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ABOUT ONE METHOD FOR ESTIMATING THE ROOTS OF TRANSCENDENTAL EQUATIONS

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Abstract: Were considered transcendental equations with trigonometric and hyperbolic functions. Were obtained two-sided estimates for all their roots.

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1. Introduction

Transcendental equations often arise when solving spectral problems for differential equations, for example, [1]. In the paper [2] were studied equation

$$\cos\mu\sinh\mu + \sin\mu\cosh\mu = 0 \tag{1}$$

(or $\tan\mu = -\tanh\mu$) and others. For positive roots of equation (1) were obtained formula $\mu_k = -\pi/4 + \pi k + \varepsilon_k$, where $\varepsilon_k > 0$, $\lim\limits_{k\to\infty} \varepsilon_k = 0$.

In this paper we consider a more general equations than equation (1), and we obtain more accurate two-sided estimates for their roots.

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2. Transcendental equations

Consider equation

$$\tan(az) = -\tanh(bz), \quad a,\, b > 0. \tag{2}$$

Theorem 1. *The equation (2) has a countable set of roots which consists of zero, real numbers*

$$z_k^{(1),(2)} = \pm\Big(-\frac{\pi}{4a} + \frac{\pi}{a}k + \varepsilon_k\Big), \quad \frac{1}{4a}e^{\pi/2(1-b/a)}e^{-2\pi k} < \varepsilon_k < \frac{\pi}{2a}e^{\pi/2}e^{-2\pi k},$$

and purely imaginary numbers

$$z_k^{(3),(4)} = \pm i\Big(-\frac{\pi}{4b} + \frac{\pi}{b}k + \varepsilon'_k\Big), \quad \frac{1}{4b}e^{\pi/2(1-a/b)}e^{-2\pi k} < \varepsilon'_k < \frac{\pi}{2b}e^{\pi/2}e^{-2\pi k},$$

where $k = 1, 2, \dots$.

Proof. Obviously $z = 0$ is a root of (2). Let $z = x + iy$, $z \neq 0$.

Case 1. Let $y = 0$ then

$$\tan(ax) = -\tanh(bx). \tag{3}$$

We see from the graphics of functions $f_1(x) = \tan(ax)$ and $f_2(x) = -\tanh(bx)$ that equation (3) has a single root x_k in each interval $\big(-\pi/(2a) + \pi k/a, \pi k/a\big)$ and

$$x_k = -\frac{\pi}{4a} + \frac{\pi}{a}k + \varepsilon_k,$$

where $\varepsilon_k > 0$, $\varepsilon_{k+1} < \varepsilon_k$, $\varepsilon_1 < \pi/(4a)$, $k = 1,\, 2,\, \dots$.

Then for the values x_k we have

$$1 + \tan(ax_k) = 1 - \tanh(bx_k),$$

$$\tan\frac{\pi}{4} + \tan\big(-\frac{\pi}{4} + \pi k + a\varepsilon_k\big) = 1 - \tanh(s + b\varepsilon_k),$$

where $s = -\pi/4 + \pi k$. Then

$$\tan\frac{\pi}{4} - \tan\big(\frac{\pi}{4} - a\varepsilon_k\big) = 1 - \frac{\tanh s + \tanh(b\varepsilon_k)}{1 + \tanh s \cdot \tanh(b\varepsilon_k)},$$

$$\frac{\sin(a\varepsilon_k)}{\cos\pi/4\,\cos(\pi/4 - a\varepsilon_k)} = \frac{(1-\tanh s)(1-\tanh(b\varepsilon_k))}{1 + \tanh s \cdot \tanh(b\varepsilon_k)}.$$

The left side of the equation is bounded from below and from above. On the one hand we have

$$\frac{\sin(a\varepsilon_k)}{\cos\pi/4\,\cos(\pi/4-a\varepsilon_k)} > \frac{\frac{2\sqrt{2}}{\pi}a\varepsilon_k}{\frac{\sqrt{2}}{2}\cdot 1} = \frac{4a\varepsilon_k}{\pi}, \text{ if } 0<\varepsilon_k<\pi/(4a).$$

Then

$$\varepsilon_k < \frac{\pi}{4a}\frac{(1-\tanh s)(1-\tanh(b\varepsilon_k))}{1+\tanh s\cdot\tanh(b\varepsilon_k)} < \frac{\pi}{4a}\frac{2e^{-2s}\cdot 1}{1} <$$
$$< \frac{\pi}{2a}e^{-2s} = \frac{\pi}{2a}e^{\pi/2-2\pi k}. \tag{4}$$

On the other hand we have

$$\frac{\sin(a\varepsilon_k)}{\cos\pi/4\,\cos(\pi/4-a\varepsilon_k)} < \frac{a\varepsilon_k}{\left(\frac{\sqrt{2}}{2}\right)^2} = 2a\varepsilon_k.$$

Then

$$\varepsilon_k > \frac{1}{2a}\frac{(1-\tanh s)(1-\tanh(b\varepsilon_k))}{1+\tanh s\cdot\tanh(b\varepsilon_k)} > \frac{1}{2a}\frac{e^{-2s}e^{-2b\varepsilon_k}}{2} >$$
$$> \frac{1}{4a}e^{\pi/2-2\pi k-b\pi/(2a)} = \frac{1}{4a}e^{\pi/2(1-b/a)}e^{-2\pi k}. \tag{5}$$

In obtaining estimates (4) and (5) were used obvious inequality $e^{-2x} < 1-\tanh x < 2e^{-2x}$, $x>0$.

So

$$x_k = -\frac{\pi}{4a} + \frac{\pi}{a}k + \varepsilon_k,$$

where $1/(4a)e^{\pi/2(1-b/a)}e^{-2\pi k} < \varepsilon_k < \pi/(2a)e^{\pi/2}e^{-2\pi k}$.

Case 2. If $x=0$ then $\tan(aiy) = -\tanh(biy)$ or $\tanh(ay) = -\tan(by)$. In this case we obtain

$$y_k = -\frac{\pi}{4b} + \frac{\pi}{b}k + \varepsilon'_k,$$

where $1/(8b)e^{\pi/2}e^{-2\pi k} < \varepsilon'_k < \pi/(2b)e^{\pi/2}e^{-2\pi k}$, $k=1,2,\ldots$.

Case 3. It can be shown that equation (2) has no other complex roots $x = x+iy$ except those found in Case 2. It is proved similarly [2]. □

Next, consider equation

$$\cos(az)\cosh(bz) = 1, \quad a,\, b > 0. \tag{6}$$

In his book [3], Rayleigh found 6 positive roots of the simpler equation $\cos m\cosh m = 1$ and obtained an approximate formula for large values $m_k \approx \pi k + \pi/2$.

Theorem 2. *The equation (6) has a countable set of roots which consists of zero, real numbers* $\pm z_k$ *and purely imaginary numbers* $\pm iz_k$*, where*

$$z_k = \frac{\pi}{2a} + \frac{\pi}{a}k + (-1)^{k-1}\varepsilon_k,$$

where

$$\frac{1}{a}e^{-b\pi/2a}\, e^{-2b\pi n/a} < \varepsilon_{2n} < \frac{\pi}{a\sqrt{2}}e^{-b\pi/4a}\, e^{-2b\pi n/a},$$

$$\frac{1}{a}e^{-3b\pi/4a}\, e^{-b\pi(2n-1)/a} < \varepsilon_{2n-1} < \frac{\pi}{a\sqrt{2}}e^{-b\pi/2a}\, e^{-b\pi(2n-1)/a},$$

$n = 1, 2, \ldots$.

Proof. Obviously $z = 0$ is a root of (6). Let $z = x + iy$, $z \neq 0$.

Case 1. Let $y = 0$ then

$$\cos(ax)\cosh(bx) = 1. \tag{7}$$

We see from the graphics of functions $f_1(x) = \cos(ax)$ and $f_2(x) = 1/\cosh(bx)$ that equation (7) has next roots:

$$x_k = \frac{\pi}{2a} + \frac{\pi}{a}k + (-1)^k\varepsilon_k,$$

where $\varepsilon_k > 0$, $\varepsilon_{k+1} < \varepsilon_k$, $\varepsilon_1 < \pi/(4a)$, $k = 1, 2, \ldots$.

Then we substitute the values x_k into (7):

$$\cos\left(\frac{\pi}{2} + \pi k + (-1)^{k-1}a\varepsilon_k\right) = \frac{1}{\cosh(\frac{b\pi}{2a} + \frac{b\pi k}{a} + (-1)^{k-1}b\varepsilon_k)},$$

$$\sin(a\varepsilon_k) = \frac{1}{\cosh(s + (-1)^{k-1}b\varepsilon_k)}, \quad s = \frac{b\pi}{2a} + \frac{b\pi k}{a}.$$

i) If $k = 2n$ (even number) then

$$\sin(a\varepsilon_{2n}) = \frac{1}{\cosh(s - b\varepsilon_{2n})}.$$

On the one hand we have

$$\sin(a\varepsilon_{2n}) > \frac{2a\sqrt{2}}{\pi}\varepsilon_{2n}, \quad \text{because} \quad \varepsilon_k < \frac{\pi}{4a} \quad \text{for all} \quad k.$$

Then

$$\varepsilon_{2n} < \frac{\pi}{2a\sqrt{2}}\,\frac{1}{\cosh(s - b\varepsilon_{2n})} < \frac{\pi}{2a\sqrt{2}}\,\frac{1}{\cosh(s - b\pi/4a)} <$$

$$< \frac{\pi}{a\sqrt{2}e^{s-b\pi/4a}} = \frac{\pi}{a\sqrt{2}}e^{-b\pi/4a}e^{-2b\pi n/a}.$$

On the other hand we have $\sin(a\varepsilon_{2n}) < a\varepsilon_{2n}$. Then

$$\varepsilon_{2n} > \frac{1}{a}\frac{1}{\cosh(s-b\varepsilon_{2n})} > \frac{1}{a\cosh s} = \frac{1}{a}e^{-b\pi/2a}e^{-2b\pi n/a}.$$

And we obtain the estimate

$$\frac{1}{a}e^{-b\pi/2a}e^{-2b\pi n/a} < \varepsilon_{2n} < \frac{\pi}{a\sqrt{2}}e^{-b\pi/4a}e^{-2b\pi n/a}.$$

ii) If $k = 2n-1$ (odd number) then

$$\sin(a\varepsilon_{2n-1}) = \frac{1}{\cosh(s+b\varepsilon_{2n-1})}.$$

On the one hand we have

$$\sin(a\varepsilon_{2n-1}) > \frac{2a\sqrt{2}}{\pi}\varepsilon_{2n-1}.$$

Then

$$\varepsilon_{2n-1} < \frac{\pi}{2a\sqrt{2}}\frac{1}{\cosh(s+b\varepsilon_{2n-1})} < \frac{\pi}{2a\sqrt{2}}\frac{1}{\cosh s} = \frac{\pi}{a\sqrt{2}}e^{-b\pi/2a}\,e^{-b\pi(2n-1)/a}.$$

On the other hand we have $\sin(a\varepsilon_{2n-1}) < a\varepsilon_{2n-1}$. Then

$$\varepsilon_{2n} > \frac{1}{a}\frac{1}{\cosh(s+b\varepsilon_{2n-1})} > \frac{1}{a\cosh(s+b\pi/4a)} = \frac{1}{a}e^{-3b\pi/4a}e^{-b\pi(2n-1)/a}.$$

And we obtain the inequality

$$\frac{1}{a}e^{-3b\pi/4a}e^{-b\pi(2n-1)/a} < \varepsilon_{2n-1} < \frac{\pi}{a\sqrt{2}}e^{-b\pi/2a}\,e^{-b\pi(2n-1)/a}.$$

Case 2. If $x = 0$ then $\cos(iay)\cosh(iby) = 1$ or $\cos(ay)\cosh(by) = 1$ and we have case 1.

Case 3. Now we prove that equation (6) doesn't have other complex roots. Let $z = x + iy$, $x \neq 0$, $y \neq 0$. From (6) we have

$$\begin{cases} \cos(ax)\cos(by)\cosh(ay)\cosh(bx) + \sin(ax)\sin(by)\sinh(ay)\sinh(bx) = 1, \\ \sin(ax)\cos(by)\sinh(ay)\cosh(bx) - \cos(ax)\sin(by)\cosh(ay)\sinh(bx) = 0 \end{cases} \tag{8}$$

or

$$\begin{cases} \cos(ax-by)\cosh(ay+bx)+\cos(ax+by)\cosh(ay-bx)=2, \\ \sin(ax+by)\sinh(ay-bx)+\sin(ax-by)\sinh(ay+bx)=0. \end{cases}$$

If we indicate $ax-by=n$, $ay+bx=m$, $ax+by=p$, $ay-bx=t$ then

$$\begin{cases} \cos n\,\cosh m+\cos p\,\cosh t=2, \\ \sin p\,\sinh t+\sin n\,\sinh m=0. \end{cases}$$

From last system we receive $(\cosh t-\cos p)^2=(\cos n-\cosh m)^2$. Further $\cosh t-\cos p=\cos n-\cosh m$ or $\cosh t-\cos p=\cosh m-\cos n$. In the first case we have $\cosh t+\cosh m=\cos p+\cos n$ and $\cosh t+\cosh m\geq 2$, $\cos p+\cos n\leq 2$ ie $\cosh t=\cosh m=\cos p=\cos n=1$ then $x=y=0$.

In the second case we have $\cosh t-\cosh m=\cos p-\cos n$ ie $\sinh(t+m)/2\cdot\sinh(t-m)/2=\sin(n+p)/2\cdot\sin(n-p)/2$.

Then

$$\sinh(ay)\sinh(bx)=\sin(ax)\sin(by). \tag{9}$$

We can verify that the values $x=\pi n/a$ are not solutions of the system (8) for any y. Therefore, we can obtain the equivalent equation from (9):

$$\frac{a\sinh(bx)}{b\sin(ax)}=\frac{a\sin(by)}{b\sinh(ay)}. \tag{10}$$

It can be proved that for function $f(x)=\frac{a\sinh(bx)}{b\sin(ax)}$ with $x\neq\pi n/a$ we have $|f(x)|>1$, but for function $g(y)=\frac{a\sin(by)}{b\sinh(ay)}$ with $y\neq 0$ we have $|g(y)|<1$. So equation (10) doesn't have roots. □

3. Corollary

As we see from these equations this method of estimating of roots can be applied to equations with trigonometric and hyperbolic functions.

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