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ON THE DIAMOND HEAT KERNEL RELATED TO THE ∇^k OPERATOR

Wanchak Satsanit
Department of Mathematics
Faculty of Science
Maejo University
Chiangmai, 50290, THAILAND

Abstract: In this paper, we introduce a Fourier transform method in distribution theory to obtained exact solution of some partial differential equation in n dimension. It was found that the solution of such equation obtained a diamond heat kernel which related to the heat equation.

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1. Introduction

Considering the ultra-hyperbolic operator iterated $k-$ times defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k, \quad (1.1)$$

$p+q=n$. S.E. Trione[8] has shown that the generalized function $R_{2k}(x)$ defined by (2.4) is the unique elementary solution of the operator $\Box^k$ that is $\Box^k R_{2k}(x) = \delta(x)$. Also M. Aguirre Tellez[1] has proved that $R_{2k}(x)$ exists only if n is odd with p odd and q even or only if n is even with p odd and q odd.

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In 1996, A. Kananthai [3] first introduced the operator $\diamondsuit^k$ and named Diamond operator and defined by

$$\diamondsuit^k = \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k . \tag{1.2}$$

The operator $\diamondsuit^k$ can be written as the product of the operators in the form

$$\diamondsuit^k = \triangle^k \square^k = \square^k \triangle^k, \tag{1.3}$$

where $\triangle^k$ is the Laplacian operator iterated $k-$ times and defined by

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right)^k, \tag{1.4}$$

and $\square^k$ is the ultra-hyperbolic operator iterated $k-$ times and defined by(1.1). A.Kananthai [3] has shown that the convolution $(-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ is an elementary solution of the operator $\diamondsuit^k$, that is

$$\diamondsuit^k((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) = \delta(x), \tag{1.5}$$

where $\delta(x)$ is Dirac-delta distribution and the function $R^e_{2k}(x)$ and $R^H_{2k}(v)$ are defined by (2.6) and (2.4) respectively with $\alpha = 2k, k$ is nonnegative integer.

In 2002,the operator $\oplus^k$ first introduced by A. Kananthai, S. Suantai and V. Longani ,[4] and the operator $\oplus^k$ can be written in the form

$$\begin{aligned} \oplus^k &= \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right)^k \\ &= \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \left[\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k \\ &\quad \left[\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k , \end{aligned} \tag{1.6}$$

where $p + q = n$ is the dimension of $\mathbb{R}^n$, $i = \sqrt{-1}$ and k is the positive integer.

Let us denote the operator

$$L_1 = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}$$

and

$$L_2 = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}.$$

Thus the equation(1.6) can be written as

$$\oplus^k = \diamondsuit^k L_1^k L_2^k.$$

They studied the elementary solution or Green function of the operator $\oplus^k$ and then such a solution is related to the solution of the wave equation and the Laplacian equation.

Next, W. Satsanit [7] first introduced the $\bigtriangledown^k$ operator and is defined by

$$\begin{aligned} \bigtriangledown^k &= \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right)^k \\ &= \left[\left(\frac{\triangle + \Box}{2}\right)^2 + \left(\frac{\triangle - \Box}{2}\right)^2\right]^k \\ &= \left(\frac{\triangle^2 + \Box^2}{2}\right)^k, \end{aligned} \tag{1.7}$$

where Δ^2 and $\Box^2$ defined by (1.4)and (1.1) respectively with $k = 2$.

It is well known that for the heat equation [2]

$$\frac{\partial}{\partial t} u(x,t) = c^2 \triangle u(x,t) \tag{1.8}$$

with the initial condition

$$u(x,0) = f(x),$$

where $\triangle$ is the Laplace operator and defined by(1.4) with $k = 1, f(x)$ is a continuous function and $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0, \infty)$, we obtain

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4c^2 t}\right) f(y) dy \tag{1.9}$$

as the solution of (1.8). The equation (1.9) can be written

$$u(x,t) = E(x,t) * f(x),$$

where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4c^2 t}\right). \tag{1.10}$$

$E(x,t)$ is called the heat kernel, $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$. Moreover, we obtain $E(x,t) \to \delta$ as $t \to 0$ where δ is the Dirac delta distribution.
Next, K. Nonlaopon and A. Kananthai [5] present the equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Box u(x,t), \tag{1.11}$$

where $\Box$ is the ultra-hyperbolic operator defined by (1.1) with $k = 1$. They obtained the ultra-hyperbolic heat kernel

$$E(x,t) = \frac{i^q}{(4c^2\pi t)^{n/2}} exp\left(\frac{\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2}{4c^2 t}\right), \tag{1.12}$$

where $p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$ and $i = \sqrt{-1}$.

Now, the purpose of this work has been studied the equation

$$\frac{\partial}{\partial t} \Diamond^k u(x,t) - c^2 \oplus^k u(x,t) = 0, \tag{1.13}$$

with the initial condition

$$\Diamond^k u(x,0) = f(x),$$

where $(x,t) = (x_1, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty), t$ is a time, c is a positive constant, $u(x,t)$ is an unknown function and $f(x)$ is a given generalized function for $x \in \mathbb{R}^n$. By the fourier transform method, we obtain

$$u(x,t) = ((-1)^k R_{2k}^e(x) * R_{2k}^H(x)) * E(x,t) * f(x)$$

as a solution of (1.13), where

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 + \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right)^k + i(\xi, x)\right] d\xi. \tag{1.14}$$

and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed $t > 0$. The convolution $((-1)^k R_{2k}^e(x) * R_{2k}^H(\upsilon)) * E(x,t)$ is called the Diamond Heat Kernel and will be studied all properties in details. Before proceeding definitions and some important concepts are needed.

2. Preliminaries

Definition 2.1 Let $f(x) \in \mathrm{L}_1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)\, dx \tag{2.1}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ and $dx = dx_1\, dx_2 \ldots dx_n$.

Also, the inverse of Fourier transform defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f}(\xi)\, d\xi. \tag{2.2}$$

If f is a distribution with compact supports[9] then the equation (2.1) can be written as

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \left\langle f(x), e^{-i(\xi,x)} \right\rangle.$$

Definition 2.2 The spectrum of the kernel $E(x,t)$ of (1.9) is the bounded support of the Fourier transform $\widehat{E(\xi,t)}$ for any fixed $t > 0$.

Definition 2.3 Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be a point in $\mathbb{R}^n$ and be denoted by

$$\Gamma_+ = \{\xi \in \mathbb{R}^n : \xi_1^2 + \xi_2^2 + \ldots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \ldots - \xi_{p+q}^2 > 0 \text{ and } \xi_1 > 0\}$$

the set of an interior of the forward cone, and $\overline{\Gamma}_+$ denotes the closure of Γ_+.

Let Ω be a spectrum of $E(x,t)$ defined by definition 2.2 for any fixed $t > 0$ and $\Omega \subset \overline{\Gamma}_+$. Let $\widehat{E(\xi,t)}$ be the Fourier transform of $E(x,t)$ and defined by

$$\widehat{E(\xi,t)} = \begin{cases} \frac{1}{(2\pi)^{n/2}} \exp\left(c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 + \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right)^k\right) & \text{for } \xi \in \Gamma_+, \\ 0 & \text{for } \xi \notin \Gamma_+. \end{cases} \tag{2.3}$$

Definition 2.4 Let $x = (x_1, x_2, ..., x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$ and written as

$$\upsilon = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2,$$

where $p+q=n$ is the dimension of the space $\mathbb{R}^n$.

Let $\Gamma_+=\{x\in\mathbb{R}^n : x_1>0 \text{ and } \upsilon>0\}$ be the interior of the forward cone and $\overline{\Gamma}_+$ denote its closure. For any complex number α, define the function

$$R^H_\alpha(\upsilon)=\begin{cases}\frac{\upsilon^{\frac{\alpha-n}{2}}}{K_n(\alpha)}, & \text{for } x\in\Gamma_+,\\ 0, & \text{for } x\notin\Gamma_+,\end{cases} \tag{2.4}$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha)=\frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{2+\alpha-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2})\Gamma(\frac{p-\alpha}{2})}. \tag{2.5}$$

The function $R^H_\alpha(\upsilon)$ is called the ultra-hyperbolic kernel of Marcel Riesz and was introduced by Y. Nozaki [6].

It is well known that $R^H_\alpha(\upsilon)$ is an ordinary function if $Re(\alpha)\geq n$ and it is a distribution of α if $Re(\alpha)<n$. Let supp $R^H_\alpha(\upsilon)$ denote the support of $R^H_\alpha(\upsilon)$ and suppose supp $R^H_\alpha(\upsilon)\subset\bar{\Gamma}_+$, that is supp $R^H_\alpha(\upsilon)$ is compact.

Definition 2.5 Let $x=(x_1,x_2,\ldots,x_n)$ be a point of $\mathbb{R}^n$ and $|x|=(x_1^2+x_2^2+\cdots+x_n^2)^{1/2}$. The elliptic kernel of Marcel Riesz defined by

$$R^e_\alpha(x)=\frac{|x|^{\alpha-n}}{W_n(\alpha)}, \tag{2.6}$$

where

$$W_n(\alpha)=\frac{\pi^{\frac{n}{2}}2^\alpha\Gamma(\alpha/2)}{\Gamma\left(\frac{n-\alpha}{2}\right)}, \tag{2.7}$$

with α is a complex parameter and n the dimension of $\mathbb{R}^n$. It can be shown that $R^e_{-2k}(x)=(-1)^k\triangle^k\delta(x)$ where $\triangle^k$ is defined by (1.4). It follows that $R^e_0(x)=\delta(x)$; [8]. The function $R^e_{2k}(x)$ is called the elliptic kernel of Marcel Riesz.

Lemma 2.1. *The function $R^H_{2k}(\upsilon)$ and $(-1)^kR^e_{2k}(x)$ are the elementary solutions of the operator $\Box^k$ and $\triangle^k$ respectively, where $\Box^k$ and $\triangle^k$ are defined by (1.4) and (1.3) respectively. That is $\Box^kR^H_{2k}(\upsilon)=\delta(x)$ and $\triangle^k(-1)^kR^e_{2k}(x)=\delta(x)$.*

Proof. [See 1,8, p.31, see 9,p.31].

Lemma 2.2. *The convolution $R^H_{2k}(\upsilon)*(-1)^kR^e_{2k}(x)$ is an elementary solution of the operator $\diamondsuit^k$ iterated $k-$ times and is defined by (1.1),that is*

$$\diamondsuit^k((-1)^kR^e_{2k}(x)*R^H_{2k}(x))=\delta(x). \tag{2.8}$$

Proof. [See 3, p.33].

Lemma 2.3. *(The Fourier transform of $\nabla^k\delta$)*

$$\mathcal{F}\nabla^k\delta = \frac{(-1)^{2k}}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^2+\left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^2\right]^k, \tag{2.9}$$

where $\mathcal{F}$ is the Fourier transform defined by (2.1) and if the norm of ξ is given by $\|\xi\| = \left(\xi_1^2+\xi_2^2+\ldots+\xi_n^2\right)^{1/2}$, we obtain

$$\mathcal{F}\nabla^k\delta \leq \frac{M}{(2\pi)^{n/2}}\|\xi\|^{4k}.$$

Since M is constant thus $\mathcal{F}\nabla^k\delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution, thus

$$\nabla^k\delta = \mathcal{F}^{-1}\frac{(-1)^{3k}}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^2+\left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^2\right]^k. \tag{2.10}$$

Proof.

By Eq. (2.10), we have

$$\begin{aligned}
\mathcal{F}\nabla^k\delta &= \frac{1}{(2\pi)^{n/2}}\left\langle\nabla^k\delta, e^{-i(\xi,x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, \nabla^k e^{-(i\xi,x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, \nabla^{k-1}\nabla e^{-i(\xi,x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, \nabla^{k-1}\left(\frac{1}{2}\triangle^2+\frac{1}{2}\Box^2\right)e^{-i(\xi,x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, \nabla^{k-1}\left(\frac{1}{2}\triangle^2\right)e^{-i(\xi,x)}\right\rangle \\
&\quad+\frac{1}{(2\pi)^{n/2}}\left\langle\delta, \nabla^{k-1}\left(\frac{1}{2}\Box^2\right)e^{-i(\xi,x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, \nabla^{k-1}\frac{1}{2}(-1)^2\left(\sum_{i=1}^{n}\xi_i^2\right)^2 e^{(-i\xi,x)}\right\rangle \\
&\quad+\frac{1}{(2\pi)^{n/2}}\left\langle\delta, \nabla^{k-1}\frac{1}{2}(-1)^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)-\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)\right]^2 e^{-i(\xi,x)}\right\rangle
\end{aligned}$$

$$= \frac{1}{(2\pi)^{n/2}}\left\langle \delta, \nabla^{k-1}\frac{1}{2}(-1)^2\left[\left(\left(\sum_{i=1}^{p}\xi_i^2\right) - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)\right)^2 + \left(\sum_{i=1}^{n}\xi_i^2\right)^2\right] e^{-i(\xi,x)}\right\rangle$$

$$= \frac{(-1)^2}{(2\pi)^{n/2}}\left\langle \delta, \nabla^{k-1}\left[\left(\sum_{i=1}^{p}\xi_i^2\right) + \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)\right]^2 e^{-i(\xi,x)}\right\rangle.$$

By keeping on operator ∇ with $k-1$ times, we obtain

$$\mathcal{F}\nabla^k\delta = \frac{(-1)^{2k}}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^2 + \left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^2\right]^k.$$

Thus,

$$\begin{aligned}
|\mathcal{F}\nabla^k\delta| &= \frac{1}{(2\pi)^{n/2}}\left|\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^2 + \left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^2\right|^k \\
&\leq \frac{M}{(2\pi)^{n/2}}\left|\xi_1^2+\ldots+\xi_n^2\right|^k \\
&\leq \frac{M}{(2\pi)^{n/2}}\|\xi\|^{4k},
\end{aligned}$$

where M is positive constant and $\|\xi\| = \left(\xi_1^2+\xi_2^2+\ldots+\xi_n^2\right)^{1/2}$, $\xi_i (i=1,2,\ldots,n) \in \mathbb{R}$. Hence we obtain $\mathcal{F}\nabla^k\delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution. Since $\mathcal{F}$ is $1-1$ transformation from the space $\mathcal{S}'$ of the tempered distribution to the real space $\mathbb{R}$, thus

$$\nabla^k\delta = \mathcal{F}^{-1}\frac{(-1)^{3k}}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^2 + \left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^2\right]^k.$$

That completes the proof.

Lemma 2.4. *Let L be the operator and defined by*

$$\mathrm{L} = \frac{\partial}{\partial t} - c^2\nabla^k, \tag{2.11}$$

where ∇^k is the operator iterated k times defined by (1.7) for $t \in (0,\infty)$ and c

is a positive constant. We obtain

$$E(x,t)=\frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k+i(\xi,x)\right]d\xi. \tag{2.12}$$

as an elementary solution of (2.11)in the spectrum $\Omega\subset\mathbb{R}^n$.

Proof. Let

$$\mathrm{L}E(x,t)=\delta(x,t),$$

where $E(x,t)$ is an elementary solution of operator L and δ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t}E(x,t)-c^2\left(\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2+\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2\right)^k E(x,t)=\delta(x)\delta(t). \tag{2.13}$$

If we apply the Fourier transform defined by (2.1) to both sides of the equation, we obtain

$$\frac{\partial}{\partial t}\widehat{E(\xi,t)}-c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k\widehat{E(\xi,t)}=\frac{1}{(2\pi)^{n/2}}\delta(t).$$

Thus

$$\widehat{E(\xi,t)}=\frac{H(t)}{(2\pi)^{n/2}}\exp\left(c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k\right),$$

where $H(t)$ is the Heaviside function. Since $H(t)=1$ for $t>0$. Therefore,

$$\widehat{E(\xi,t)}=\frac{1}{(2\pi)^{n/2}}\exp\left(c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k\right).$$

By inverse fourier transform method, we obtained

$$\begin{aligned}E(x,t)&=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{i(\xi,x)}\widehat{E(\xi,t)}\,d\xi\\&=\frac{1}{(2\pi)^{n/2}}\int_\Omega e^{i(\xi,x)}\widehat{E(\xi,t)}\,d\xi\end{aligned}$$

or

$$E(x,t)=\frac{1}{(2\pi)^n}\int_\Omega \exp\left(c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k+i(\xi,x)\right)d\xi. \tag{2.14}$$

That complete the proof.

Definition 2.6 We can extend $E(x,t)$ to $\mathbb{R}^n\times\mathbb{R}$ by setting

$$E(x,t)=\begin{cases}\dfrac{1}{(2\pi)^n}\displaystyle\int_\Omega \exp\left(c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k\right), & \text{for } t>0\\ 0, & \text{for } t\leq 0.\end{cases}$$

3. Main Results

Theorem 3.1. *Given the equation*

$$\frac{\partial}{\partial t}\left(\diamondsuit^k u(x,t)\right)-c^2\oplus^k u(x,t)=0, \tag{3.1}$$

with the initial condition

$$\diamondsuit^k u(x,0)=f(x), \tag{3.2}$$

where $(x,t)=(x_1,\ldots,x_n,t)\in\mathbb{R}^n\times(0,\infty), t$ is a time, c is a positive constant, $u(x,t)$ is an unknown function and $f(x)$ is a given generalized function for $x\in\mathbb{R}^n$. and $\diamondsuit^k$ is Diamond operator iterated k times. Then we obtain

$$u(x,t)=((-1)^kR^e_{2k}(x)*R^H_{2k}(x))*E(x,t)*f(x) \tag{3.3}$$

as a solution of (3.1) where $R^H_{2k}(\upsilon)$ and $R^e_{2k}(x)$ are given by (2.4)and (2.6) respectively and $E(x,t)$ is given by (2.9).

Proof. The equation(3.1) can be written in the form

$$\frac{\partial}{\partial t}\left(\diamondsuit^k u(x,t)\right)-c^2\diamondsuit^k\nabla^k u(x,t)=0. \tag{3.4}$$

Let $w(x,t) = \Diamond^k u(x,t)$. The above equation can be written as

$$\frac{\partial}{\partial t} w(x,t) - c^2 \nabla^k w(x,t) = 0. \tag{3.5}$$

Applying the $n-$ dimensional Fourier transform to both sides of (3.5)and by Lemma 2.3 we obtain

$$\widehat{w}(\xi,t) = K(\xi) \exp\left(c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^2 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \right)^k \right) \tag{3.6}$$

where $K(\xi)$ is constant and $\widehat{u}(\xi,0) = K(\xi)$.Now, by (2.11) we have

$$K(\xi) = \widehat{w}(\xi,0) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)\, dx \tag{3.7}$$

and by the inversion in (2.2), (2.13) and (2.14) we obtain

$$w(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{w}(\xi,t)\, d\xi$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\xi,x)} e^{-i(\xi,y)} f(y) \exp\left(c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^2 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \right)^k \right) dy\, d\xi.$$

Thus

$$w(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\xi,x-y)} \exp\left(c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^2 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \right)^k \right) f(y)\, dy\, d\xi$$

or

$$w(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left(c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2 \right)^2 + \left(\sum_{j=p+1}^{p+q} \xi_j^2 \right)^2 \right)^k + i(\xi, x-y) \right) f(y)\, dy\, d\xi. \tag{3.8}$$

Set

$$E(x,t)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\exp\left(c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k+i(\xi,x)\right)d\xi. \tag{3.9}$$

We choose $\Omega\subset\mathbb{R}^n$ be the spectrum of $E(x,t)$ and by (2.6), we have

$$\begin{aligned}E(x,t)&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\exp\left(c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k+i(\xi,x)\right)d\xi\\&=\frac{1}{(2\pi)^n}\int_{\Omega}\exp\left(c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k+i(\xi,x)\right)d\xi.\end{aligned} \tag{3.10}$$

Thus (3.8) can be written in the convolution form

$$w(x,t)=E(x,t)*f(x). \tag{3.11}$$

Since $E(x,t)$ exists, then

$$\begin{aligned}\lim_{t\to0}E(x,t)&=\frac{1}{(2\pi)^n}\int_{\Omega}e^{i(\xi,x)}\,d\xi\\&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{i(\xi,x)}\,d\xi\\&=\delta(x),\quad\text{for}\quad x\in\mathbb{R}^n.\end{aligned} \tag{3.12}$$

See [2, p.396].

Substituting $w(x,t)=\diamondsuit^k u(x,t)$ into (3.11), we obtain

$$\diamondsuit^k u(x,t)=E(x,t)*f(x) \tag{3.13}$$

Taking convolution both sides of (3.13) by the function $((-1)^kR^e_{2k}(x)*R^H_{2k}(\upsilon))$, we obtain

$$((-1)^kR^e_{2k}(x)*R^H_{2k}(\upsilon))*\diamondsuit^k u(x,t)=((-1)^kR^e_{2k}(x)*R^H_{2k}(\upsilon))*E(x,t)*f(x).$$

By properties of convolution

$$\diamondsuit^k((-1)^kR^e_{2k}(x)*R^H_{2k}(\upsilon))*u(x,t)=((-1)^kR^e_{2k}(x)*R^H_{2k}(\upsilon))*E(x,t)*f(x).$$

By Lemma 2.2, we obtain

$$\delta * u(x,t) = ((-1)^k R^e_{2k}(x) * R^H_{2k}(\upsilon)) * E(x,t) * f(x).$$

or

$$u(x,t) = ((-1)^k R^e_{2k}(x) * R^H_{2k}(\upsilon)) * E(x,t) * f(x). \tag{3.14}$$

is a solution of (3.1). Now, from (3.4) and by the continuity of convolution,

$$\begin{aligned}\lim_{t\to 0} \Diamond^k u(x,t) &= \lim_{t\to 0} (E(x,t) * f(x)) \\ &= \delta * f(x) \\ &= f(x)\end{aligned}$$

Theorem 3.2. *(The properties of the Diamond Heat Kernel*

$$((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) * E(x,t))$$

(1) $((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) * E(x,t)$ *exists and is tempered distribution.*

(2) $((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) * E(x,t) \in \mathbb{C}^\infty$ *the space of continuous function with infinitely differentiable.*

(3) $\lim_{t\to 0} \big((-1)^k R^e_{2k}(x) * R^H_{2k}(x) * E(x,t)\big) = ((-1)^k R^e_{2k}(x) * R^H_{2k}(x))$.

(4) $((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) * E(x,t)$*is bounded and*

$$|E(x,t)| \leq \frac{2^{2-n}}{\pi^{n/2}} \frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}, \text{ for } t > 0,$$

where $M(t)$ *is a function of* t *in the spectrum* Ω *and* Γ *denote the Gamma function. Thus* $E(x,t)$ *is bounded for any fixed* $t > 0$.

(5) $\frac{\partial}{\partial t}\Diamond^k((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) * E(x,t) - c^2\nabla^k((-1)^k R^e_{2k}(x) * R^H_{2k}(x))$ $*E(x,t) = 0$

Proof. (1) Since $E(x,t)$ is a tempered distribution and $((-1)^k R^e_{2k}(x) * R^H_{2k}(x))$ also a tempered distribution with compact support. Then

$$((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) * E(x,t)$$

exists and is a tempered distribution.
(2) We have

$$\frac{\partial^n}{\partial x^n}\Big(((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) * E(x,t)\Big) = ((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) * \frac{\partial^n}{\partial x^n}E(x,t)$$

Since $E(x,t)$ is infinitely differentiable. Thus $((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) * E(x,t) \in \mathbb{C}^\infty$.
(3) By the continuity of the convolution,

$$((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) * E(x,t) \to ((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) * \delta \ \ as \ \ t \to 0.$$

Thus

$$\lim_{t\to 0}\Big(((-1)^k R^e_{2k}(x) * R^H_{2k}(x)) * E(x,t)\Big) = ((-1)^k R^e_{2k}(x) * R^H_{2k}(x))$$

(4) We have

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \exp\left(c^2 t\left(\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 + \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right)^k\right) + i(\xi,x)\right) d\xi.$$

$$|E(x,t)| \leq \frac{1}{(2\pi)^n}\int_\Omega \exp\left(c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^2 + \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)\right)^2\right)^k d\xi.$$

By changing to bipolar coordinates

$$\xi_1 = r\omega_1,\ \xi_2 = r\omega_2, \ldots,\ \xi_p = r\omega_p \quad \text{and}$$

$$\xi_{p+1} = s\omega_{p+1},\ \xi_{p+2} = s\omega_{p+2}, \ldots,\ \xi_{p+q} = s\omega_{p+q}$$

where $\sum_{i=1}^{p}\omega_i^2 = 1$ and $\sum_{j=p+1}^{p+q}\omega_j^2 = 1$. Thus

$$|E(x,t)| \leq \frac{1}{(2\pi)^n}\int_\Omega \exp\left[c^2 t\left(r^2+s^2\right)\right] r^{p-1}s^{q-1}\,dr\,ds\,d\Omega_p\,d\Omega_q,$$

where $d\xi = r^{p-1}s^{q-1}\,dr\,ds\,d\Omega_p\,d\Omega_q$, $d\Omega_p$ and Ω_q are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Since $\Omega \subset \mathbb{R}^n$ is the spectrum of

$E(x,t)$ and we suppose $0 \le r \le R$ and $0 \le s \le T$ where R and T are constants. Thus we obtain

$$\begin{aligned}|E(x,t)| &\le \frac{\Omega_p\,\Omega_q}{(2\pi)^n}\int_0^R\int_0^T \exp\left[c^2t\left(r^2+s^2\right)\right] r^{p-1}s^{q-1}\,ds\,dr\\ &= \frac{\Omega_p\,\Omega_q}{(2\pi)^n}M(t) \quad \text{for any fixed } t>0 \ \text{ in the spectrum } \Omega\\ &= \frac{2^{2-n}}{\pi^{n/2}}\frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})},\end{aligned}$$

where

$$M(t) = \int_0^R\int_0^T \exp\left[c^2t\left(r^2+s^2\right)\right] r^{p-1}s^{q-1}\,ds\,dr$$

is a function of t, $\Omega_p = \dfrac{2\pi^{p/2}}{\Gamma\left(\frac{p}{2}\right)}$ and $\Omega_q = \dfrac{2\pi^{p/2}}{\Gamma\left(\frac{q}{2}\right)}$. Thus, for any fixed $t > 0$, $E(x,t)$ is bounded. Since $((-1)^kR^e_{2k}(x)*R^H_{2k}(x))$ is a tempered distribution with compact support that is $(-1)^kR^e_{2k}(x)*R^H_2(\upsilon)$ is bounded. Thus $(-1)^kR^e_{2k}(x)*R^H_2(x)*E(x,t)$ is bounded.

(5) Since

$$\begin{aligned}\Diamond^k\left(((-1)^kR^e_{2k}(x)*R^H_{2k}(x))*E(x,t)\right) &= \Diamond^k((-1)^kR^e_{2k}(x)*R^H_{2k}(x))*E(x,t)\\ &= \delta*E(x,t)\\ &= E(x,t).\end{aligned}$$

Thus

$$\frac{\partial}{\partial t}\Diamond^k\left((-1)^kR^e_{2k}(x)*R^H_{2k}(x))*E(x,t)\right) = \frac{\partial}{\partial t}E(x,t). \tag{3.15}$$

And

$$\begin{aligned}\oplus^k\left(((-1)^kR^e_{2k}(x)*R^H_{2k}(x))*E(x,t)\right) &= \nabla^k\left(\Diamond^k((-1)^kR^e_{2k}(x)*R^H_{2k}(x))*E(x,t)\right)\\ &= \nabla^k\left(\delta*E(x,t)\right)\\ &= \nabla^kE(x,t).\end{aligned} \tag{3.16}$$

By (3.8), (3.9), we get

$$\frac{\partial}{\partial t}\Diamond^k\left((-1)^kR^e_{2k}(x)*R^H_{2k}(x))*E(x,t)\right) - c^2\oplus^k\left((-1)^kR^e_{2k}(x)*R^H_{2k}(x))*E(x,t)\right) = 0$$

This complete the proof.

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