SUBMERSION OF SEMI-INVARIANT SUBMANIFOLDS OF LORENTZIAN PARA-SASAKIAN MANIFOLDS

Vibha Srivastava\textsuperscript{1}§, P.N. Pandey\textsuperscript{2}
\textsuperscript{1,2}Department of Mathematics
University of Allahabad
Allahabad, 211002, INDIA

Abstract: In this paper, we discuss submersion of semi-invariant submanifolds of Lorentzian para-Sasakian manifolds and derive some results on its geometry. We also derive some curvature relations.

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1. Introduction

In 1966, O’Neill [1] initiated the study of Riemannian submersion. Semi-Riemannian submersion was introduced by O’Neill in [2]. In 1981, Kobayashi [3] studied CR-Submanifold of Sasakian manifold whereas Benjancu [4, 5] introduced the CR-submanifolds of Kähler manifold. Submersion of CR-submanifolds of nearly trans-Sasakian manifold were studied by Jamali and shahid [6]. Submersion of semi-invariant submanifolds of trans-Sasakian manifold were studied by Jamali et al [7]. In [8], Matsumoto introduced the notion of Lorentzian para-Sasakian manifold. In [9], the authors defined the same notion independently and they obtained many results about this type of manifold. In this paper we studied submersion of semi-invariant submanifolds of Lorentzian para-Sasakian manifolds.

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§Correspondence author
2. Preliminaries

Let $\overline{M}$ be an $n$–dimensional Lorentzian manifold with a tensor field $\phi$ of type $(1, 1)$, a vector field $\xi$ and a 1- form $\eta$ which satisfy

\[
\phi^2 X = X + \eta(X) \xi, \quad \eta(\xi) = -1, 
\]

\[
g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), 
\]

\[
g(X, \xi) = \eta(X), 
\]

\[
g(X, \phi Y) = g(\phi X, Y), 
\]

for any vector fields $X, Y$ tangents to $\overline{M}$, it is called Lorentzian almost para-contact manifold [8]. Also in a Lorentzian almost para-contact structure the following relations hold :

\[
\phi(\xi) = 0, \quad \eta(\phi X) = 0, \quad \text{rank } \phi = n - 1.
\]

A Lorentzian almost para-contact manifold $\overline{M}$ is called Lorentzian para-Sasakian ($LP$–Sasakian) manifold if [10]

\[
(\nabla_X \phi) Y = g(X, Y) \xi + \eta(Y) X + 2\eta(X) \eta(Y) \xi. 
\]

**Definition 1.** A $m$–dimensional Riemannian submanifold $M$ of a Lorentzian para-Sasakian manifold $\overline{M}$ is called a semi-invariant submanifold if $\xi$ is tangent to $M$ and it is endowed with a pair of orthogonal differentiable distributions $(D, D^\perp)$, which satisfies (1) $TM = D \oplus D^\perp \oplus \xi$, where $\oplus$ denotes the orthogonal direct sum, (2) the distribution $D_x : x \mapsto D \subset T_x M$ is invariant under $\phi$ i.e $\phi D_x \subset D_x$ for each $x \in M$ (3) the orthogonal complementary distribution $D^\perp : x \mapsto D^\perp \subset T_x M$ of the distribution $D$ on $M$ is totally real i.e $\phi D^\perp \subset T_x^\perp M$ where $T_x M$ and $T_x^\perp M$ are the tangent space and the normal space of $M$ at $x$ respectively.

Let the dimension of $D$ (resp. $D^\perp$) be $2p$ (resp. $q$), where $2p + q = m - 1$. If $p = 0$ (resp. $q = 0$) the submanifold $M$ becomes anti-invariant (resp. invariant)submanifold. A generic submanifold $M$ satisfies $D^\perp = \dim T_x^\perp M$. A submanifold is called proper if it is neither invariant nor anti-invariant. It is easy to see that any hypersurface to which the vector field $\xi$ is tangent is a typical example of semi-invariant submanifold. Where $D$ and $D^\perp$ are the horizontal and vertical distribution respectively. Let $\nabla$ (resp. $\nabla$) be the covariant differentiation with respect to Levi-Civita connection on $\overline{M}$ (resp. $M$).
The Gauss and Weingarten formulas for $M$ are respectively given by
\[ \nabla_X Y = \nabla_X Y + \sigma (X, Y), \tag{6} \]
and
\[ \nabla_X N = -A_N X + \nabla^\perp_X N, \tag{7} \]
for $X, Y \in TM, N \in T^\perp M$, where $\sigma$ (resp. $A$) is the second fundamental form (resp. tensor) of $M$ in $\overline{M}$, and $\nabla$ denote the operator of the normal connection.

The projection of $TM$ to $D$ and $D^\perp$ are denoted by $h$ and $v$ respectively i.e, for any $X \in TM$ we have
\[ X = \sigma X + vX + \eta (X) \xi. \tag{9} \]
The normal bundle to $M$ has the decomposition
\[ T^\perp M = \phi D^\perp \oplus n_1, \tag{10} \]
where $g (\phi D^\perp, n_1) = 0$. For any $U \in T^\perp M$, we put
\[ U = nU + mU, \tag{11} \]
where $nU \in \phi D^\perp, mU \in n_1$. From the above equation we have
\[ \phi U = \phi nU + \phi mU, U \in T^\perp M, \phi nU \in D^\perp, \phi mU \in n_1. \tag{12} \]

**Definition 2.** Let $M$ be a semi-invariant submanifold of a Lorentzian para-Sasakian manifolds $\overline{M}$ and $M'$ be an Lorentzian manifold with structure $(\phi', \xi', \eta', g')$. Assume that there is a submersion $\pi : M \to M'$ such that (i) $D^\perp = \ker \pi_* : TM \to TM'$ is the tangent mapping to $\pi$, (ii) $\pi_* : D_p \oplus \{ \xi \} \to T_{\pi(p)}M'$ is an isometry for each $p \in M$ which satisfies $\pi_* \circ \phi = \phi' \circ \pi_*; \eta = \eta' \circ \pi_*; \pi_* (\xi_p) = \xi'_p$, where $T_{\pi(p)}M'$ denotes the tangent space of $M'$ at $\pi(p)$.

A vector $X$ on $M$ is said to be basic if, $X \in D_p \oplus \xi$ and $X$ is $\pi$–related to a vector field on $M'$ i.e there exists a vector field $X_* \in TM'$ such that $\pi_* (X_p) = X_{*\pi(p)}$ for each $p \in M$. Note that, by condition (ii) of the above definition 2, we have that the structural vector field $\xi$ is a basic vector field.

**Lemma 3.** Let $X, Y$ be basic vector fields on $M$. Then
(i) $g(X, Y) = g' (X_*, Y_*) \circ \pi$,
(ii) the component $\sigma ([X, Y]) + \eta ([X, Y] \xi) = [X_*, Y_*]$,
(iii) $[U, X] \in D^\perp$ for any $U \in D^\perp$,
(iv) $\sigma (\nabla_X Y) + \eta (\nabla_X Y) \xi$, is a basic vector field corresponding to $\nabla^*_{X_*} Y_*$, where $\nabla^*$ denote the Levi-Civita connection on $M'$. 

For basic vector fields on $M$, we define the operator $\tilde{\nabla}^*$ corresponding to $\nabla^*$ by setting $\tilde{\nabla}_X^*Y = \sigma([X,Y]) + \eta([X,Y] \xi)$ for $X,Y \in (D_p \oplus \{\xi\})$. By $(iv)$ of Lemma 3, $\tilde{\nabla}_X^*Y$ is a basic vector field and we have

$$\pi_*(\tilde{\nabla}_X^*Y) = \nabla_{\pi_*X}^*Y_*$.$$

(13)

Define the tensor field $C$ by

$$\nabla_XY = \tilde{\nabla}_X^*Y + C(X,Y),$$

(14)

$X,Y \in (D_p \oplus \{\xi\})$, where $C(X,Y)$ is the verticle part of $\nabla_XY$. It is known that $C$ is skew-symmetric and satisfies

$$C(X,Y) = \frac{1}{2}v[X,Y],$$

(15)

where $X,Y \in (D_p \oplus \{\xi\})$.

The curvature tensor $R, R^*$ of the connection $\nabla, \nabla^*$ on $M$ and $M'$ respectively are related by

$$R(X,Y,Z,W) = R^*(X_*,Y_*,Z_*,W_*) - g(C(Y,Z),C(X,W))$$

$$+ g(C(X,Z),C(Y,W)) + 2g(C(X,Y),C(Z,W))$$

(16)

for $X,Y,Z,W \in (D_p \oplus \{\xi\})$, where $\pi_*X = X_*, \pi_*Y = Y_*, \pi_*Z = Z_*$ and $\pi_*W = W_* \in \chi M'$. For Lorentzian para-Sasakian manifold $\overline{M}$ we prove

**Proposition 4.** Let $\pi : M \to M'$ be a submersion of semi-invariant submanifold of a Lorentzian para-Sasakian manifold $\overline{M}$ onto a Lorentzian manifold $M'$. Then we have

$$\left(\tilde{\nabla}_X^*\phi\right)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

(17)

$$C(X,\phi Y) = \phi n\sigma(X,Y),$$

(18)

$$\phi C(X,Y) = n\sigma(X,\phi Y),$$

(19)

$$\phi m\sigma(X,Y) = m\sigma(X,\phi Y),$$

(20)

for any $X,Y \in (D_p \oplus \{\xi\})$.

**Proof.** For any $X,Y \in (D \oplus \{\xi\})$ and by using Gauss formula (6), decomposition equation (11) and (14), we obtain

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) = \nabla_X Y + n\sigma(X,Y) + m\sigma(X,Y)$$
\[
\tilde{\nabla}_X^* Y + C(X, Y) + n\sigma(X, Y) + m\sigma(X, Y). \tag{21}
\]

And
\[
\phi\tilde{\nabla}_X Y = \phi\tilde{\nabla}_X^* Y + \phi C(X, Y) + \phi n\sigma(X, Y) + \phi m\sigma(X, Y). \tag{22}
\]

Putting \(Y = \phi Y\) in equation (21), we get
\[
\tilde{\nabla}_X \phi Y = \tilde{\nabla}_X^* \phi Y + C(X, \phi Y) + n\sigma(X, \phi Y) + m\sigma(X, \phi Y). \tag{23}
\]

Using the definition of Lorentzian para-Sasakian manifold, we find
\[
\left(\tilde{\nabla}_X^* \phi\right) Y = \tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y = g(X, Y) \xi + \eta(Y) X + 2\eta(X) \eta(Y) \xi. \tag{24}
\]

Substituting (22) and (23) in (24), we get
\[
\tilde{\nabla}_X^* \phi Y + C(X, \phi Y) + n\sigma(X, \phi Y) + m\sigma(X, \phi Y) - \phi \tilde{\nabla}_X^* Y - \phi C(X, Y) - \phi n\sigma(X, Y) - \phi m\sigma(X, Y)
= g(X, Y) \xi + \eta(Y) X + 2\eta(X) \eta(Y) \xi. \tag{25}
\]

Comparing the components of \((D \oplus \{\xi\}), D^⊥, \phi D^⊥\) and \(n_1\) respectively on both sides in the above equation, we get the required results.

**Proposition 5.** Let \(\pi : M \to M'\) be a submersion of semi-invariant submanifold of a Lorentzian para-Sasakian manifold \(\overline{M}\) onto a Lorentzian manifold \(M'\). Then \(M'\) is also a Lorentzian para-Sasakian manifold.

**Proof.** From equation (17) of proposition (4), we have
\[
\left(\tilde{\nabla}_X^* \phi\right) Y = g(X, Y) \xi + \eta(Y) X + 2\eta(X) \eta(Y) \xi. \tag{26}
\]

Applying \(\pi_*\) to the above equation and using Lemma 3, equation (13) and definition of submersion, we derive
\[
\left(\tilde{\nabla}_{X_*}^* \phi'\right) Y_* = g'(X_*, Y_*) \xi' + \eta'(Y_*) X' + 2\eta'(X_*) \eta'(Y_*) \xi'. \tag{27}
\]

Hence \(M'\) is a Lorentzian para-Sasakian manifold.

**Proposition 6.** Let \(\pi : M \to M'\) be a submersion of semi-invariant submanifold of a Lorentzian para-Sasakian manifold \(\overline{M}\) onto a Lorentzian manifold \(M'\). Then

\(i\) \(n\sigma(X, \phi Y) + n\sigma(\phi X, Y) = 0, \)

\(ii\) \(n\sigma(\phi X, \phi Y) = -n\sigma(Y, X), \)

\(iii\) \(m\sigma(\phi X, \phi Y) = m\sigma(X, Y), \)

\(iv\) \(C(\phi X, \phi Y) = -C(X, Y), \)

for any \(X, Y \in (D \oplus \{\xi\}).\)
Proof. (i) Interchanging $X$ and $Y$ in equation (19) gives

$$\phi C (Y, X) = n\sigma (Y, \phi X) = n\sigma (\phi X, Y),$$

(28)

Then

$$n\sigma (X, \phi Y) + n\sigma (\phi X, Y) = \phi C (X, Y) + \phi C (Y, X) = \phi C (X, Y) - \phi C (X, Y) = 0.$$

(ii) Putting $X = \phi X$ in (19), we get

$$n\sigma (\phi X, \phi Y) = \phi C (\phi X, Y) = -\phi C (Y, \phi X).$$

(29)

Using (18) in (29), we deduce

$$n\sigma (\phi X, \phi Y) = -\phi C (Y, \phi X) = -\phi (\phi n\sigma (Y, X)) = -\phi^2 n\sigma (Y, X)$$

$$= -n\sigma (Y, X) - \eta (\sigma (X, Y)) \xi = -n\sigma (Y, X).$$

(iii) Putting $X = \phi X$ in (20) and using again the same equation, we find

$$m\sigma (\phi X, \phi Y) = \phi m\sigma (\phi X, Y) = \phi m\sigma (Y, \phi X) = \phi^2 m\sigma (Y, X) = m\sigma (X, Y).$$

(iv) Putting $X = \phi X$ in (18) and then using (19) yields

$$C (\phi X, \phi Y) = \phi n\sigma (\phi X, Y) = \phi n\sigma (Y, \phi X) = \phi^2 C (Y, X)$$

$$= C (Y, X) + \eta (C (Y, X)) \xi = -C (X, Y).$$

\[ \square \]

3. Curvature Relation

**Proposition 7.** Let $\pi : M \to M'$ be a submersion of semi-invariant submanifold of a Lorentzian para-Sasakian manifold $\overline{M}$ onto a Lorentzian manifold $M'$. Then the $\phi$–sectional curvature of $\overline{M}$ and $M'$ are related by

$$\overline{B} (X, Y) = B' (X, Y) - 2g (n\sigma (X, X), n\sigma (Y, Y)).$$

where $X, Y \in (D \oplus \{\xi\})$.

Proof. We know

$$\overline{B} (X, Y) = \overline{R} (X, \phi X, \phi Y, Y).$$

Putting $Y = \phi X, Z = \phi Y, W = Y$ in Gauss equation

$$\overline{R} (X, Y, Z, W) = R (X, Y, Z, W) - g (\sigma (X, W), \sigma (Y, Z))$$
we get

\[ g((X, Z), \sigma(Y, W)) + g((X, \sigma(Y)), \sigma(\phi(X, Y))) \]
Applying $\phi$ on both side of equation (19), we obtain
\[ C(X, Y) = \phi n\sigma(X, \phi Y). \] (32)

Using equation (32) in equation (31) give
\[
R(X, \phi X, \phi Y, Y) = R^*(X_*, \phi' X_*, \phi' Y_*, Y_*) - \|n\sigma(X, Y)\|^2 + \|n\sigma(X, \phi Y)\|^2
- 2g(n\sigma(X, X), n\sigma(Y, Y)).
\]

Putting the value of $R(X, \phi X, \phi Y, Y)$ in (30), we obtain
\[
R(X, \phi X, \phi Y, Y) = R^*(X_*, \phi' X_*, \phi' Y_*, Y_*) - 2g(n\sigma(X, X), n\sigma(Y, Y))
\]
or
\[
\overline{B}(X, Y) = B'(X_*, Y_*) - 2g(n\sigma(X, X), n\sigma(Y, Y)). \tag{33}
\]

**Proposition 8.** Let $\pi : M \to M'$ be a submersion of semi-invariant submanifold of a Lorentzian para- Sasakian manifold $\overline{M}$ onto a Lorentzian manifold $M'$. Then the $\phi$–sectional curvature of $\overline{M}$ and $M'$ are related by
\[
\overline{H}(X) = H'(X_*) - 2\|n\sigma(X, X)\|^2,
\]
where $X, Y \in (D \oplus \{\xi\})$.

**Proof.** Putting $X = Y$ in equation (33) we obtain
\[
\overline{B}(X, X) = \overline{H}(X) = H'(X_*) - 2g(n\sigma(X, X), n\sigma(X, X))
= H'(X_*) - 2\|n\sigma(X, X)\|^2.
\]

Thus we get
\[
\overline{H}(X) = H'(X_*) - 2\|n\sigma(X, X)\|^2. \tag*{\Box}
\]

**Application:** Lorentzian para-Sasakian manifolds are used in the theory of Relativity and Newton’s law of gravitational field.
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References


