

NONHOLONOMIC FRAMES FOR FINSLER SPACES WITH A SPECIAL QUARTIC METRIC

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Abstract: The unified formalism which considers the motion of charged particles in an external electromagnetic field can be studied with the help of a nonholonomic Finsler frame which can be defined in quadric spaces also. Quartic spaces are members of a bigger class of the Finsler spaces with (α, β) metric. However, it's not known that whether a Finsler space with a special quartic metric have such a nonholonomic frame or not and also there is no direct method available yet to determine a nonholonomic frame for these spaces. The present study attempts to test that the indicatrix of special (α, β) quartic metric of form $F = \{\alpha^2(\alpha^2 + \epsilon\beta^2)\}^{1/4}$ is strictly convex. Further, it provides the nonholonomic frame for Finsler spaces equipped with a special quartic metric if it is strictly convex in nature.

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1. Introduction

In 1926, Vranceanu[16] has introduced the concept of a nonholonomic space

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which is more general than a Riemannian space. In a Finsler space with (α, β) metric, Matsumoto [11] introduces an important relation between the original Riemannian metric a_{ij} and the Finsler metric g_{ij} defined as

$$g_{ij}(x, \dot{x}) = \rho a_{ij}(x) + \rho_0 b_i(x) b_j(x) + \rho_{-1} (b_i(x) \dot{x}_i + b_j(x) \dot{x}_j) + \rho_{-2} \dot{x}_i \dot{x}_j, \quad (1)$$

where ρ , ρ_0 , ρ_{-1} and ρ_{-2} are some Finsler invariants but from this formula, we can not see any direct information about some properties of Finsler metric g_{ij} to which Riemannian metric a_{ij} posses. In [8], Holland introduces a new relation between Finsler metric g_{ij} and the Riemannian metric a_{ij} during the study of unified formalism that uses a nonholonomic Finsler frame on space-time arising from the consideration of a charged particle moving in an external electromagnetic field.

In fact, Ingarden [9] was first to point out that the Lorentz force law, in this case, could be written as geodesic equations on a Finsler space called Randers space [14]. In [4, 5] the physicist Beil has studied a generalized Lagrange metric and defines a relation between Generalized Lagrange metric g_{ij} and the Lorentz metric. In [1], Anastasiei and Shimada have studied a class of generalized Lagrange metrics and the resulting metric is called Beil metric. In this direction some work has been done in [6, 7] with Generalized Lagrange spaces with (α, β) -metrics and Finsler spaces with (α, β) -metrics. The idea we shall use from the above-mentioned papers are to view a Beil metric as a Finsler deformation of an original Riemannian metric. Then we can find a nonholonomic frame that generalizes the frame used by Beil in [4]. In [15], Shanker and Baby has also evaluated a nonholonomic frame for general (α, β) -metric.

Miron and Izumi have also studied nonholonomic Finsler frames and the induced Finsler connection in [12] for the so-called strongly non-Riemannian Finsler spaces. M. Matsumoto has also studied these nonholonomic frames, in [10], where he has called such frames the Miron frames of a strongly non-Riemannian Finsler space. The Miron frame is a natural generalization of the Berwald frame for a two-dimensional Finsler space or the Moor frame for a Finsler space of dimension three. In [13], Miron, Tavakol, Balan and Roxburgh have also defined the geometry of space-time and generalized Lagrange-Gauge theory.

For a Finsler space with (α, β) -metric, the fundamental tensor field might be thought as the result of two Finsler deformations. Then one can determine a corresponding frame for each of these two Finsler deformations. Consequently, a nonholonomic frame for a Finsler space with (α, β) -metric will appear as a product of two Finsler frames formerly determined.

In [2, 3], Antonelli and Bucataru have also investigated a nonholonomic

frame for two important classes of Finsler spaces that are Randers and Kropina spaces. As quartic spaces are members of a bigger class of the Finsler spaces with (α, β) special quartic metric, it appears a natural question: does a Finsler space with a special quartic metric have such a nonholonomic frame? As the fundamental tensor of a Finsler space with (α, β) -metric is not so easy to handle with, we didn't find so far, a direct method to determine a nonholonomic frame for these spaces.

In this paper, we consider a special (α, β) quartic metric

$$F = \{\alpha^2(\alpha^2 + \epsilon\beta^2)\}^{1/4},$$

where $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i$ and we investigate the condition that the indicatrix of this special quartic metric is strictly convex as well as find the nonholonomic frames for Finsler spaces equipped with a special quartic metric.

2. Preliminaries

An important class of Finsler spaces is the class of Finsler spaces with (α, β) -metrics [11]. In [14], the physicist Randers introduces a concept of Finsler space with (α, β) -metric, later on, it is known as the Randers space. The other notable Finsler spaces with (α, β) -metrics are Kropina space, Generalized Kropina space and Matsumoto space.

Definition 1. A Finsler space $F^n = (M, F(x, y))$ is called with (α, β) -metric if there exists a 2-homogeneous function L of two variables such that the Finsler metric $F : TM \rightarrow R$ is given by:

$$F^2(x, y) = L(\alpha(x, y), \beta(x, y)), \quad (2)$$

where $\alpha^2(x, y) = a_{ij}(x)y^iy^j$, α is a Riemannian metric on M and $\beta(x, y) = b_i(x)y^i$, β is a 1-form on M .

For a Finsler space with (α, β) -metric $F^2(x, y) = L(\alpha(x, y), \beta(x, y))$, we have the following Finsler invariants [10]:

$$\begin{aligned} \rho &= \frac{1}{2\alpha} \frac{\partial L}{\partial \alpha}, \rho_0 = \frac{1}{2} \frac{\partial^2 L}{\partial \beta^2}, \rho_{-1} = \frac{1}{2\alpha} \frac{\partial^2 L}{\partial \alpha \partial \beta}, \\ \rho_{-2} &= \frac{1}{2\alpha^2} \left(\frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha} \frac{\partial L}{\partial \alpha} \right). \end{aligned} \quad (3)$$

For a Finsler space with (α, β) -metric, we have,

$$\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0. \quad (4)$$

With respect to these notations, the fundamental metric tensor g_{ij} of a Finsler space with (α, β) -metric is given by [10]:

$$g_{ij}(x, y) = \rho a_{ij}(x) + \rho_0 b_i(x) b_j(x) + \rho_{-1} \{b_i(x) y_j + b_j(x) y_i\} + \rho_{-2} y_i y_j. \quad (5)$$

The metric tensor g_{ij} of a Lagrange space with (α, β) -metric can be arranged into the form:

$$g_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}} (\rho_{-1} b_i + \rho_{-2} y_i) (\rho_{-1} b_j + \rho_{-2} y_j) + \frac{1}{\rho_{-2}} (\rho_0 \rho_{-2} - \rho_{-1}^2) b_i b_j. \quad (6)$$

From (6), we can see that g_{ij} is the result of two Finsler deformations

$$a_{ij} \mapsto h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}} (\rho_{-1} b_i + \rho_{-2} y_i) (\rho_{-1} b_j + \rho_{-2} y_j) \quad (7)$$

$$\text{and } h_{ij} \mapsto g_{ij} = h_{ij} + \frac{1}{\rho_{-2}} (\rho_0 \rho_{-2} - \rho_{-1}^2) b_i b_j. \quad (8)$$

The nonholonomic Finsler frame that corresponds to the first deformation (7) is,

according to the theorem (7.9.1) in [9], given by

$$X_j^i = \sqrt{\rho} \delta_j^i - \frac{1}{B^2} \left(\sqrt{\rho} \pm \sqrt{\rho + \frac{B^2}{\rho_{-2}}} \right) (\rho_{-1} b^i + \rho_{-2} y^i) (\rho_{-1} b_j + \rho_{-2} y_j), \quad (9)$$

where $B^2 = a_{ij} (\rho_{-1} b^i + \rho_{-2} y^i) (\rho_{-1} b^j + \rho_{-2} y^j) = \rho_{-1}^2 b^2 + \beta \rho_{-1} \rho_{-2}$. The metric tensors a_{ij} and h_{ij} are related by

$$h_{ij} = X_i^k X_j^l a_{kl}. \quad (10)$$

According to the theorem (7.9.1) in [9], the nonholonomic Finsler frame that corresponds to the second deformation (8), is given by

$$Y_j^i = \delta_j^i - \frac{1}{C^2} \left(1 \pm \sqrt{1 + \frac{\rho_{-2} C^2}{\rho_0 \rho_{-2} - \rho_{-1}^2}} \right) b^i b_j, \quad (11)$$

where,

$$C^2 = h_{ij} b^i b^j = \rho b^2 + \frac{1}{\rho_{-2}} (\rho_{-1} b^2 + \rho_{-2} \beta)^2. \quad (12)$$

The metric tensors h_{ij} and g_{ij} are related by the formula:

$$g_{mn} = Y_m^i Y_n^j h_{ij}. \quad (13)$$

From (10) and (13), we have that $V_m^k = X_i^k Y_m^i$, with X_i^k given by (9) and Y_m^i given by (11), is a nonholonomic Finsler frame of the Finsler space with (α, β) -metric.

3. The Condition for Space to be Positive Definite

We consider the Finsler space (M, F) , where $F = \{\alpha^2(\alpha^2 + \epsilon\beta^2)\}^{1/4}$, $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i(x)y^i$. Since the case of either $\beta = 0$ or $\epsilon = 0$ implies that $F = \alpha$, therefore, we mainly consider the case of $\beta \neq 0$ and positive ϵ throughout the paper. Putting $s = \beta/\alpha$, we get

$$F = \alpha\phi(s) \quad (14)$$

Thus, we have,

$$\phi(s) = (1 + \epsilon s^2)^{1/4}. \quad (15)$$

We know that the fundamental tensor g_{ij} is defines by $g_{ij} := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$. We check whether the matrix (g_{ij}) of a Finsler space (M, F) is positive definite or not.

We have a tool to make a judgment for the positive definiteness of the matrix (g_{ij}) . It is the following Chern and Shen's lemma:

Lemma 3.1. $F = \alpha\phi(\beta/\alpha)$ is a Minkowski norm for any Riemannian metric α and 1- form β with $\|\beta\|_\alpha < b_0$ if and only if $\phi = \phi(s)$ satisfies the following conditions:

$$\phi(s) > 0, \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad (16)$$

where s and b are arbitrary numbers with $|s| \leq b < b_0$.

The first condition in (16) is trivially satisfied and we have,

$$\Phi'(s) = \frac{\epsilon s}{2(1 + \epsilon s^2)^{3/4}} \quad \text{and} \quad \Phi''(s) = \frac{\epsilon(2 - \epsilon s^2)}{4(1 + \epsilon s^2)^{7/4}}.$$

Now,

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s)$$

$$= \frac{1}{4(1 + \epsilon s^2)^{7/4}} \{3\epsilon^2 s^4 + \epsilon(4 - \epsilon b^2)s^2 + 2\epsilon b^2 + 4\}$$

Therefore, it follows that the inequality $3\epsilon^2 s^4 + \epsilon(4 - \epsilon b^2)s^2 + 2\epsilon b^2 + 4 > 0$ holds for any s such that $s^2 \leq b^2$ if and only if

$$\epsilon b^2 \leq 4 \quad (17)$$

or

$$\epsilon b^2 > 4 \quad \text{and} \quad \epsilon^2 b^4 - 32\epsilon b^2 - 32 < 0 \quad (18)$$

So we get

Theorem 2. *The metric (g_{ij}) of the Finsler space (M, F) , where $F = \alpha(1 + \epsilon s^2)^{1/4}$, is positive definite if and only if β satisfies the inequality given in equations (17) or (18).*

4. Nonholonomic Frames for Finsler Space with a Special (α, β) -Quartic Metric

Let us consider a Finsler space $F^n = (M, F)$ with special (α, β) -metric $F = \alpha(1 + \epsilon s^2)^{1/4}$, where $s = \beta/\alpha$, then the Finsler invariants (3), are given by:

$$\begin{aligned} \rho &= \frac{2 + \epsilon s^2}{2\sqrt{(1 + \epsilon s^2)^3}}, \quad \rho_0 = \frac{\epsilon}{2\sqrt{(1 + \epsilon s^2)^3}}, \quad \rho_{-1} = \frac{\epsilon^2 s^3}{2\alpha\sqrt{(1 + \epsilon s^2)^3}}, \\ \rho_{-2} &= \frac{-\epsilon^2 s^4}{2\alpha^2\sqrt{(1 + \epsilon s^2)^3}}, \quad B^2 = \frac{\epsilon^2 s^3}{4\alpha(1 + \epsilon s^2)^6} \left\{ 2b^2\sqrt{(1 + \epsilon s^2)^3} - \epsilon^2 s^5 \right\}. \end{aligned} \quad (19)$$

Using equation (19) in equations (9), (11) and (12), we get:

$$X_j^i = \sqrt{\frac{2 + \epsilon^2}{2\sqrt{2 + \epsilon^2}}} \left[\delta_j^i - \frac{1}{B^2} \left\{ 1 \pm D(b^i - \frac{s}{\alpha}y^i)(b_j - \frac{s}{\alpha}y_j) \right\} \right], \quad (20)$$

$$Y_j^i = \delta_j^i - \frac{1}{C^2} \left[1 \pm \sqrt{1 + \frac{C^2 2\sqrt{(1 + \epsilon s^2)^3}}{1 + 2\epsilon s^2}} \right] b^i b_j, \quad (21)$$

where

$$D = \sqrt{1 - \frac{\alpha \left\{ 2b^2\sqrt{(1 + \epsilon s^2)^3} - \epsilon^2 s^5 \right\}}{s(1 + \epsilon s^2)^3(2 + \epsilon s^2)}}$$

and

$$C^2 = \frac{(1 + \epsilon s^2)(2 + \epsilon s^2) - s^2 \epsilon^2 (b^2 - s^2)^2}{2\sqrt{(1 + \epsilon s^2)^3}}.$$

Hence, we have the following:

Theorem 3. Let $F^n = (M, F)$ be a Finsler space with (α, β) -metric $F = \{\alpha^2(\alpha^2 + \epsilon\beta^2)\}^{1/4}$, then its nonholonomic frame is $V_m^k = X_i^k Y_m^i$, where X_i^k and Y_m^i are given by (20) and (21) respectively.

When $\epsilon = 1$ then, we have,

$$L = F^2 = \alpha^2(1 + s^2)^{1/2},$$

$$\begin{aligned} \rho &= \frac{2 + s^2}{2\sqrt{(1 + s^2)^3}}, \rho_0 = \frac{1}{2\sqrt{(1 + s^2)^3}}, \rho_{-1} = \frac{s^3}{2\alpha\sqrt{(1 + s^2)^3}}, \\ \rho_{-2} &= \frac{-s^4}{2\alpha^2\sqrt{(1 + s^2)^3}}, B^2 = \frac{s^3}{4\alpha(1 + s^2)^6} \left\{ 2b^2\sqrt{(1 + s^2)^3} - s^5 \right\}. \end{aligned} \quad (22)$$

Using equation (22) in equations (9), (11) and (12), we get:

$$X_j^i = \sqrt{\frac{\sqrt{3}}{2}} \left[\delta_j^i - \frac{1}{B^2} \left\{ 1 \pm D(b^i - \frac{s}{\alpha}y^i)(b_j - \frac{s}{\alpha}y_j) \right\} \right], \quad (23)$$

$$Y_j^i = \delta_j^i - \frac{1}{C^2} \left[1 \pm \sqrt{1 + \frac{2C^2\sqrt{(1 + s^2)^3}}{1 + 2s^2}} \right] b^i b_j, \quad (24)$$

where,

$$D = \sqrt{1 - \frac{\alpha \left\{ 2b^2\sqrt{(1 + s^2)^3} - s^5 \right\}}{s(1 + s^2)^3(2 + s^2)}}$$

and

$$C^2 = \frac{(1 + s^2)(2 + s^2) - s^2(b^2 - s^2)^2}{2\sqrt{(1 + s^2)^3}}.$$

Hence, we have the following:

Corollary 4.1. Let $F^n = (M, F)$ be a Finsler space with (α, β) -metric $F = \{\alpha^2(\alpha^2 + \beta^2)\}^{1/4}$, then its nonholonomic frame is $V_m^k = X_i^k Y_m^i$, where X_i^k and Y_m^i are given by (23) and (24) respectively.

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