy and their long run controlled territory, supposing a certain level of military intervention.

This article adds up to the hypothesis that one of the main causes in the recent surge of violence in Mexico has to do with the use of intervention strategies (cf. [15], [2], [1], [12], [16], [7] and [13]). This idea is not new, it has been known that the prohibition of addictive substances is correlated with increase in turfs and violence between criminal organizations (cf. [17] and [19]).

The present article provides a theoretical analysis of a model in which we propose a logical sequence for the hypothesis. In particular, we believe that, if we consider the drug cartels as profit maximizing agents with the constraint that their profits come from the sales they can make in the territory they control, then they are motivated to increase their control over a proportion of the zone as big as possible. If these incentives are fixed, then a policy of intervention will lead to an increase of warfare to recover territory and protect what is controlled.

The rest of the article is organized as follows: in Section 2 we provide the basis of the model, along with its assumptions. The Nash equilibria are presented in Section 3, and an application of the model and its results is provided in Section 4. In this application, we use data from a nation survey of victimization in Mexico to observe the relation between military intervention and gunfighting in the cities. Our final remarks can be found in Section 5.

2. The Model

In this section we develop a competition model of drug organizations fighting for control over a territory. The control over the territory is then diminished by an exogenous force that can represent a military intervention taking over and securing the zone.

The competition between drug organizations is played in continuous time from period 0 to terminal time T, in which there is a terminal payoff, depending on the portion of the territory under control. In this model, two players are considered, that is, two profit maximizing drug cartels. They are interested in gaining control over the highest portion possible of a fixed territory M which they can exploit to obtain a constant revenue rate π_i for every instant $\mathrm{d}t$. We can imagine that some territories are more valuable for these organizations than others; for example, a border city would present a higher level of revenue rate due to the access to a bigger market (think of Mexican cities like Juárez). As we will see, a higher revenue rate from the territory holding means a greater incentive for the criminal organizations to increase warfare expenditures.

Naturally, the control over a city by player 1 is opposed by a similar objective by player 2. Since they operate in an illegal framework, it is impossible for them to gain territory by legal means (e.g. by advertising or by buying franchises or physical spaces like retail stores to supply their product). Instead, they should turn to violence to gain control of a share of their territory. This creates a direct negative externality over society in general. This externality might justify the existence of a strong exogenous force that takes away the territory of the cartels. For example, society can demand the presence of military intervention to recover occupied territory from the drug dealers.

Thus, let $x_i(t)$ represent the share of the territory that player $i \in \{1, 2\}$ is holding at time $t \in [0, T]$. At each instant, both cartels decide their warfare expenditure level $u_i(t)$, directed towards the share they do not control (i.e. $1 - x_i(t)$). Agent i's level of control over the territory is therefore driven by the differential equation

$$\dot{x}_i(t) = \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = k_i u_i(t)[1 - x_i(t)] - k_j u_j x_i - \delta_i x_i, \qquad x_i(0) = x_i^0, \tag{1}$$

for $i \in \{1, 2\}$. Here, k_i is a constant that represents the effectiveness of player i's warfare activities. Note that while player i's share of control of the territory increases according to her warfare expenditures, but is diminished by player j's expenditures and a constant externality δ_i, that can represent military intervention. The fact that there is a different rate δ_i for every player is a generalization in which the external forces can affect each cartel in a different way. These external forces can be military interventions or the creation of tougher controls in the case of border cities.

Player i can have therefore two kinds of competitive advantage against player $j \neq i$. The first is given by their warfare capacity, reflected in their relative weight of k_i against k_j. If $k_i = k_j$, then both cartels have the same *offensive* capacity. The second comes in the form of how much *defensive* power the cartel has against external forces, relative to its rival. This is measured by the difference between the δ_i against δ_j. A higher level of the externality rate means the cartel is more vulnerable to them.

Note that the territory is divided in three domains: one for each cartel and a third that is occupied by the external forces. This can mean the government or another abstraction of the society, which has taken control of the city from the cartels. There is no explicit description of the efforts taken to obtain this control nor if such actions were deliberate or not, the only explicit characteristic about the exogenous force is that they are constant and non-negative, such that if $u_i(t) = u_j(t) = 0$ for all t, then the territory will be occupied by it in the long run.

Unfortunately, this does not mean that a cartel's control can be completely eliminated by means of increasing the amount of military intervention. Note that whenever $x_i = 0$, the dynamics in (1) yield that $\dot{x}_i(t) = k_i u_i(t)$ which is positive unless $u_i(t) = 0$.

This is a differential game known as *Lanchester model.* The name comes from the work of Frederick Lanchester [11], developed for warfare and later reimagined for a business competition framework (see [3], [9], [8] and [18]). In this work we consider a mixed approach, in which the drug cartels' main objetive is profit maximization, which can be modeled for each $i \in \{1, 2\}$ as

$$\max_{u_i} J_i = \int_0^T \left[M\pi_i x_i - \frac{u_i^2}{2} \right] \mathrm{d}t + g_i x_i(T), \tag{2}$$

where π_i represents the marginal revenue from holding the proportion x_i of the territory M during a period of size period $\mathrm{d}t$ and g_i is the terminal payoff for the share of the market held at the end of the game. Note that, although warfare is the available action for a player to increment their presence in the market and therefore increase her profit, it has a cost. Moreover, the cost function is modeled to be convex in u_i, so it is unfeasible for any player to increase the warfare expenditures infinitely to obtain more land control.

It appears that under this model, the natural policy variable is the size of the externalities for the drug cartels, which can be interpreted as military expenditures aimed to remove the presence of such organizations from the territory. However, as we will see, this idea can prove to be counterproductive whenever the profitability of the territory is kept constant.

The concept of solution for the present non-cooperative game is a Nash equilibrium, i.e. a set of strategies (u_i^*, u_j^*), for which $J_i(u_i^*, u_j^*) \geq J_j(u_i, u_j^*)$ for any $i \in \{1, 2\}$, $j \neq i$ and any u_i in the space of feasible actions U_i. This solution is given as a function of time, which means that the result should apply for any moment in the period $t \in [0, T]$, for which none of the players has incentives to profitably deviate from the strategy.

In section 3, we examine the Nash equilibria that leads us to our main result.

3. Nash equilibria

In this section we find an open-loop solution of the differential game. We arrive to this solution by assuming that the cartels have perfect foresight of their own and their opponent's strategy plan. This yields a selection of a strategy that is indistinguishable from a plan that is made before the start of the game and implemented without the need of immediate feedback. The result is computationally more efficient without significant loss in accuracy.

We follow the work of Wang and Wu [23], in which the authors develop a numerical algorithm to find a feedback Nash equilibrium. This work shows only a simple open-loop solution to the problem. For this, consider the hamiltonian (see [24], Theorem 2.1.3) of the problem of maximizing (2) subject to the state dynamics in (1) represented by

$$\begin{aligned}\mathcal{H}_i = M\pi_i x_i - \frac{u_i^2}{2} + \lambda_i[k_i u_i(t)(1 - x_i(t)) - (k_j u_j + \delta_i)x_i]\\ + \phi_i[k_j u_j(1 - x_i) - (k_i u_i + \delta_i)x_j],\end{aligned}$$

with necessary conditions

$$\begin{aligned}\dot{\lambda}_i(t) &= -\frac{\partial \mathcal{H}_i}{\partial x_i} - \frac{\partial \mathcal{H}_i}{\partial u_j}\frac{\partial u_j}{\partial x_i},\\ \dot{\phi}_i(t) &= -\frac{\partial \mathcal{H}_i}{\partial x_j} - \frac{\partial \mathcal{H}_i}{\partial u_j}\frac{\partial u_j}{\partial x_j},\\ \lambda_i(T) &= g_i,\end{aligned}$$

for $i = 1, 2$ and $j \neq i$. Here, λ_i and ϕ_i are auxiliary costate variables that can be interpreted as the marginal effect in the profits of small increments in the control of territory. This is a relaxation of the problem's restrictions set by the state variable's dynamics. Note though that we are using x_j as the territory controlled by firm $j \neq i$.

Each player can thus find their optimal strategy u_i^* by using first order conditions. Let $\partial \mathcal{H}_i / \partial u_i(t) = 0$. This yields an optimal strategy for player i:

$$u_i^*(t) = k_i[\lambda_i(1 - x_i(t)) - \phi_i x_j(t)]$$

Note that for each cartel i, the condition $\lambda_i > \phi_i$ should apply, since increasing their own share of territory is more profitable than increasing their rival's, therefore the warfare action rate for player i does not turn into a negative value during the development of the game. In an open loop strategy, it verifies that $\partial u_i / \partial x_i = 0$ which implies that the necessary conditions for the costate variables are $\dot{\lambda}_i(t) = -\partial \mathcal{H}_i / \partial x_i$ and $\dot{\phi}_i(t) = -\partial \mathcal{H}_h / \partial x_j$, in other words,

$$\begin{aligned} \dot{\lambda}_i &= -M\pi_i + \lambda_i(t)[k_i u_i(t) + k_j u_j(t) + \delta_i], \\ \dot{\phi}_i &= \phi_i(t)[k_j u_j + k_i u_i + \delta_i] \end{aligned}$$

and

$$\lambda_i(T) = g_i$$

The costate variable ϕ_i has a terminal condition $\phi_i(T) = 0$, and therefore the differential equation that describes its motion yields that $\phi_i(t) = 0$ for all $t \in [0, T]$. Note that this means that the optimal strategy of player i is simply

$$u_i^*(t) = k_i \lambda_i(t)[1 - x_i(t)], \tag{3}$$

if $\lambda_i(t) > 0$ for all t in the period of the game. By inserting the control variables in (3) in the state and costate dynamics, we have a system of differential equations which can be solved with the use of numeric methods (see [23]), to find the pair (u_1^*, u_2^*) that constitutes a Nash equilibrium, i.e. for which no cartel has incentives to profitably deviate, considering the actions of their rivals.

Considering boundary conditions of the problem, we introduce an adjustment variable for the costate λ_i. Let $v = k_1 u_1 + k_2 u_2$, then we adjust the difference between the function g_i and the value of λ_t in the terminal time with

$$\alpha[g_i - \lambda(T)] \exp\left\{ \int_0^T [v(\varphi) + \delta_i] \mathrm{d}\varphi \right\},$$

where $\alpha \in (0,1)$ is a scale parameter, therefore

$$\lambda_i(t) = \left[-M\pi_i \int_0^t \exp\left\{\int_0^\tau [v(\varphi)+\delta_i(\varphi)]\mathrm{d}\varphi\right\}\mathrm{d}\tau + \lambda_i(0)\right] \times \exp\left\{\int_0^T [v(\varphi)+\delta_i(\varphi)]\mathrm{d}\varphi\right\}, \tag{4}$$

is the value of the marginal effect in the profits by increasing control in the territory of the cartel i at time t. This yields a terminal value for $\lambda(T)$ of

$$\lambda_i(T) = -M\pi_i \int_0^T \exp\left\{\int_t^T [v(\varphi)+\delta_i(\varphi)]\mathrm{d}\varphi\right\}\mathrm{d}\tau + \lambda_i(0)\exp\left\{\int_0^T [v(\varphi)+\delta_i(\varphi)]\mathrm{d}\varphi\right\}. \tag{5}$$

Note that an appropriate estimate of the auxiliary variable requires a proper guess of the initial values of it, since an unitary change of $\lambda_i(0)$ modifies the final value in $\exp\left\{\int_0^T [v(\varphi)+\delta(\varphi)]\mathrm{d}\varphi\right\}$.

An estimate for $\lambda_i(0)$ can be obtained by using (5) to derive it and expressing $\lambda_i(T) = g_i$. Since the government intervention has not started, it is natural to assume that $\delta_i(0) = 0$, and therefore

$$\lambda_i(0) = M\pi_i \int_0^T \exp\left\{-\int_0^\tau v_i(\varphi)\mathrm{d}\varphi)\right\}\mathrm{d}\tau + g_i \exp\left\{-\int_0^T v_i(\varphi)\mathrm{d}\varphi\right\},$$

for $i = 1, 2$. Note that this value still depends on the value of v_i, which can be estimated by setting $v_i = k_1\bar{u}_1 + k_2\bar{u}_2$ where $\bar{u}_i$ is the average measure of narcotraffic warfare.

By inserting (4) in equation (3) and by supposing that all M, π_i and v_i are constants, the value of the warfare expenditures depend solely on the exogenous variable δ_i.

It could be argued that, if we knew had at our disposition sufficient data on the dynamics of the intervention efforts, we could try to estimate the warfare efforts of the cartels. The dynamics of $\delta = \delta_1 + \delta_2$ could be then either taken as an external information to feed the system or as a reaction to the actions of the cartels. Unfortunately, a proper modeling of the dynamics of the intervention should state clearly the main drivers of the actions. For example, if the context of the model is a democracy, the public opinion could play an important role on the decisions made by the government about the efforts in intervention. Nevertheless, it is not clear what the correct approach should be about the incentives of the government and an attempt to include them in the model appears to have little improvement in the conclusions of the work, therefore in this article we assume that the dynamics of δ_i are of an exogenous nature.

4. Application to Mexico Data

According to the analysis made in previous sections, higher rates of military intervention as operations to control illegal drug trade can yield more violence from cartels due to a higher pressure to control the territory. This means that we would expect to see that the cities with more military intervention would also present higher rates of gunfighting violence. In this section we present the relation between these variables across cities of Mexico.

Mexico started in December 2006 a "war against drug related violence". One of the first states where the army was deployed was Michoacán, which observed a dramatic increase in homicide rates [4]. Further deployments of military interventions resulted in higher rates of homicides throughout the country [6]. Figure 1 show the dramatic increase in deaths by homicide recorded in the years after 2006, which suggest relation from military intervention strategies.

Figure 2 presents the results of the National Survey of Victimization and Perception of Public Security [5] made by Mexico's National Institute of Statistics and Geography (INEGI). In this survey, the subject is asked about their perception of security issues, including their knowledge about government efforts made to control illegal drug trade. The survey is made in metropolitan areas but in this work it is classified by municipalities. This yields a map in which the states with big territory and few municipalities (such as Baja California Sur in the western peninsula, with more than 75 thousand square kilometers of extension and just 5 municipalities) can show a somehow dramatic presence of military intervention compared to the states with more municipalities (like Oaxaca, with 93 thousand square kilometers of extension, but with 570 municipalities).

The municipalities that do not contain urban areas are therefore not included in the survey and their data is not shown. The choropleth in figure 2 shows in the whitest color the municipalities with either nonexistent military presence or those municipalities that were not reached by the survey. The colors in the map are progressively darker, in indication of a higher presence of military, as an answer to the question "Do you have knowledge about (fighting

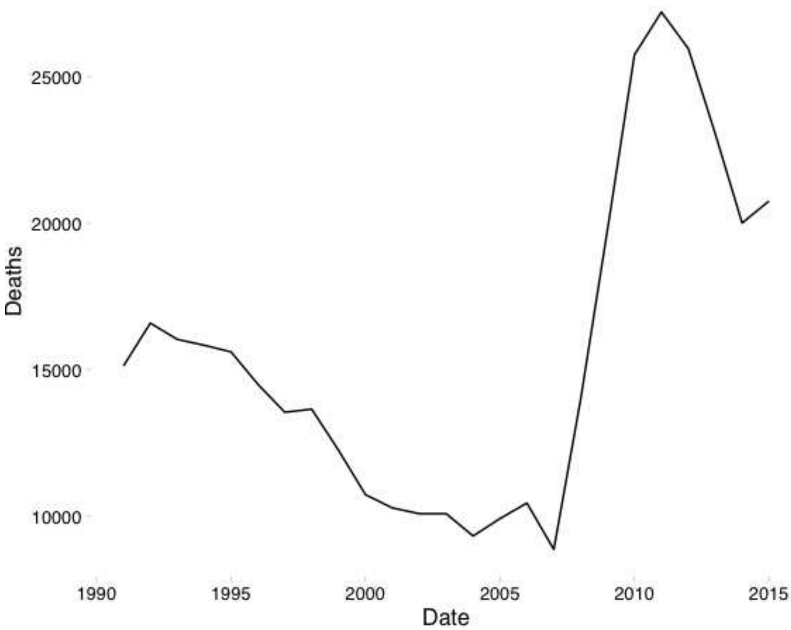

Figure 1: Deaths by homicide from 1990 to 2015. Source: INEGI.

illegal drug traffic) actions done in 2015 in your municipality?".

Naturally, the result of the surveys is not a direct measure of the intervention of programs to combat illegal drug traffic or its violence, but it is a more complete dataset and allows us to have a better picture of the concentration of the criminal activity in regions. For example, consider figure 3, that shows the perceived presence of illegal drug dealers in the municipality. It would be expected to see, under our model, higher rates of activity in the same zones as the military intervention.

The causal relation can be thought of, naturally, in both ways: the presence of military intervention yields more conspicuous competition activities by the criminal gangs, and the turfs can make them more noticeable to ordinary citizens (and therefore they are more likely to answer yes to the question "Do you have knowledge about these criminal activities in your municipality?"). Otherwise, it can be thought that the presence of drug dealing activities attract the attention of intervention agencies.

A simple logistic regression is made, in which we set $P(Y = 1|X)$ as the probability of perceiving the presence of drug dealing activity. Here, Y is a binary variable, which takes the value 1 if the answer to the question in the

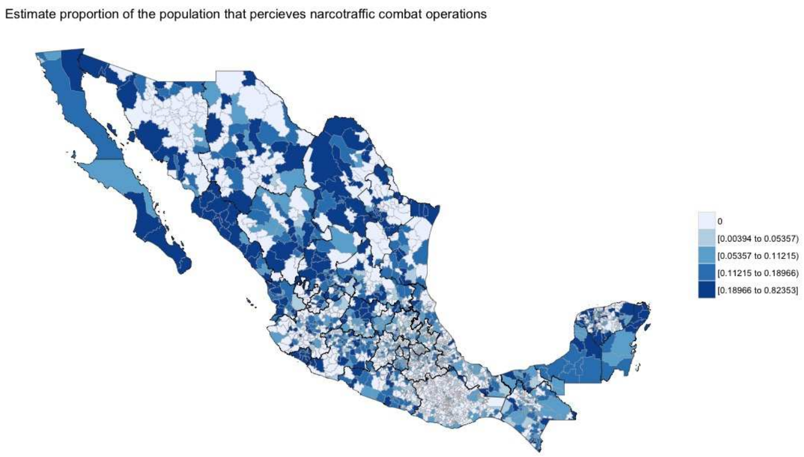

Figure 2: Estimated military intervention by municipality

survey is yes and 0 otherwise. The variable X is also a binary variable, which answers the question about the presence of military intervention in the municipality. In other words, X represents the answers in figure 2, while Y the ones in figure 3. We estimate the logistic regression model as

$$P(Y=1|X)=\frac{e^{\beta_0+\beta_1 X}}{1+e^{\beta_0+\beta_1 X}} \tag{6}$$

Where β_0 and β_1 are the regression parameters. Under this model, and given the survey data, we estimate that $\beta_0=-1.25$ and $\beta_1=0.1504$. That is, the presence of military intervention can increase up to 15% the probability of perception of drug dealing in the municipality, but there is a baseline level of military presence that could still mean a zero probability of drug dealing. These results show standard errors of 0.0223 and 0.0090, respectively and p-values close to zero.

Consider that X represents gunfighting activity in the municipality, then applying the model in equation (6) we are making a logistic regression between the data in figure 2 and the one shown in figure 4.

The model shows values of the intercept and the regression coefficient of

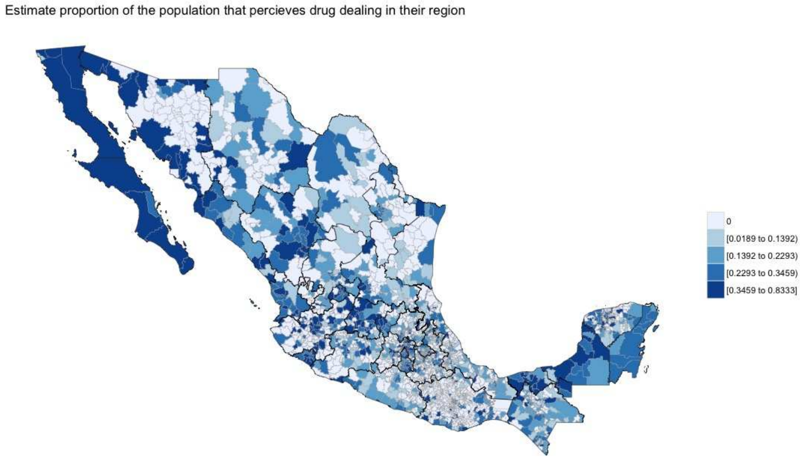

Figure 3: Perceived presence of drug dealing by municipality

-1.59 and .12 respectively, both with p-values close to zero and standard errors of 0.0244 and 0.0489, respectively. These results suggest a positive relation between the presence of military intervention and gunfight, or at least with the probability of answering yes to the question about perception of these activities in the municipality.

5. Final Remarks

We consider the use of a competition model to explain the increase in homicide rates with an intervention strategy. In this model, the agents in a differential game use violence to settle the turfs, in face of competition and military intervention.

The analysis in this article adds up to the available literature that supports the hypothesis that a military intervention strategy to limit the control of territory of drug cartels lead to more violence in the area. This hypothesis is consistent with the evidence, which is discussed in this article.

Further research can be done in the development of the game theoretical

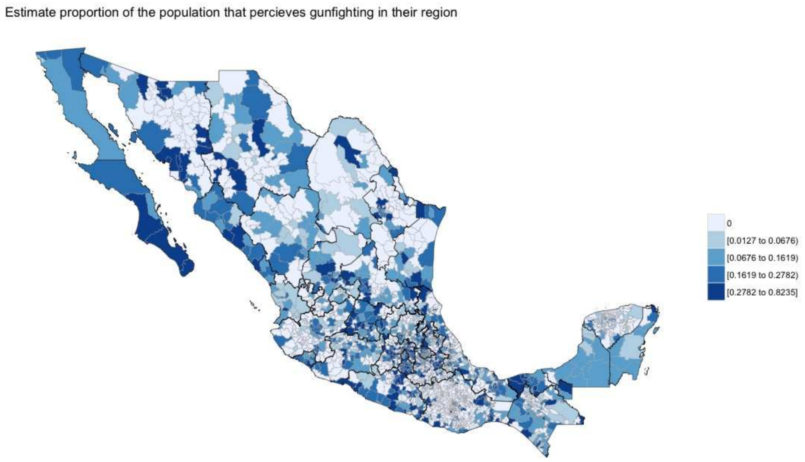

Figure 4: Perceived presence of gunfighting by municipality

model by setting the military agent as a player. It is not clear what the set of actions and the payment functions should be for this kind of player in this particular context, but ideally, a payment function should reflect at least indirectly the social welfare that implies minimizing the presence of criminal organizations and the warfare activity they are involved in. Another incentive that can be considered to drive a government to increase it's expenditure on public safety is economic growth (consider the analysis by [10]). Needless to say, the deterministic framework of this paper requires stochastic extensions considering: stochastic future improvements in technology for military intervention, stochastic homicide rate from military intervention, stochastic growth rate of criminal groups, etc. Regarding this issue, the stochastic optimal control approach can be more suitable by introducing mixed diffusion jump processes for modeling the dynamics of these variables (cf. [22], [20], and [21]).

An interesting observation of the results of this analysis is that the sum of warfare expenditures can be minimized by changing the context of the model. After all, it is essential for the present model that the control variable represents warfare expenditures instead of the marketing efforts from other dynamic competition models. This means that the game is in essence completely different in a context where the firms have legal ways of competing.

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