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APPROXIMATE GENERALIZED CUBIC MAPPINGS IN MODULAR SPACES

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Abstract: In this article, we investigate an alternative generalized Hyers–Ulam stability theorem of a modified cubic functional equation in a modular space X_ρ without using the Fatou property on the modular function ρ.

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Key Words: generalized Hyers–Ulam stability; modular spaces; Fatou property; convex modular.

1. Introduction

In 1940, S.M. Ulam [14] raised the question concerning the stability of group homomorphisms: Let G be a group and let G' be a metric group with the metric $d(\cdot,\ \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G \to G'$ satisfies the inequality

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $F : G \to G'$ with $d(f(x), F(x)) < \varepsilon$ for all $x \in G$? D.H. Hyers [8] has solved the problem of Ulam for the case of additive mappings in 1941. The result was generalized by T.Aoki [1] in 1950, by Th.M. Rassias [11] in 1978, by J.M. Rassias [10] in 1992, and by P. Găvruta [4] in 1994. Over the past few decades, many mathematicians have

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published on various generalized Hyers–Ulam stability functional equations [2, 3, 13].

Now, we recall some basic definitions and remarks of modular spaces with modular functions, which are primitive notions corresponding to norms or metrics, as in the followings [5, 6, 15].

Definition 1. Let χ be a linear space.

(a) A function $\rho : \chi \to [0, \infty]$ is called a modular if for arbitrary $x, y \in \chi$,

(1) $\rho(x) = 0$ if and only if $x = 0$,

(2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$,

(3) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for any scalars α, β, where $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,

(b) alternatively, if (3) is replaced by

(3)' $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for every scalars α, β, where $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$,

then we say that ρ is a convex modular.

We remark a modular ρ defines a corresponding modular space, i.e., the linear space χ_ρ given by

$$\chi_\rho = \{x \in \chi : \rho(\lambda x) \to 0 \quad \text{as} \quad \lambda \to 0\}.$$

Let ρ be a convex modular. Then, the modular space χ_ρ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho = \inf\{\lambda > 0 : \rho\Big(\frac{x}{\lambda}\Big) \leq 1\}.$$

Remark 2. (a) If ρ is a modular on χ, we note that $\rho(tx)$ is an increasing function in $t \geq 0$ for each fixed $x \in \chi$, that is, $\rho(ax) \leq \rho(bx)$ whenever $0 \leq a < b$. (b) If ρ is a convex modular on χ, then $\rho(\alpha x) \leq \alpha\rho(x)$ for all $x \in \chi$ and $0 \leq \alpha \leq 1$. Moreover, we see that $\rho(\alpha x) \leq |\alpha|\rho(x)$ for all $x \in \chi$ and $|\alpha| \leq 1$. (c) In general, we note that $\rho\big(\sum_{i=1}^n \alpha_i x_i\big) \leq \sum_{i=1}^n \alpha_i \rho(x_i)$ for all $x_i \in \chi$ and $\alpha_i \geq 0$ $(i = 1, \cdots, n)$ whenever $0 < \sum_{i=1}^n \alpha_i := \alpha \leq 1$ [6]. (d) Consequently, we lead to $\rho\big(\sum_{i=1}^n \alpha_i x_i\big) \leq \sum_{i=1}^n |\alpha_i| \rho(x_i)$ for all $x_i \in \chi$ and $0 < \sum_{i=1}^n |\alpha_i| := \alpha \leq 1$.

Definition 3. Let χ_ρ be a modular space and let $\{x_n\}$ be a sequence in χ_ρ. Then,

(1) $\{x_n\}$ is ρ-convergent to $x \in \chi_\rho$ and write $x_n \stackrel{\rho}{\to} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(2) $\{x_n\}$ is called ρ-Cauchy in χ_ρ if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(3) A subset K of χ_ρ is called ρ-complete if and only if any ρ-Cauchy sequence is ρ-convergent to an element in K.

They say that the modular ρ has the Fatou property if and only if $\rho(x) \leq \liminf_{n\to\infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ-convergent to x. A modular function ρ is said to satisfy the Δ_2-condition if there exists $\kappa > 0$ such that $\rho(2x) \leq \kappa\rho(x)$ for all $x \in \chi_\rho$.

In 2014, G. Sadeghi [12] has established generalized Hyers–Ulam stability via the fixed point method of a generalized Jensen functional equation $f(rx + sy) = rg(x) + sh(y)$ in convex modular spaces with the Fatou property satisfying the Δ_2-condition with $0 < \kappa \leq 2$. In [15], the authors have presented the generalized Hyers–Ulam stability of quadratic functional equations via the extensive studies of fixed point theory in the framework of modular spaces whose modulars are convex, lower semicontinuous but do not satisfy any relatives of Δ_2-conditions.

In 2007, Najati [9] investigated the following cubic functional equation

$$f(ax + y) + f(ax - y) = af(x + y) + af(x - y) + 2(a^3 - a)f(x) \quad (1)$$

for a fixed positive integer $a \geq 2$. In 2014, H. Koh and D. Kang [7] investigated following functional equation

$$\begin{aligned} &f(ax + by) + f(ax - by) \\ &\quad = ab^2[f(x + y) + f(x - y)] + 2a(a^2 - b^2)f(x), \end{aligned} \quad (2)$$

for all $x, y \in X$ and fixed $a, b \in \mathbb{Z}(a \geq 2,\ b \geq 1,\ a \neq b)$.

In this article, we first present generalized Hyers–Ulam stability via direct method of the equation (2) in modular spaces without using the Fatou property and $\rho(ax) \leq \kappa\rho(x)$, and then alternatively study generalized Hyers–Ulam stability via direct method of the equation (2) in modular spaces using necessarily $\rho(ax) \leq \kappa\rho(x)$ without the Fatou property.

2. Generalized Hyers–Ulam Stability of Eq. (2).

For notational convenience, we let the abbreviation Df as follows:

$$Df(x, y) \ := \ f(ax + by) + f(ax - by) - ab^2[f(x + y) + f(x - y)]$$

$$-2a(a^2 - b^2)f(x)$$

for all x, y in a linear space X. In the following, we present a generalized Hyers–Ulam stability via direct method of the equation (2) in modular spaces without using the Fatou property as in the following.

Theorem 4. *Let X be a linear space and χ_ρ a ρ-complete convex modular space. If a mapping $f : X \to \chi_\rho$ satisfies*

$$\rho(Df(x, y)) \leq \phi(x, y) \tag{3}$$

and $\phi : X^2 \to [0, \infty)$ is a mapping such that

$$\Phi(x, y) := \sum_{j=0}^{\infty} \frac{\phi(a^j x, a^j y)}{a^{3(j+1)}} < \infty \tag{4}$$

for all $x, y \in X$, then there exists a unique cubic mapping $F_1 : X \to \chi_\rho$ which satisfies the equation (2) and

$$\rho(f(x) - F_1(x)) \leq \frac{1}{2}\Phi(x, 0) \tag{5}$$

for all $x \in X$.

Proof. Putting $y = 0$ in (3), we obtain

$$\rho(Df(x, 0)) = \rho(2f(ax) - 2a^3 f(x)) \leq \phi(x, 0), \tag{6}$$

and so

$$\rho(f(x) - \frac{f(ax)}{a^3}) \leq \frac{1}{2a^3}\rho(2f(ax) - 2a^3 f(x)) \leq \frac{1}{2a^3}\phi(x, 0)$$

for all $x \in X$. Since $\sum_{j=0}^{n-1} \frac{1}{2a^{3(j+1)}} \leq 1$, we prove the following functional inequality

$$\begin{aligned} \rho(f(x) - \frac{f(a^n x)}{a^{3n}}) &= \rho\Big[\sum_{j=0}^{n-1} (\frac{f(a^j x)}{a^{3j}} - \frac{f(a^{j+1} x)}{a^{3(j+1)}})\Big] \\ &= \rho\Big[\sum_{j=0}^{n-1} \frac{1}{2a^{3(j+1)}}\big(2a^3 f(a^j x) - 2f(a^{j+1} x)\big)\Big] \\ &\leq \sum_{j=0}^{n-1} \frac{1}{2a^{3(j+1)}}\rho\Big(\big(2a^3 f(a^j x) - 2f(a^{j+1} x)\big)\Big) \end{aligned} \tag{7}$$

$$= \sum_{j=0}^{n-1} \frac{1}{2a^{3(j+1)}} \rho\Big(Df(a^j x, 0)\Big)$$
$$\leq \frac{1}{2} \sum_{j=0}^{n-1} \frac{\phi(a^j x, 0)}{a^{3(j+1)}}$$

for all $x \in X$.

Now, replacing x by $a^m x$ in (7), we have

$$\rho\Big(\frac{f(a^m x)}{a^{3m}} - \frac{f(a^{m+n} x)}{a^{3(m+n)}}\Big) \leq \frac{1}{2} \sum_{j=m}^{m+n-1} \frac{\phi(a^j x, 0)}{a^{3(j+1)}} \tag{8}$$

which converges to zero as $m \to \infty$ by the assumption (4). Thus the above inequality implies that the sequence $\{\frac{f(a^n x)}{a^{3n}}\}$ is ρ-Cauchy for all $x \in X$ and so it is convergent in χ_ρ since the space χ_ρ is ρ-complete.

Thus, we may define a mapping $F_1 : X \to \chi_\rho$ as

$$F_1(x) := \rho - \lim_{n\to\infty} \frac{f(a^n x)}{a^{3n}} \Longleftrightarrow \lim_{n\to\infty} \rho\Big(\frac{f(a^n x)}{a^{3n}} - F_1(x)\Big) = 0,$$

for all $x \in X$.

Now, we claim the mapping F_1 satisfies the equation (2). If we set $(x, y) := (a^n x, a^n y)$ in (3), and then divide the resulting inequality by a^{3n}, we get

$$\rho\Big(\frac{Df(a^n x, a^n y)}{a^{3n}}\Big) \leq \frac{\rho(Df(a^n x, a^n y))}{a^{3n}} \leq \frac{\phi(a^n x, a^n y)}{a^{3n}}$$

for all $x, y \in X$. Thus, for any fixed $L \geq 2a^3 + 4ab^2 + 3$ subject to $\frac{2ab^2}{L} + \frac{|2a(a^2-b^2)|}{L} + \frac{3}{L} \leq 1$, we lead to

$$\begin{aligned}
&\rho\Big(\frac{1}{L} DF_1(x, y)\Big) \\
&= \rho\Big(\frac{1}{L} DF_1(x, y) - \frac{Df(a^n x, a^n y)}{L \cdot a^{3n}} + \frac{Df(a^n x, a^n y)}{L \cdot a^{3n}}\Big) \\
&\leq \frac{1}{L}\rho\Big(F_1\big(ax + by\big) - \frac{f\big(a^n(ax + by)\big)}{a^{3n}}\Big) + \frac{1}{L}\rho\Big(F_1\big(ax - by\big) - \frac{f\big(a^n(ax - by)\big)}{a^{3n}}\Big) \\
&+ \frac{ab^2}{L}\rho\Big(F_1\big(x + y\big) - \frac{f\big(a^n(x + y)\big)}{a^{3n}}\Big) + \frac{ab^2}{L}\rho\Big(F_1\big(x - y\big) - \frac{f\big(a^n(x - y)\big)}{a^{3n}}\Big) \\
&+ \frac{|2a(a^2 - b^2)|}{L}\rho\Big(F_1(x) - \frac{f(a^n x)}{a^{3n}}\Big) + \frac{1}{L}\rho\Big(\frac{Df\big(a^n x, a^n y\big)}{a^{3n}}\Big)
\end{aligned}$$

for all $x, y \in X$ and all positive integers n by Remark 2. Taking the limit as $n \to \infty$, one obtains $\rho(\frac{1}{L}DF_1(x,y)) = 0$, and so $DF_1(x,y) = 0$ for all $x, y \in X$. Hence F_1 satisfies the equation (2) and so it is cubic.

On the other hand, since $\sum_{i=0}^{n} \frac{1}{2a^{3(i+1)}} + \frac{1}{a^3} \le 1$ for all $n \in \mathbb{N}$, it follows from (6) and Remark 2 that

$$\begin{aligned}
&\rho(f(x) - F_1(x)) \\
&= \rho\left(\sum_{i=0}^{n} \frac{1}{2a^{3(i+1)}}\Big(2a^3 f(a^i x) - 2f(a^{i+1}x)\Big) + \frac{f(a^{n+1}x)}{a^{3(n+1)}} - \frac{F_1(ax)}{a^3}\right) \\
&\le \sum_{i=0}^{n} \frac{1}{2a^{3(i+1)}}\rho\Big(Df(a^i x, 0)\Big) + \frac{1}{a^3}\rho\Big(\frac{f(a^{n+1}x)}{a^{3n}} - F_1(ax)\Big) \\
&\le \frac{1}{2}\sum_{i=0}^{n} \frac{1}{a^{3(i+1)}}\phi(a^i x, 0) + \frac{1}{a^3}\rho\Big(\frac{f(a^{n+1}x)}{a^{3n}} - F_1(ax)\Big),
\end{aligned}$$

without applying the Fatou property of the modular ρ for all $x \in X$ and all $n \in \mathbb{N}$, from which we obtain the approximation of f by the cubic mapping F_1 as follows

$$\rho(f(x) - F_1(x)) \le \sum_{i=0}^{\infty} \frac{1}{2a^{3(i+1)}}\phi(a^i x, 0) = \frac{1}{2}\Phi(x, 0)$$

for all $x \in X$ by taking $n \to \infty$ in the last inequality.

To show the uniqueness of F_1, we assume that there exists a cubic mapping $G_1 : X \to \chi_\rho$ which satisfies the inequality

$$\rho(f(x) - G_1(x)) \le \sum_{j=0}^{\infty} \frac{\phi(a^j x, 0)}{2a^{3(j+1)}} = \frac{1}{2}\Phi(x, 0)$$

for all $x \in X$, but suppose $F_1(x_0) \neq G_1(x_0)$ for some $x_0 \in X$. Then there exists a positive constant $\varepsilon > 0$ such that $\varepsilon < \rho(F_1(x_0) - G_1(x_0))$. For such given $\varepsilon > 0$, it follows from (4) that there is a positive integer $n_0 \in \mathbb{N}$ such that $\sum_{j=n_0}^{\infty} \frac{\phi(a^j x_0, 0)}{a^{3(j+1)}} < \varepsilon$. Since F_1 and G_1 are cubic mappings, we see from the equality $F_1(a^{n_0}x_0) = a^{3n_0}F_1(x_0)$ and $G_1(a^{n_0}x_0) = a^{3n_0}G_1(x_0)$ that

$$\begin{aligned}
\varepsilon &< \rho(F_1(x_0) - G_1(x_0)) \\
&= \rho\Big(\frac{F_1(a^{n_0}x_0) - f(a^{n_0}x_0)}{a^{3n_0}} + \frac{f(a^{n_0}x_0) - G_1(a^{n_0}x_0)}{a^{3n_0}}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{a^{3n_0}}\rho\Big(F_1(a^{n_0}x_0)-f(a^{n_0}x_0)\Big)+\frac{1}{a^{3n_0}}\rho\Big(f(a^{n_0}x_0)-G_1(a^{n_0}x_0)\Big)\\
&\le \frac{1}{a^{3n_0}}\sum_{j=0}^{\infty}\frac{\phi(a^{j+n_0}x_0,0)}{a^{3(j+1)}}\\
&= \sum_{j=n_0}^{\infty}\frac{\phi(a^{j}x_0,0)}{a^{3(j+1)}}<\varepsilon,
\end{aligned}
$$

which leads a contradiction. Hence the mapping F_1 is a unique cubic mapping near f satisfying the approximation (5) in the modular space χ_ρ. □

As corollary of Theorem 4, we obtain the following stability result of the equation (2), which generalizes stability result in normed spaces.

Corollary 5. *Suppose X is a normed space and χ_ρ is a ρ-complete convex modular space. For given real numbers $\varepsilon_i > 0$ and $0 \le r_i < 3$ $(i = 1, 2)$. Suppose that a mapping $f : X \to \chi_\rho$ satisfies*

$$\rho(Df(x,y)) \le \varepsilon_1\|x\|^{r_1} + \varepsilon_2\|y\|^{r_2}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $F_1 : X \to \chi_\rho$ such that

$$\rho(f(x) - F_1(x)) \le \frac{\varepsilon_1\|x\|^{r_1}}{2(a^3 - a^{r_1})}$$

for all $x \in X$.

We observe that if the modular ρ satisfies the Δ_2-condition, then $\kappa \ge 1$ for nontrivial modular ρ, and that $\kappa \ge 2$ for nontrivial convex modular ρ. See references [5, 6, 12, 15]. Similarly, we apply the assumption $\rho(ax) \le \kappa\rho(x)$ for all $x \in \chi_\rho$ to the following theorem to prove an alternative stability result of functional equation (2), where we note $\kappa \ge a$ for nontrivial convex modular ρ.

Theorem 6. *Suppose X is a linear space and χ_ρ is a ρ-complete convex modular space with $\rho(ax) \le \kappa\rho(x)$ for all $x \in \chi_\rho$. If there exists a function $\varphi : X^2 \to [0, \infty)$ for which a mapping $f : X \to \chi_\rho$ satisfies*

$$
\begin{aligned}
\rho(Df(x,y)) \quad &\le \quad \varphi(x,y), \quad \text{and} && (9)\\
\sum_{j=1}^{\infty}\frac{\kappa^{4j-3}}{a^j}\varphi\Big(\frac{x}{a^j},\frac{y}{a^j}\Big) \quad &:= \quad \Psi(x,y) < \infty && (10)
\end{aligned}
$$

for all $x, y \in X$, then there exists a unique cubic mapping $F_2 : X \to \chi_\rho$ satisfies the equation (2) and

$$\rho(f(x) - F_2(x)) \leq \frac{\kappa}{2a}\Psi(x, 0) \tag{11}$$

for all $x \in X$.

Proof. It follows from (6) that

$$\rho(f(x) - a^3 f(\frac{x}{a})) \leq \frac{1}{2}\varphi(\frac{x}{a}, 0)$$

for all $x \in X$. Thus, we obtain inequality by the convexity of the modular ρ

$$\begin{aligned}\rho(f(x) - a^6 f(\frac{x}{a^2})) &\leq \frac{1}{a}\rho\Big(af(x) - a^4 f(\frac{x}{a})\Big) + \frac{1}{a^2}\rho\Big(a^5 f(\frac{x}{a}) - a^8 f(\frac{x}{a^2})\Big) \\ &\leq \frac{\kappa}{a} \cdot \frac{1}{2}\varphi\Big(\frac{x}{a}, 0\Big) + \frac{\kappa^5}{a^2} \cdot \frac{1}{2}\varphi\Big(\frac{x}{a^2}, 0\Big)\end{aligned}$$

for all $x \in X$. Then using the repeating process for any $n \geq 2$, we prove the following functional inequality

$$\rho(f(x) - a^{3n} f(\frac{x}{a^n})) \leq \frac{1}{2}\sum_{j=1}^{n} \frac{\kappa^{4j-3}}{a^j}\varphi(\frac{x}{a^j}, 0) \tag{12}$$

for all $x \in X$. In fact, it is true for $j = 2$. Assume that the inequality (12) holds true for n. Thus, using the convexity of the modular ρ, we deduce

$$\begin{aligned}&\rho(f(x) - a^{3(n+1)} f(\frac{x}{a^{n+1}})) \\ &= \rho\Big(\frac{1}{a}\big(af(x) - a \cdot a^3 f(\frac{x}{a})\big) + \frac{1}{a}\big(a \cdot a^3 f(\frac{x}{a}) - a \cdot a^{3(n+1)} f(\frac{x}{a^{n+1}})\big)\Big) \\ &\leq \frac{\kappa}{a}\rho\Big(f(x) - a^3 f(\frac{x}{a})\Big) + \frac{\kappa^4}{a}\rho\Big(f(\frac{x}{a}) - a^{3n} f(\frac{x}{a^{n+1}})\Big) \\ &\leq \frac{\kappa}{a} \cdot \frac{1}{2}\varphi\Big(\frac{x}{a}, 0\Big) + \frac{\kappa^4}{a} \cdot \frac{1}{2}\sum_{j=1}^{n} \frac{\kappa^{4j-3}}{a^j}\varphi\Big(\frac{x}{a^{j+1}}, 0\Big) \\ &= \frac{\kappa}{2a}\varphi\Big(\frac{x}{a}, 0\Big) + \frac{1}{2}\sum_{j=1}^{n} \frac{\kappa^{4j+1}}{a^{j+1}}\varphi\Big(\frac{x}{a^{j+1}}, 0\Big) \\ &= \frac{1}{2}\sum_{j=1}^{n+1} \frac{\kappa^{4j-3}}{a^j}\varphi\Big(\frac{x}{a^j}, 0\Big),\end{aligned}$$

which proves (12) for $n+1$. Now, replacing x by $a^{-m}x$ in (12), we have

$$\begin{aligned}
\rho\Big(a^{3m}f(\frac{x}{a^m}) - a^{3(m+n)}f(\frac{x}{a^{m+n}})\Big) &\le \kappa^{3m}\rho\Big(f(\frac{x}{a^m}) - a^{3n}f(\frac{x}{a^{m+n}})\Big) \\
&\le \kappa^{3m}\cdot\frac{1}{2}\sum_{j=1}^{n}\frac{\kappa^{4j-3}}{a^j}\varphi\Big(\frac{x}{a^{j+m}},0\Big) \\
&\le \kappa^{3m}\cdot\frac{1}{2}\sum_{j=1}^{n}\frac{\kappa^{4j-3}}{a^j}\varphi\Big(\frac{x}{a^{j+m}},0\Big)\cdot\frac{\kappa^m}{a^m} \\
&= \frac{1}{2}\sum_{j=1}^{n}\frac{\kappa^{4(j+m)-3}}{a^{j+m}}\varphi\Big(\frac{x}{a^{j+m}},0\Big) \\
&= \frac{1}{2}\sum_{j=m+1}^{m+n}\frac{\kappa^{4j-3}}{a^j}\varphi\Big(\frac{x}{a^j},0\Big),
\end{aligned}$$

which converges to zero as $m\to\infty$ by the assumption (10). Thus, the sequence $\{a^{3n}f(\frac{x}{a^n})\}$ is ρ-Cauchy for all $x\in X$ and so it is ρ-convergent in χ_ρ since the space χ_ρ is ρ-complete.

Thus, we may define a mapping $F_2 : X \to \chi_\rho$ as

$$F_2(x) := \rho - \lim_{n\to\infty} a^{3n}f(\frac{x}{a^n}) \iff \lim_{n\to\infty}\rho\Big(a^{3n}f(\frac{x}{a^n}) - F_2(x)\Big) = 0,$$

for all $x\in X$. Then, by the $\rho(ax)\le\kappa\rho(x)$ without using the Fatou property, we can see the following inequality

$$\begin{aligned}
\rho(f(x)-F_2(x)) &\le \frac{1}{a}\rho\Big(af(x) - a\cdot a^{3n}f(\frac{x}{a^n}) + a\cdot a^{3n}f(\frac{x}{a^n}) - aF_2(x)\Big) \\
&\le \frac{\kappa}{a}\rho\Big(f(x) - a^{3n}f(\frac{x}{a^n})\Big) + \frac{\kappa}{a}\rho\Big(a^{3n}f(\frac{x}{a^n}) - F_2(x)\Big) \\
&\le \frac{\kappa}{a}\cdot\frac{1}{2}\sum_{j=1}^{n}\frac{\kappa^{4j-3}}{a^j}\varphi(\frac{x}{a^j},0) + \frac{\kappa}{a}\rho\Big(a^{3n}f(\frac{x}{a^n}) - F_2(x)\Big) \\
&\le \frac{1}{2}\sum_{j=1}^{\infty}\frac{\kappa^{4j-2}}{a^{j+1}}\varphi(\frac{x}{a^j},0) \\
&= \frac{\kappa}{2a}\Psi(x,0)
\end{aligned}$$

by taking $n\to\infty$, which yields the approximation (11).

Now, we claim the mapping F_2 satisfies the equation (2). Setting $(x,y) := (a^{-n}x, a^{-n}y)$ in (9), and multiplying the resulting inequality by a^{3n}, we get

$$\begin{aligned}\rho(a^{3n}Df(a^{-n}x,a^{-n}y)) &\le \kappa^{3n}\varphi(a^{-n}x,a^{-n}y)\\ &\le \kappa^{3n}\varphi(a^{-n}x,a^{-n}y)\cdot\frac{\kappa^n}{a^n}\\ &= \frac{\kappa^{4n}}{a^n}\varphi(a^{-n}x,a^{-n}y),\end{aligned}$$

which tends to zero as $n\to\infty$ for all $x,y\in X$. Thus, it follows from Remark 2 that for any fixed $L\ge 2a^3+4ab^2+3$,

$$\begin{aligned}&\rho(\frac{1}{L}DF_2(x,y))\\ &=\rho\Big(\frac{1}{L}DF_2(x,y)-\frac{1}{L}a^{3n}Df(\frac{x}{a^n},\frac{y}{a^n})+\frac{1}{L}a^{3n}Df(\frac{x}{a^n},\frac{y}{a^n})\Big)\\ &\le\frac{1}{L}\rho\Big(F_2(ax+by)-a^{3n}f(\frac{ax+by}{a^n})\Big)+\frac{1}{L}\rho\Big(F_2(ax-by)-a^{3n}f(\frac{ax-by}{a^n})\Big)\\ &\quad+\frac{ab^2}{L}\rho\Big(F_2(x+y)-a^{3n}f(\frac{x+y}{a^n})\Big)+\frac{ab^2}{L}\rho\Big(F_2(x-y)-a^{3n}f(\frac{x-y}{a^n})\Big)\\ &\quad+\frac{|2a(a^2-b^2)|}{L}\rho\Big(F_2(x)-a^{3n}f(\frac{x}{a^n})\Big)+\frac{1}{L}\rho(a^{3n}Df(\frac{x}{a^n},\frac{y}{a^n})\Big)\end{aligned}$$

for all $x,y\in X$ and all positive integers n. Taking the limit as $n\to\infty$, one obtains $\rho(\frac{1}{L}DF_2(x,y))=0$, and thus $DF_2(x,y)=0$ for all $x,y\in X$. Hence F_2 satisfies the equation (2), and so it is cubic.

To show the uniqueness of F_2, we assume that there exists a cubic mapping $G_2 : X\to\chi_\rho$ which satisfies the inequality

$$\rho(f(x)-G_2(x)) \le \frac{\kappa}{2a}\sum_{j=1}^{\infty}\frac{\kappa^{4j-3}}{a^j}\varphi(\frac{x}{a^j},0)$$

for all $x\in X$. Since F_2 and G_2 are cubic mappings, we see from the equality $a^{3n}F_2(a^{-n}x)=F_2(x)$ and $a^{3n}G_2(a^{-n}x)=G_2(x)$ that

$$\begin{aligned}\rho(G_2(x)-F_2(x)) =&\rho\Big(\frac{a^{3(n+1)}}{a^3}\big(G_2(\frac{x}{a^n})-f(\frac{x}{a^n})\big)+\frac{a^{3(n+1)}}{a^3}\big(f(\frac{x}{a^n})-F_2(\frac{x}{a^n})\big)\Big)\\ \le&\frac{\kappa^{3(n+1)}}{a^3}\rho\Big(G_2(\frac{x}{a^n})-f(\frac{x}{a^n})\Big)+\frac{\kappa^{3(n+1)}}{a^3}\rho\Big(f(\frac{x}{a^n})-F_2(\frac{x}{a^n})\Big)\\ \le&\frac{\kappa}{a}\cdot\frac{\kappa^{3(n+1)}}{a^3}\sum_{j=1}^{\infty}\frac{\kappa^{4j-3}}{a^j}\varphi(\frac{x}{a^{j+n}},0)\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\kappa}{a} \cdot \frac{\kappa^3}{a^3} \sum_{j=1}^{\infty} \frac{\kappa^{4(j+n)-3}}{a^{j+n}} \varphi(\frac{x}{a^{j+n}}, 0) \\
&= \frac{\kappa^4}{a^4} \sum_{j=n+1}^{\infty} \frac{\kappa^{4j-3}}{a^j} \varphi(\frac{x}{a^j}, 0)
\end{aligned}
$$

which tends to zero as $n \to \infty$ for all $x \in X$. Hence the mapping F_2 is a unique cubic mapping satisfying (11). ☐

Remark 7. In Theorem 6 if χ_ρ is a Banach space with norm ρ, and so $\rho(ax) = a\rho(x)$, $\kappa := a$, then we see from (9) and (10) that there exists a unique cubic mapping $F_2 : X \to \chi_\rho$, defined as $F_2(x) = \lim_{n\to\infty} a^{3n} f(\frac{x}{a^n})$, $x \in X$, which satisfies the equation (2) and

$$
\rho(f(x) - F_2(x)) \le \frac{1}{2} \sum_{j=1}^{\infty} a^{3j} \varphi(\frac{x}{a^j}, 0)
$$

for all $x \in X$, which is an implicit alternative stability result in the paper [7, 9].

As corollary of Theorem 6, we obtain the following stability result of the equation (2), which generalize stability result in normed spaces.

Corollary 8. *Suppose X is a normed space and χ_ρ is a ρ-complete convex modular space with $\rho(ax) \le \kappa\rho(x)$. For given real numbers $\varepsilon_i > 0$ and $\kappa^4 < a^{1+r_i}$ $(i = 1, 2)$, if a mapping $f : X \to \chi_\rho$ satisfies*

$$
\rho(Df(x, y)) \le \varepsilon_1 \|x\|^{r_1} + \varepsilon_2 \|y\|^{r_2}
$$

for all $x, y \in X$, then there exists a unique cubic mapping $F_2 : X \to \chi_\rho$ such that

$$
\rho(f(x) - F_2(x)) \le \frac{\kappa^2 \varepsilon_1 \|x\|^{r_1}}{2a(a^{1+r_1} - \kappa^4)}
$$

for all $x \in X$.

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