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ON NUMERICAL SOLUTION OF MULTI-TERMS FRACTIONAL DIFFERENTIAL EQUATIONS USING SHIFTED CHEBYSHEV POLYNOMIALS

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Abstract: This work provides numerical solution to multi-term differential equation of fractional order by collocation method using the Chebyshev polynomials based functions. The fractional derivative are used in Caputo's sense. The method assumed an approximate solution of the forms of shifted Chebyshev polynomials based functions. The assumed approximate solution is now substitute into the multi-term differential equations of fractional order. After a careful implementation of fractional order differential, we collect the equation at some suitable points and solve it together with boundary conditions to obtain a system of easily solvable linear or nor linear algebraic equations. Numerical examples of multi-order fractional differential equations (MOFDEs) are present to illustrate the method. The results converge to the exact solutions after some iterations and hence it revealed that proposed method is very effective and simple. Thats exposed the validity and applicability of the method.

AMS Subject Classification: 65N14
Key Words: Chebyshev polynomials, Caputo's fractional derivative, Multi Order fractional differential equation

1. Introduction

The study of fractional differential equations in fractional calculus has been

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main focus of many disciplines due to their frequent appearance in various applications like image processing, earthquake and biomedical engineering , viscoelasticity [4], finance [11], hydrology [5] and control system [1], [8]. Consequently, considerable attention has been given to the solution of fractional differential equations of physical interest. In numerical simulation, it is important to find out the exact solutions of fractional differential equations but unfortunately we do not have appropriate method to find the exact solution of many types of fractional differential equations. The situation become more complicated when we use MOFDEs because we cannot find their analytical solutions so approximations and numerical techniques must be used. Such type of approximations and other numerical techniques help us to understand the mechanism and complexity of the problem [20]. To find out such types of approximation is not an easy work and sometimes impossible to approximate the solution. Recently, several numerical methods have been used to solve the fractional differential equations such as variation iteration method [23], homotopy perturbation method [24], Collocation method [2], [17], Galerkin Finite Element Method [10], Laplace transform and Fourier transform methods [15], [7] and other methods [3], [6], [9], [12], [13], [14], [16], [18], [19], [21], [22].

Solving the MOFDEs by different numerical methods had been of great interest for researchers. Ibtisam (2011) used the homotopy analysis technique to solve the multi-order fractional integro differential equation. M. M. Khader (2011) carried out implementation to solve nonlinear MOFDEs by developing operational matrix and using Lengendre polynomials. Taiwo (2013) used decomposition method for approximation of MOFDEs. Davood (2013) solved MOFDEs of boundary value problems by reducing the multi terms of fractional differential into a system of algebraic equations. In this paper, we implement shifted Chebyshev polynomials based functions to solve MOFDEs by collocation method. It collects the formulated equation at some specific points together with boundary conditions to get the approximate solution.

The organization of this paper is as follows: In Section 2, we introduce some definitions regarding to fractional derivatives and MOFDEs. In Section 3, we introduce chebyshev polynomials as well as shifted chebyshev polynomials and find the approximate formula for fractional derivative. In Section 4, we give the procedure for solving MOFDEs. In Section 5, we present three numerical examples to show the validity and efficiency of the method by comparison of exact and approximate solutions. Finally in last section, we give some remarks about calculations and graphs in our paper.

2. Preliminaries and Notations

Definition 1. The Caputo's fractional derivative w.r.t x of order α denoted by $\frac{d^{(\alpha)}f(x)}{dx}$ and defined as follows:

$$\frac{d^{(\alpha)}f(x)}{dx} = \frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}}dt$$

where $m-1<\alpha<m,\ m\in N\ x>0,\ \alpha>0$

For the function of the type $f(x)=x^n, n\in W=\{0,1,2...\}$.The Caputo's fractional derivative is defined as:

$$\frac{d^{(\alpha)}x^2}{dx} = \begin{cases} 0, & for \quad n\in W \quad n<\lceil\alpha\rceil \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha}, & for \quad n\in W \quad n\geq\lceil\alpha\rceil \end{cases}$$

Here $\lceil\alpha\rceil$ denotes the ceiling value of α. It the smallest integer greater than or equal to α. Moreover if $f(x)=c$ is any constant function then its the Caputo's fractional derivative will be zero like the integer order derivative i.e $\frac{d^{(\alpha)}c}{dx}=0$. Amongst the main properties of Caputo's fractional derivative,the most important property is the linearity property i.e.

$$\frac{d^{(\alpha)}}{dx}(\lambda f(x)+\mu g(x)) = \lambda\frac{d^{(\alpha)}}{dx}f(x)+\mu\frac{d^{(\alpha)}}{dx}g(x)$$

where λ and μ are constants.

In this paper, we use the following type of MOFDE ,

$$D^{\alpha}u(x) = F(x,u(x),D^{\beta_1}u(x),...D^{\beta_m}u(x)) \tag{1}$$

with the following initial conditions;

$$u^k(0)=d_k, k=0,1,2...n \tag{2}$$

Where $n<\alpha\leq n+1, 0<\beta_1<\beta_2<...\beta_m<\alpha$ and D^{α} denotes the Caputo's derivative of order α. F should be non-linear in general.

3. Definitions and Properties of Chebyshev Polynomials

Many authors and researcher use the Chebyshev polynomials not only for the solution of ODEs but also they use for solving the FDEs. The Chebyshev polynomials are a sequence of orthogonal polynomials and widely used in many areas of numerical analysis for simulation like least square approximation, uniform approximation and in spectral or pseudospectral methods. The Chebyshev polynomials are also very important in approximation theory for interpolation. One of the advantages of Chebyshev polynomials is the approximation of a function $f(x)$ by a polynomial $p(x)$ that gives a uniform and accurate description in the real interval $[a, b]$. Another advantage is to approximate a function $f(x)$ in terms of series expansion that form the basis of the FDEs [2], [5].

The recurrence formula for Chebyshev polynomials (also known as Chebyshev polynomials of first kind) on the interval $[1, -1]$ is as follows:

$$T_0(x) = 1, T_1(x) = x$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad , n = 1, 2...$$

The orthogonality condition is

$$\int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}} dx = \begin{cases} \Pi, & for \quad i = j = 0 \\ \frac{\Pi}{2}, & for \quad i = j \neq 0 \\ 0, & for \quad i \neq j \end{cases}$$

The analytic form of Chebyshev polynomials $T_n(x)$ of degree n over the interval $[-1, 1]$ is given by ;

$$T_n(x) = n \sum_{i=0}^{\lceil \frac{n}{2} \rceil} (-1)^i 2^{(n-2i-1)} \frac{(n-i-1)!}{(i)!(n-2i)} x^{n-2i}$$

where $\lceil \frac{n}{2} \rceil$ denotes the integer part of $\frac{n}{2}$.

In order to use these polynomials,it is difficult to work over the interval $[-1, 1]$.For easy computations we shift these polynomials on the interval $[0, 1]$ and define the so called shifted Chebyshev polynomials by introducing the change of variable $x = 2t - 1$. The recurrence formula of the shifted Chebyshev polynomials $T^*_n(t)$ are defined as :

$$T^*_n(t) = T_n(2t - 1) = T_{2n}(\sqrt{t})$$

where

$$T_0^*(x) = 1 \quad , T_1^*(x) = 2x - 1$$

$$T_{n+1}^* = (4x-2)T_n^* - T_{n-1}^* \quad , \quad n = 1, 2, 3...$$

The analytic form of shifted Chebyshev polynomials $T_n^*(t)$ of degree n is given by ;

$$T_n^*(t) = n \sum_{k=0}^{n} (-1)^{n-k} \frac{2^{2k}(n+k-1)!}{(2k)!(n-k)!} t^k \tag{3}$$

where $T_n^*(0) = (-1)^n$ *and* $T_n^*(1) = 1$

The orthogonality conditions for the shifted Chebyshev polynomial are given by,

$$\int_0^1 \frac{T_i^*(x)T_j^*(x)}{\sqrt{x-x^2}} dx = \delta_{i,j} h_j$$

where $h_k = \frac{b_k}{2}\Pi$ with $b_0 = 1, b_k = 1, k \geq 1$.

we can approximate the solution of the equation of the type defined in (1) with the boundary conditions in (2) as the infinite series of shifted Chebyshev polynomials as,

$$x(t) = \sum_{i=0}^{\infty} c_i T_i^*(t) \tag{4}$$

The coefficients c_i are given by;

$$c_i = \begin{cases} \frac{1}{\Pi} \int_0^1 \frac{x(t)T_0^*(t)}{\sqrt{t-t^2}} dt, & for \quad i = 0 \\ \\ \frac{2}{\Pi} \int_0^1 \frac{x(t)T_i^*(t)}{\sqrt{t-t^2}} dt, & for \quad i = 1, 2... \end{cases}$$

We consider the first $m+1$ terms of the shifted Chebyshev polynomials. So Eq.(4) will take the form :

$$x_m(t) = \sum_{i=0}^{m} c_i T_i^*(t) \tag{5}$$

Theorem 2. *Let $x_m(t)$ be the approximate solution of DFDE as defined in Eq.(5) and also suppose that $\alpha > 0$, then*

$$\frac{d^{\alpha}x_m(t)}{dx} = \sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} c_i\Omega_{i,k}^{(\alpha)}t^{k-\alpha}$$

where $\Omega_{i,k}^{(\alpha)}$ *is weight function and defined as*

$$\Omega_{i,k}^{(\alpha)} = (-1)^{i-k}\frac{2^{2k}i(i+k-1)!\Gamma(k+1)}{(i-k)!(2k)!\Gamma(k+1-\alpha)}$$

Proof. Due to linearity property of Caputo's fractional derivative we can write Eq.(5) as

$$\frac{d^{\alpha}x_m(t)}{dx} = \sum_{i=\lceil\alpha\rceil}^{m} c_i\frac{d^{\alpha}}{dx}(T_i^{*}(t)) \tag{6}$$

From definition (1), $\frac{d^{\alpha}}{dx}(T_i^{*}(t)) = 0$,$\forall \;\; i = 0, 1, ...\lceil\alpha\rceil - 1, \alpha > 0$

Again apply property of linearity on the analytic form of the shifted Chebyshev polynomials described in Eq.(3) ,we have

$$\frac{d^{\alpha}}{dx}(T_i^{*}(t)) = i\sum_{k=\lceil\alpha\rceil}^{i}(-1)^{i-k}\frac{2^{2k}(i+k-1)!}{(i-k)!(2k)!}\frac{d^{\alpha}}{dx}(t^k) \;\; \forall, i = \lceil\alpha\rceil + \lceil\alpha\rceil + 1....m$$

Apply definition (1),we have

$$\frac{d^{\alpha}}{dx}(T_i^{*}(t)) = i\sum_{k=\lceil\alpha\rceil}^{i}(-1)^{i-k}\frac{2^{2k}(i+k-1)!\Gamma(k+1)}{(i-k)!(2k)!\Gamma(k+1-\alpha)}t^{k-\alpha} \tag{7}$$

Using values from Eq.(7) into Eq.(6), we have

$$\frac{d^{\alpha}x_m(t)}{dx} = \sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} c_i(-1)^{i-k}\frac{2^{2k}i(i+k-1)!\Gamma(k+1)}{(i-k)!(2k)!\Gamma(k+1-\alpha)}t^{k-\alpha}$$

as desired. □

4. Procedure for the Solution of MOFDEs

Consider the MOFDE of the type given in section (2). In order to use the Chebyshev polynomials collection method, we first approximate $u(x)$ as;

$$u_m(x) = \sum_{i=0}^{m} y_i T_i^*(x) \tag{8}$$

Employing the Theorem (2) and Eq.(8) into the Eq.(1), we have

$$\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} y_i \Omega_{i,k}^{(\alpha)} x^{k-\alpha} = f(x, \sum_{i=0}^{m} y_i T_i^*(x), \sum_{i=\lceil\beta_1\rceil}^{m} \sum_{k=\lceil\beta_1\rceil}^{i} y_i \Omega_{i,k}^{(\beta_1)} x^{k-\beta_1}$$

$$... \sum_{i=\lceil\beta_m\rceil}^{m} \sum_{k=\lceil\beta_m\rceil}^{i} y_i \Omega_{i,k}^{(\beta_m)} x^{k-\beta_m}) \tag{9}$$

By using the collocation method we collocate the above Eq.(9) at the points $x_p, p = 0, 1, ...m - \lceil\alpha\rceil$ i.e total number of collocated points will be $(m+1-\lceil\alpha\rceil)$. For appropriate collocated points we use the roots of the shifted Chebyshev polynomial $T^*_{m+1-\lceil\alpha\rceil}(x)$.

$$\sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=\lceil\alpha\rceil}^{i} y_i \Omega_{i,k}^{(\alpha)} x_p^{k-\alpha} = f(x_p, \sum_{i=0}^{m} y_i T_i^*(x_p), \sum_{i=\lceil\beta_1\rceil}^{m} \sum_{k=\lceil\beta_1\rceil}^{i} y_i \Omega_{i,k}^{(\beta_1)} x_p^{k-\beta_1}$$

$$... \sum_{i=\lceil\beta_m\rceil}^{m} \sum_{k=\lceil\beta_m\rceil}^{i} y_i \Omega_{i,k}^{(\beta_m)} x_p^{k-\beta_m}) \tag{10}$$

Employing the Eq.(8) in the boundary conditions Eq.(2), we can obtain $\lceil\alpha\rceil$ equations as follows :

$$\sum_{i=0}^{m} y_i T_i^k(0) = d_k, k = 0, 1, 2...n \tag{11}$$

From Eq.(10) we get $(m+1-\lceil\alpha\rceil)$ algebraic equations and from Eq. (11) we get $\lceil\alpha\rceil$ algebraic equations so we get total $(m+1)$ linear or non-linear algebraic equations which can easily be solved with the help of matrices for unknowns $c_n, n = 0, 1, 2...m$ to calculate the approximate solution $y_m(x)$.

5. Numerical Implementations

In this section, we solve numerically the MOFDEs using shifted Chebyshev polynomials. In the following examples, absolute values of errors $E_i = |u_{ext} - u_{apx}|,\ \ for\ i = 1,\ 2,\ 3$ between the exact values u_{ext} and the approximate value u_{apx} are present at different values of m.

Example 5.1. Consider the following MOFDE;

$$D^{2.5}u(x) + xD^{1.9} + u(x) = \frac{6}{\Gamma(1.5)}x^{.5} + \frac{6}{\Gamma(2.1)}x^3 + x$$

with the initial conditions;

$$u(0) = 0, u^{'}(0) = 0, u^{''}(0) = 0 \tag{12}$$

Here $\alpha = 2.5\ and\ \beta = 1.9$. We implement the suggested method with $m = 4$ and we approximate the solution as:

$$u_4(x) = \sum_{i=0}^{4} y_i T_i^*(x)$$

The exact solution of this problem is $x^3 + x$. Using Eq. (10) we have

$$\sum_{i=3}^{4}\sum_{k=3}^{i} y_i \Omega_{i,k}^{(2.5)} x_p^{k-2.5} + x_p(\sum_{i=2}^{4}\sum_{k=2}^{i} y_i \Omega_{i,k}^{(1.9)} x_p^{k-1.9} + \sum_{i=0}^{4} y_i T_i(x_p) = F_1 \tag{13}$$

where

$$F_1 = \frac{6}{\Gamma(1.5)}x_p^{.5} + \frac{6}{\Gamma(2.1)}x_p^3 + x_p$$

with $p = 0, 1$ where x_p are the roots of shifted Chebyshev polynomial $T_2^*(x)$ and their values are;

$$x_0 = 0.8536, x_1 = 0.1464$$

By using Eq.(11) and (12) we have

$$y_0 - y_1 + y_2 - y_3 + y_4 = 0 \tag{14}$$

$$2y_1 - 8y_2 + 18y_3 - 32y_4 = 0 \tag{15}$$

$$16y_2 - 96y_3 + 320y_4 = 0 \tag{16}$$

Solving Eq.(13) together with Eq.(14),(15) and (16) we find the approximate values and its absolute error values E_1 are given in table (1) for $m = 4$. Similarly the values of absolute errors E_2 and E_3 are calculated for $m = 5$ and $m = 6$.

Table 1: Values of absolute errors E_i for $i = 1, 2, 3$ at $m = 4, 5, 6$ respectively and $\alpha = 2.5$ and $\beta = 1.9$

x	$E_1 = \lvert u_{ext} - u_{apx}\rvert$	$E_2 = \lvert u_{ext} - u_{apx}\rvert$	$E_3 = \lvert u_{ext} - u_{apx}\rvert$
0.0000	$1.362514\ e^{-04}$	$2.214035\ e^{-05}$	$9.365794\ e^{-06}$
0.1000	$5.421578\ e^{-04}$	$4.042176\ e^{-05}$	$4.012453\ e^{-06}$
0.2000	$2.824691\ e^{-03}$	$4.391758\ e^{-04}$	$2.396754\ e^{-05}$
0.3000	$1.102450\ e^{-03}$	$6.972043\ e^{-04}$	$7.854316\ e^{-05}$
0.4000	$1.014759\ e^{-03}$	$2.317964\ e^{-04}$	$6.635247\ e^{-05}$
0.5000	$2.349720\ e^{-03}$	$3.042176\ e^{-04}$	$4.754167\ e^{-05}$
0.6000	$2.457291\ e^{-03}$	$8.443720\ e^{-04}$	$7.301460\ e^{-05}$
0.7000	$1.041350\ e^{-03}$	$4.436679\ e^{-04}$	$4.042631\ e^{-05}$
0.8000	$1.143628\ e^{-03}$	$6.146279\ e^{-04}$	$4.138567\ e^{-05}$
0.9000	$1.972435\ e^{-03}$	$4.652413\ e^{-04}$	$3.399746\ e^{-05}$
1.000	$1.472016\ e^{-03}$	$7.785214\ e^{-04}$	$1.142276\ e^{-05}$

Example 5.2. Consider the following MOFDE;

$$D^2u(x) + sin(x)D^{1/2}u(x) + xu(x) = g(x)$$

where

$$g(x) = x^9 - x^8 + 56x^6 - 42x^5 + sin(x)[\frac{32768}{6435}x^{15/2} - \frac{2048}{429}x^{13/2}]$$

with the initial conditions

$$u(0) = u'(0) = 0 \tag{17}$$

The exact solution of the problem is $u(x) = x^8 - x^7$ Let $m = 3$ and in this example,we have $\alpha = 2$ and $\beta_1 = 0.5$ we approximate the solution as:

$$u_3(x) = \sum_{i=0}^{3} y_i T_i^*(x)$$

Using Eq.(10) we have

$$\sum_{i=2}^{3}\sum_{k=2}^{i} y_i \Omega_{i,k}^{(2)} x_p^{(k-2)} + sin(x)\sum_{i=1}^{3}\sum_{k=1}^{i} y_i \Omega_{i,k}^{(0.5)} x_p^{(k-.5)} + x_p u(x) = g(x_p) \tag{18}$$

where

$$g(x_p) = x_p^9 - x_p^8 + 56x_p^6 - 42x_p^5 + sin(x_p)[\frac{32768}{6435}x_p^{15/2} - \frac{2048}{429}x_p^{13/2}]$$

With $p = 0, 1$ where x_p are the roots of shifted Chebyshev polynomial $T_2^*(x)$ and their values are;

$$x_0 = 0.8536, x_1 = 0.1464$$

By using Eq.(11) and (17) we have

$$y_0 - y_1 + y_2 - y_3 = 0 \tag{19}$$

$$2y_1 - 8y_2 + 18y_3 = 0 \tag{20}$$

Solving Eq.(18) together with Eq.(19) and (20) we find the approximate values and its absolute error values E_1 are given in table (2) for $m = 3$. Similarly absolute error values for $m = 4$ and $m = 5$ are calculated as E_2 and E_3.

Table 2: Values of absolute errors E_i $\ for\ i = 1,\ 2,\ 3$ at $m = 3,\ 4,\ 5$ respectively and $\alpha = 2$ and $\beta^1 = 0.5$

x	$E_1 = \|u_{ext} - u_{apx}\|$	$E_2 = \|u_{ext} - u_{apx}\|$	$E_3 = \|u_{ext} - u_{apx}\|$
0.0000	5.012447 e^{-02}	5.457362 e^{-03}	5.251764 e^{-04}
0.1000	2.321145 e^{-02}	2.452197 e^{-03}	2.521476 e^{-04}
0.2000	4.421731 e^{-02}	1.632597 e^{-03}	5.142769 e^{-04}
0.3000	2.452107 e^{-02}	4.452173 e^{-03}	8.289741 e^{-04}
0.4000	7.632417 e^{-02}	8.012483 e^{-03}	9.248573 e^{-04}
0.5000	6.415273 e^{-02}	4.369754 e^{-03}	3.452169 e^{-04}
0.6000	4.314724 e^{-02}	5.452176 e^{-03}	5.485201 e^{-04}
0.7000	4.142731 e^{-02}	3.673214 e^{-03}	6.129357 e^{-04}
0.8000	6.174996 e^{-02}	1.938271 e^{-03}	2.968571 e^{-04}
0.9000	3.247510 e^{-02}	2.794513 e^{-03}	5.685742 e^{-04}
1.000	2.367241 e^{-02}	6.193752 e^{-03}	7.857464 e^{-04}

Example 5.3. Consider the following MOFDE;

$$D^2u(x) + x^{1/2}D^{1.234}u(x) + x^{1/3}Du(x) + x^{1/4}D^{.333}u(x) + x^{1/5}u(x) = g(x)$$

where
$$g(x) = -1 - \frac{x^{1.266}}{\Gamma(1.766)} - x^{4/3} - \frac{x^{1.817}}{\Gamma(2.766)} + x^{1/5}(2 - \frac{x^2}{2})$$
with the initial conditions
$$u(0) = 2, u'(0) = 0 \tag{21}$$

The exact solution to this problem is $u(x) = 2 - \frac{x^2}{2}$
Let $m = 3$, $\alpha = 2$, $\beta^1 = 1.234, \beta^2 = 1, \beta^3 = .333$
we approximate the solution as:
$$u_3(x) = \sum_{i=0}^{3} y_i T_i^*(x)$$

Using Eq.(10) we have
$$\sum_{i=2}^{3}\sum_{k=2}^{i} y_i \Omega_{i,k}^{(2)} x_p^{(k-2)} + x^{1/2}\sum_{i=2}^{3}\sum_{k=2}^{i} y_i \Omega_{i,k}^{(1.234)} x_p^{(k-1.234)} + x^{1/3}\sum_{i=1}^{3}\sum_{k=1}^{i} y_i \Omega_{i,k}^{(1)} x_p^{(k-1)}$$
$$+x^{1/4}\sum_{i=1}^{3}\sum_{k=1}^{i} y_i \Omega_{i,k}^{(.333)} x_p^{(k-.333)} + x^{1/5}\sum_{i=0}^{3} y_i T_i(x_p) = g(x_p) \tag{22}$$
Where
$$g(x) = -1 - \frac{x_p^{1.266}}{\Gamma(1.766)} - x_p^{4/3} - \frac{x_p^{1.817}}{\Gamma(2.766)} + x_p^{1/5}(2 - \frac{x_p^2}{2})$$

With $p = 0, 1$ where x_p are the roots of shifted Chebyshev polynomial $T_2^*(x)$ and their values are:
$$x_0 = 0.8536, x_1 = 0.1464$$

By using Eq.(11) and (21) we have
$$y_0 - y_1 + y_2 - y_3 = 0 \tag{23}$$

$$2y_1 - 8y_2 + 18y_3 = 0 \tag{24}$$

Solving Eq.(22) together with Eq.(23) and (24) we find the approximate values and its absolute error values E_1 are given in table (2) for $m = 3$. Similarly absolute error values for $m = 4$ and $m = 5$ are calculated as E_2 and E_3.

Table 3: Values of absolute errors $E_i \ \ for \ i = 1, \ 2, \ 3$ at $m = 3, \ 4, \ 5$ respectively and $\alpha = 2$, $\beta_1 = 1.234, \ \beta^2 = 1, \ \beta^3 = .333$

x	$E_1 = \|u_{ext} - u_{apx}\|$	$E_2 = \|u_{ext} - u_{apx}\|$	$E_3 = \|u_{ext} - u_{apx}\|$
0.0000	$1.236547 \ e^{-01}$	$4.201457 \ e^{-02}$	$4.756930 \ e^{-03}$
0.1000	$4.254869 \ e^{-01}$	$7.391725 \ e^{-02}$	$3.528640 \ e^{-03}$
0.2000	$7.754839 \ e^{-01}$	$2.493082 \ e^{-02}$	$4.004937 \ e^{-03}$
0.3000	$4.632598 \ e^{-01}$	$5.741362 \ e^{-02}$	$7.625473 \ e^{-03}$
0.4000	$6.396851 \ e^{-01}$	$9.539017 \ e^{-02}$	$5.549196 \ e^{-03}$
0.5000	$8.601243 \ e^{-01}$	$5.721934 \ e^{-02}$	$4.369258 \ e^{-03}$
0.6000	$1.968524 \ e^{-01}$	$8.583914 \ e^{-02}$	$6.134587 \ e^{-03}$
0.7000	$3.368741 \ e^{-01}$	$4.439271 \ e^{-02}$	$4.753951 \ e^{-03}$
0.8000	$4.685201 \ e^{-01}$	$5.938564 \ e^{-02}$	$5.251478 \ e^{-03}$
0.9000	$6.047896 \ e^{-01}$	$5.967438 \ e^{-02}$	$4.014259 \ e^{-03}$
1.000	$7.934271 \ e^{-01}$	$8.439507 \ e^{-02}$	$6.937518 \ e^{-03}$

6. Conclusion and Remarks

In this paper, we are implemented Chebyshev polynomials to solve multi-order differential equations of fractional order. The fractional derivative is applied in Caputo's sense. The properties of Chebyshev polynomials are used to reduce MOFDEs into easily solvable linear or nonlinear algebraic equations. Three numerically solved examples show the present method is well organized and calculated approximate values are in excellent agreement with the exact solutions and hence this approach can be solve the problem efficiently. In our

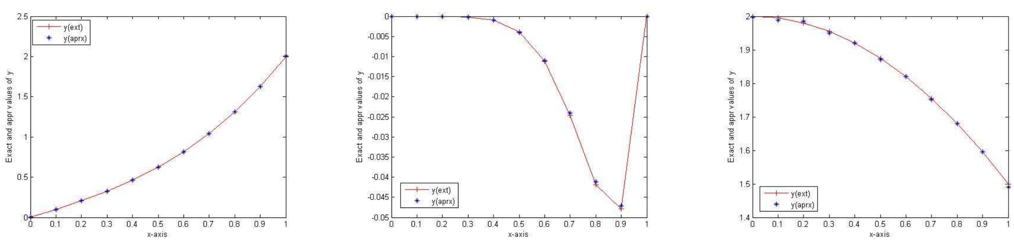

Figure 1: Graphs of the three examples by present method at different values of m, α and β

suggested method, the graphs of approximate values at indicated values of α and β converge to the graph of the exact solution. This shows that our suggested method is more effective, valid and applicable. All numerical results are obtained using MATLAB.

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