MONOPHONIC WIRELENGTH OF CIRCULANT NETWORKS INTO WHEELS AND FANS

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Abstract: This paper presents the monophonic wirelength of circulant graph into wheels and fans. Also we present a monophonic algorithm to find the monophonic wirelength of family of circulant graphs into wheels and fans. Our monophonic algorithm produces the monophonic wirelength and cover a wide range of interconnection networks.

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1. Introduction

By a graph $\Gamma = (V, E)$, we mean a connected, finite, undirected graph with neither loops nor multiple edges. For notations and terminology, refer [4]. The distance $d(x, y)$ between two vertices $x$ and $y$ in a graph $G$ is the length of the shortest path from $x$ to $y$ in $G$. An edge $x_i x_j$ is a chord of a path $x_0, x_1, x_2, \ldots, x_n$ if $j \geq i + 2$. A monophonic path is a path if it contains no chord. The length...
of the longest $x - y$ monophonic path of a graph $G$ is called the monophonic distance $d_m(x, y)$ for every $x, y$ vertices in $G$. A monophonic path from $x$ to $y$ with length $d_m(x, y)$ is called an $x - y$ monophonic. For this refer [14]. Consider a graph $H$, since other graphs or networks are embedded into it, as host graph and graphs or networks which are embedded in $H$ are called guest graph. The embedding $f$ of $G$ to $H$ is a bijective mapping from the vertex set of $G$ to the vertex set of $H$ and every edge $(x, y) \in E(G)$ is mapped to a path between $f(x)$ and $f(y)$ in $H$ For this refer [8, 11]. If we find an embedding of $G$ into $H$ which produces the minimum wirelength $WL(G, H)$, such problem is called the wirelength problem. The wirelength of an embedding $f$ of $G$ into $H$ is given as

$$WL_f(G, H) = \sum_{(x, y) \in E(G)} d_H(f(x), f(y)) = \sum_{e \in E(G)} EC_f(G, H(e)), \,$$

where $EC_f(G, H(e))$ is the maximum number of edges of $G$ that are embedded on $e$ known as the edge congestion of $f$ of $G$ into $H$.

2. Preliminaries

We use definitions, lemmas and theorems from [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] for this work.

**Definition 1.** [1] A graph denoted by $W_n$ of order $n$ is called a wheel graph if it has an outer cycle of $n-1$ vertices and these $n-1$ vertices are connected to a single vertex known as hub. Wheel graphs have an unique role in interconnection network designs and circuit layout.

**Definition 2.** [1] A graph denoted by $F_n$ of order $n$ is called a fan graph if it has an outer path of $n-1$ vertices and these $n-1$ vertices are connected to a single vertex known as the core.

**Lemma 3.** (see [11], Congestion Lemma) Let $G$ be an $k$-regular graph and let $f : G \rightarrow H$ be an embedding. Let the graph $H - E$ has the components $H_i, i=1,2$ and $G_i = f^{-1}(H_i)$ then the edge cut $E$ of $H$ has the following properties:

1. The path $P_f(f(x), f(y))$ has no edges in $E$ for every edge $(x, y) \in G_i, i = 1, 2$.

2. The path $P_f(f(x), f(y))$ has exactly one edge in $E$ for every edge $(x, y)$ in $G$ with $x \in G_1$ and $y \in G_2$. 

3. $G_1$ is a maximum subgraph of $G$.

Then $EC_f(E)$ is minimum and $EC_f(E) = k|V(G_1)| - 2|E(G_1)|$.

**Lemma 4.** (see [11], Partition Lemma) Let $f : G \rightarrow H$ be an embedding. Let $\{E_1, E_2, \cdots, E_p\}$ be a partition of $E(H)$ such that each $E_i$ is an edge cut of $H$. Then $WL_f(G, H) = \sum_{i=1}^{p} EC_f(E_i)$.

**Lemma 5.** (see [1], $k$-partition Lemma) Let $f$ be an embedding of $G$ into $H$. Let $\{E_1, E_2, \cdots, E_p\}$ be a partition of $k|E(H)|$ such that each $E_i$ is an edge cut of $H$. Then $WL_f(G, H) = \frac{1}{k} \sum_{i=1}^{p} EC_f(E_i)$.

**Maximum Subgraph Problem** (see [7]) The problem of finding a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given $m$, if

$$I_G(m) = \max_{A \subseteq V, |A| = m} |I_G(A)|,$$

where $I_G(A) = \{(u, v) \in E : u, v \in A\}$, then the problem is to find $A \in V$ such that $|A| = m$ and $I_G(m) = |I_G(A)|$.

**Definition 6.** [7] A connected undirected graph represented by $G(m, \pm S)$ where $S \subseteq \{1, 2, 3, \cdots, [m/2]\}, m \geq 3$ is said to be a circulant graph if it consists of the vertex set $V = \{0, 1, 2, \cdots, m-1\}$ and the edge set $E = \{(x, y) : |x-y| \equiv s \text{ (mod } m\text{)}, s \in S\}$.

**Theorem 7.** [7] The number of edges in a maximum subgraph on $k$ vertices of $G(n, \pm S)$ where $S \subseteq \{1, 2, 3, \cdots, j\}, 1 \leq j \leq [n/2], n \geq 3$ is given by

$$\zeta = \begin{cases} k(k-1)/2, & k \leq j+1, \\ kj - j(j+1), & j+1 < k \leq n-j, \\ \frac{1}{2}((n-k)^2 + (4j+1)k - (2j+1)n), & n-j < k < n. \end{cases}$$

**Theorem 8.** [7] A set of $k$ consecutive vertices of $G(n, \pm 1); 1 \leq k \leq n$ induces a maximum subgraph of $G(n, \pm S)$ where $S = \{1, 2, 3, \cdots, j\}, 1 \leq j \leq [n/2], n \geq 3$.

**Theorem 9.** [8] The maximum subgraph on the set of all $k$ vertices of $G(n, \{1, 2, \cdots, j\})$ for $k < j$ is complete graph on $k$ vertices.
3. Monophonic Wirelength Problem

**Definition 10.** Let $G(V, E)$ and $H(V, E)$ be finite graphs with $n$ vertices. An embedding $f_m : G \rightarrow H$ is called a monophonic embedding if $f_m$ maps each vertex of $G$ into a vertex of $H$ and each edge $(x, y)$ of $G$ is mapped to a monophonic path between $f_m(x)$ and $f_m(y)$ in $H$.

**Definition 11.** Let $f_m : G \rightarrow H$ be a monophonic embedding. The monophonic edge congestion of $f_m$ of $G$ into $H$ is the maximum number of edges of the graph $G$ that are embedded on an edge $e \in H$ and is given by $\text{MEC}_{f_m}(G, H) = \max_{e \in E} \text{MEC}_{f_m}(G, H(e))$.

The monophonic wirelength problem of a graph $G$ into $H$ is the problem of finding a monophonic embedding $f_m : G \rightarrow H$ that produces the monophonic wire length $\text{MWL}(G, H)$.

**Definition 12.** Let $f_m : G \rightarrow H$ be a monophonic embedding. The monophonic wirelength $\text{MWL}(G, H)$ of $f_m$ is given as

$$\text{MWL}_{f_m}(G, H) = \sum_{(x, y) \in E(G)} d_m(f_m(x), f_m(y)).$$

**Lemma 13.** (Monophonic Congestion Lemma) Let $G$ be a $k$-regular graph with $n$ vertices. Let $H$ be a finite graph with $n$ vertices. Let $f_m : G \rightarrow H$ be a monophonic embedding of $G$ into $H$. Let the graph $H - E_j, j = 1, 2, \cdots, p$ have the components $H_i, i = 1, 2$ and $G_i = f_m^{-1}(H_i)$, where $E_j$'s are the edge cuts of $H$, form a partition in $H$ and have the following properties:

1. For $m \geq 0$, there are $m$ edges $(x, y) \in G_i, i = 1, 2$, such that the monophonic path $P_{f_m}(f_m(x), f_m(y))$ has exactly two edges in $E_j$.

2. The monophonic path $P_{f_m}(f_m(x), f_m(y))$ has exactly one edge in $E_j$ for every $(x, y) \in G$ with $x \in G_1$ and $y \in G_2$.

where $G_1$ is a maximum subgraph of $G$. Then $\text{MEC}_{f_m}(E_j)$ is monophonic and the monophonic wirelength of $f_m$ of $G$ into $H$ is given by $\text{MWL}_{f_m}(G, H) = \sum_{i=1}^{p} \text{MEC}_{f_m}(E_j)$ where $\text{MEC}_{f_m}(E_j) = r|V(G_1)| - 2|E(G_1)| + 2m, m \geq 0$.

**Result 14.** If there are partitions $\{E_1, E_2, \cdots, E_p\}$ of $kE(H)$, then by Lemma 5 $\text{MEC}_{f_m}(E_j) = \frac{1}{k}[r|V(G_1)| - 2|E(G_1)| + 2m], m \geq 0, j = 1, 2, \cdots, p$. 
4. Monophonic Wirelength into Wheels and Fans

**Theorem 15.** Let \( f_m : G \rightarrow H \) be a monophonic embedding where \( G \) is an \( r \)-regular circulant graph \( G(n, \pm S), S \subseteq \{1, 2, 3, \ldots, [n/2]\} \) and \( H \) is the wheel graph \( W_n \). Then the wirelength of \( G \) into \( H \) induced by \( f_m \) is monophonic.

**Proof.** Excluding the hub vertex of \( W_n \), let \( P_p = \{(p-1, p), (p+1, p+2), (n-1, p), (n-1, p+1)\}, 1 \leq p \leq n-1, \) where the vertices are taken \( \text{mod}(n-1) \). Consider the edge set \( \{(p-1, p), (n-1, p)/1 \leq p \leq n-1\} \) which represents the edges of \( W_n \) exactly once. Therefore \( \{P_1, P_2, \ldots, P_{n-1}\} \) form a partition of the edge set of \( W_n \) twice.

Let \( A_{p_1} \) and \( A_{p_2} \) be the components of \( W_n - P_p \) for every \( p \). Let us take \( A_{p_1} = \{p, p+1\} \). Under the monophonic embedding \( f_m \), let \( G_{p_1} = f_m^{-1}(A_{p_1}) \) and \( G_{p_2} = f_m^{-1}(A_{p_2}) \). Then an edge of \( G \) is induced by \( G_{p_1} \). Hence each \( P_p \) satisfies the properties stated in Lemma 14. Therefore \( MECf_m(P_p) \) is monophonic and hence by Lemma 5 the wirelength of \( f_m \) from \( G \) to \( W_n \) is monophonic.

**Theorem 16.** The monophonic wirelength of an \( r \)-regular graph \( G \) with \( n \) vertices, into the wheel graph \( W_n \) is given by \( MWL[G, W_n] = WL[G, W_n] + m, m \geq 0 \).

**Proof.** As explained in Theorem 15, the cut edge \( \{P_1, P_2, \ldots, P_{n-1}\} \) is a partition of edge set of \( W_n \) twice. Hence by Lemma 5,

\[
MEC_{f_m}(E_j) = \frac{1}{2} [r|V(G_1)| - 2|E(G_1)| + 2m] = \frac{1}{2} [r|V(G_1)| - 2|E(G_1)|] + \frac{1}{2} 2m
\]

Hence \( MWL[G, W_n] = WL[G, W_n] + m, m \geq 0, j = 1, 2, \ldots, p \)

**Monophonic Embedding Algorithm I**

**Aim:** To find a monophonic embedding \( f_m : G \rightarrow H \) that produces the monophonic wirelength \( MWL f_m(G, H) \) where \( G \) is the family of circulant graph with \( 2n \) vertices of \( r \)-regular and \( H \) is the wheel graph \( W_{2n} \).

**Monophonic Algorithm:** Case (i) Name the vertices of \( G[2n, \pm S], S \subseteq \{1, 2, 3, \ldots, n\} \) as a cycle from \( 0, 1, 2, \ldots, 2n-1 \).

Case (ii) Name the vertices of \( W_{2n} \) as an outer cycle from \( 0, 1, 2, \ldots, 2n-2 \) and the hub vertex \( 2n-1 \).

Case(i).

Input: *Preimage- The family of circulant graphs \( G[2n, \{1, 2, 3, \ldots, n-1\}], n \geq 3. \)
Image- The family of wheel graphs $W_{2n}$.

Output: A monophonic embedding $f_m$ of $G[2n, \{1, 2, 3, \cdots, n-1\}]$ into $W_{2n}$ is given by $f_m(x) = x$ with monophonic wirelength.

\[
MWL[G[2n, \{1, 2, 3, \cdots, n-1\}], W_{2n}] = WL[G[2n, \{1, 2, 3, \cdots, n-1\}], W_{2n}]
+ \frac{1}{2}[(3|S| - 2)(|S|(2n - 5) - 1) + 2].
\]

**Proof.** By Theorem 15, $f_m$ is monophonic and hence the proof follows from Theorem 16, as here $k = 2$ and

\[
m = \frac{1}{2}[(3|S| - 2)(|S|(2n - 5) - 1) + 2].
\]

**Case(ii).** Input: Preimage. The family of circulant graphs

$G[2n, \{1, 2, 3, \cdots, n-2\}], \quad n \geq 4.$

Image. The family of wheel graphs $W_{2n}$.

Output: A monophonic embedding $f_m$ of $G[2n, \{1, 2, 3, \cdots, n-2\}]$ into $W_{2n}$ is given by $f_m(x) = x$ with monophonic wirelength

\[
MWL[G[2n, \{1, 2, 3, \cdots, n-2\}], W_{2n}] = WL[G(2n, \{1, 2, 3, \cdots, n-2\}), W_{2n}]
+ \frac{1}{2}[(3|S| - 2)(2|S|^2 - 1) + (4n - 1)|S| - 2)].
\]

**Proof.** By Theorem 15, $f_m$ is monophonic and hence the proof follows from Theorem 16 as $k = 2$ and $m = \frac{1}{2}[(3|S| - 2)(2|S|^2 - 1) + (4n - 1)|S| - 2)].$

**Case(iii).** Input: Preimage. The family of circulant graphs

$G[2n, \{1, 2, 3, \cdots, n\}], \quad n \geq 3.$

Image. The family of wheel graphs $W_{2n}$.

Output: A monophonic embedding $f_m$ of $G[2n, \{1, 2, 3, \cdots, n\}]$ into $W_{2n}$ is given by $f_m(x) = x$ with monophonic wirelength

\[
MWL[G[2n, \{1, 2, 3, \cdots, n\}], W_{2n}] = WL[G(2n, \{1, 2, 3, \cdots, n\}), W_{2n}]
+ (2n - 1)[(3|S|^2 - 17|S| + 25)].
\]

**Proof.** By Theorem 15, $f_m$ is monophonic and hence the proof follows from Theorem 16, as here $k = 2$ and $m = (2n - 1)[(3|S|^2 - 17|S| + 25)].$
Theorem 17. Let \( f_m : G \to H \) be a monophonic embedding where \( G \) is an \( r \)-regular circulant graph \( G(n, \pm S), S \subseteq \{1, 2, 3, \ldots, \lfloor n/2 \rfloor \} \) and \( H \) is the fan graph \( F_n \). Then the wirelength of \( G \) into \( H \) induced by \( f_m \) is monophonic.

Proof. Excluding the core vertex of \( F_n \), let

\[
R_q = \{(q - 1, q), (q + 1, q + 2), (n - 1, q), (n - 1, q + 1) : 1 \leq q \leq n - 4 \},
\]

\[
R_{n-3} = \{(n - 4, n - 3), (n - 1, n - 3), (n - 1, n - 2) \},
\]

\[
R_{n-2} = \{(1, 2), (n - 1, 0), (n - 1, 1) \},
\]

\[
R_{n-1} = \{(n - 3, n - 2), (n - 1, n - 2), (n - 1, 0) \}
\]

represents the edges of \( F_n \) exactly once. Also, the edge set

\[
\{(q + 1, q + 2), (n - 1, q + 1) : 1 \leq q \leq n - 4 \} \cup R_{n-2} \cup \{(n - 1, n - 2), (0, 1) \}
\]

represents all edges of \( F_n \) exactly once.

Therefore \( \{R_1, R_2, \ldots, R_n\} \) is a partition of the edge set of \( F_n \) twice. Let \( B_{q_1} \) and \( B_{q_2} \) be the two components of \( F_n - R_q \) for every \( 1 \leq q \leq n - 3 \). Assume \( B_{q_1} = \{q, q + 1\} \). Under the monophonic embedding \( f_m \), let \( B_{q_1} = f_m^{-1}(G_{q_1}) \) and \( B_{q_2} = f_m^{-1}(G_{q_2}) \). Then an edge of \( G \) is induced by \( G_{q_1} \). Thus each \( R_q \) satisfies the properties given in Lemma 13 and by Lemma 5, \( MEC f_m(R_q) \) is monophonic and hence the wirelength of \( f_m \) from \( G \) to \( F_n \) is monophonic.

Monophonic Embedding Algorithm II Aim: To find a monophonic embedding \( f_m : G \to H \) that produces the monophonic wirelength \( MW L f_m(G, H) \), where \( G \) is the family of circulant graph with \( 2n \) vertices of \( r \)-regular and \( H \) is the fan \( F_{2n} \).

Monophonic Algorithm: Case (i) Name the vertices of \( G[2n, \pm S], S \subseteq \{1, 2, 3, \ldots, n\} \) as a cycle from \( 0, 1, 2, \cdots, 2n - 1 \).

Case (ii) Name the vertices of \( F_{2n} \) as an outer cycle from \( 0, 1, 2, \cdots, 2n - 2 \) and the core vertex \( 2n - 1 \).

Case (i). Input: *Preimage- The family of circulant graphs

\[
G[2n, \{1, 2, 3, \cdots, n - 1\}], n \geq 3.
\]

Image. The family of fan graphs \( F_{2n} \).

Output: A monophonic embedding \( f_m \) of \( G[2n, \{1, 2, 3, \cdots, n - 1\}] \) into \( F_{2n} \) is given by \( f_m(x) = x \) with monophonic wirelength.

\[
MW L[G[2n, \{1, 2, 3, \cdots, n - 1\}], F_{2n}] = WL[G(2n, \{1, 2, 3, \cdots, n - 1\}), F_{2n}]
\]
Proof. The proof is obvious from Theorems 16 & 17 as $k = 2$ and $m = \frac{1}{3}[(4|S|^2 - 9|S| + 5)]$.

Case (i) Name the vertices of $G[2n, \pm S], S \subseteq \{1, 2, 3, \cdots, n\}$ as a cycle from $0, 1, 2, \cdots, 2n - 1$.

Case (ii) Name the vertices of $F_{2n}$ as an outer cycle from $0, 1, 2, \cdots, 2n - 2$ and the core vertex $2n - 1$.

Case (iii).

Input: *Preimage- The family of circulant graphs

$G[2n, \{1, 2, 3, \cdots, n - 1\}], n \geq 3$.

Image. The family of fan graphs $F_{2n}$.

Output: A monophonic embedding $f_m$ of $G[2n, \{1, 2, 3, \cdots, n - 2\}]$ into $F_{2n}$ is given by $f_m(x) = x$ with monophonic wirelength.

$$MWL[G[2n, \{1, 2, 3, \cdots, n - 2\}], F_{2n}] = WL[G(2n, \{1, 2, 3, \cdots, n - 2\}), F_{2n}]$$

$$+ \frac{1}{3}(n - 3)[(4|S|^2 + |S| - 6)].$$

Proof. The proof is obvious from Theorems 16 & 17 as $k = 2$ and $m = \frac{1}{3}(n - 3)[(4|S|^2 + |S| - 6)]$.

References


