NONHOLONOMIC FRAMES FOR FINSLER SPACES
WITH A SPECIAL QUARTIC METRIC

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Abstract: The unified formalism which considers the motion of charged particles in an external electromagnetic field can be studied with the help of a nonholonomic Finsler frame which can be defined in quadric spaces also. Quartic spaces are members of a bigger class of the Finsler spaces with \((\alpha, \beta)\) metric. However, it’s not known that whether a Finsler space with a special quartic metric have such a nonholonomic frame or not and also there is no direct method available yet to determine a nonholonomic frame for these spaces. The present study attempts to test that the indicatrix of special \((\alpha, \beta)\) quartic metric of form 

\[ F = \left\{ \alpha^2 (\alpha^2 + \epsilon \beta^2) \right\}^{1/4} \]

is strictly convex. Further, it provides the nonholonomic frame for Finsler spaces equipped with a special quartic metric if it is strictly convex in nature.

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1. Introduction

In 1926, Vranceanu[16] has introduced the concept of a nonholonomic space
which is more general than a Riemannian space. In a Finsler space with \((\alpha, \beta)\) metric, Matsumoto [11] introduces an important relation between the original Riemannian metric \(a_{ij}\) and the Finsler metric \(g_{ij}\) defined as

\[
g_{ij}(x, \dot{x}) = \rho a_{ij}(x) + \rho_0 b_i(x)b_j(x) + \rho_{-1}(b_i(x)\dot{x}_i + b_j(x)\dot{x}_j) + \rho_{-2}\dot{x}_i\dot{x}_j, \tag{1}
\]

where \(\rho, \rho_0, \rho_{-1}\) and \(\rho_{-2}\) are some Finsler invariants but from this formula, we can not see any direct information about some properties of Finsler metric \(g_{ij}\) to which Riemannian metric \(a_{ij}\) posses. In [8], Holland introduces a new relation between Finsler metric \(g_{ij}\) and the Riemannian metric \(a_{ij}\) during the study of unified formalism that uses a nonholonomic Finsler frame on space-time arising from the consideration of a charged particle moving in an external electromagnetic field.

In fact, Ingarden [9] was first to point out that the Lorentz force law, in this case, could be written as geodesic equations on a Finsler space called Randers space [14]. In [4, 5] the physicist Beil has studied a generalized Lagrange metric and defines a relation between Generalized Lagrange metric \(g_{ij}\) and the Lorentz metric. In [1], Anastasiei and Shimada have studied a class of generalized Lagrange metrics and the resulting metric is called Beil metric. In this direction some work has been done in [6, 7] with Generalized Lagrange spaces with \((\alpha, \beta)\)-metrics and Finsler spaces with \((\alpha, \beta)\)-metrics. The idea we shall use from the above-mentioned papers are to view a Beil metric as a Finsler deformation of an original Riemannian metric. Then we can find a nonholonomic frame that generalizes the frame used by Beil in [4]. In [15], Shanker and Baby has also evaluated a nonholonomic frame for general \((\alpha, \beta)\)-metric.

Miron and Izumi have also studied nonholonomic Finsler frames and the induced Finsler connection in [12] for the so-called strongly non-Riemannian Finsler spaces. M. Matsumoto has also studied these nonholonomic frames, in [10], where he has called such frames the Miron frames of a strongly non-Riemannian Finsler space. The Miron frame is a natural generalization of the Berwald frame for a two-dimensional Finsler space or the Moor frame for a Finsler space of dimension three. In [13], Miron, Tavakol, Balan and Roxburgh have also defined the geometry of space-time and generalized Lagrange-Gauge theory.

For a Finsler space with \((\alpha, \beta)\)-metric, the fundamental tensor field might be thought as the result of two Finsler deformations. Then one can determine a corresponding frame for each of these two Finsler deformations. Consequently, a nonholonomic frame for a Finsler space with \((\alpha, \beta)\)-metric will appear as a product of two Finsler frames formerly determined.

In [2, 3], Antonelli and Bucataru have also investigated a nonholonomic
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Frame for two important classes of Finsler spaces that are Randers and Kropina spaces. As quartic spaces are members of a bigger class of the Finsler spaces with \((\alpha, \beta)\) special quartic metric, it appears a natural question: does a Finsler space with a special quartic metric have such a nonholonomic frame? As the fundamental tensor of a Finsler space with \((\alpha, \beta)\)-metric is not so easy to handle with, we didn’t find so far, a direct method to determine a nonholonomic frame for these spaces.

In this paper, we consider a special \((\alpha, \beta)\) quartic metric

\[
F = \left\{ \alpha^2(\alpha^2 + \epsilon \beta^2) \right\}^{1/4},
\]

where \(\alpha = \sqrt{a_{ij}y^iy^j}\) and \(\beta = b_iy^i\) and we investigate the condition that the indicatrix of this special quartic metric is strictly convex as well as find the nonholonomic frames for Finsler spaces equipped with a special quartic metric.

2. Preliminaries

An important class of Finsler spaces is the class of Finsler spaces with \((\alpha, \beta)\)-metrics [11]. In [14], the physicist Randers introduces a concept of Finsler space with \((\alpha, \beta)\)-metric, later on, it is known as the Randers space. The other notable Finsler spaces with \((\alpha, \beta)\)-metrics are Kropina space, Generalized Kropina space and Matsumoto space.

Definition 1. A Finsler space \(F^n = (M, F(x, y))\) is called with \((\alpha, \beta)\)-metric if there exists a 2-homogeneous function \(L\) of two variables such that the Finsler metric \(F: TM \rightarrow \mathbb{R}\) is given by:

\[
F^2(x, y) = L(\alpha(x, y), \beta(x, y)),
\]

where \(\alpha^2(x, y) = a_{ij}(x)y^iy^j\), \(\alpha\) is a Riemannian metric on \(M\) and \(\beta(x, y) = b_i(x)y^i\), \(\beta\) is a 1-form on \(M\).

For a Finsler space with \((\alpha, \beta)\)-metric \(F^2(x, y) = L(\alpha(x, y), \beta(x, y))\), we have the following Finsler invariants [10]:

\[
\rho = \frac{1}{2\alpha} \frac{\partial L}{\partial \alpha}, \rho_0 = \frac{1}{2} \frac{\partial^2 L}{\partial \beta^2}, \rho_{-1} = \frac{1}{2\alpha} \frac{\partial^2 L}{\partial \alpha \partial \beta}, \\
\rho_{-2} = \frac{1}{2\alpha^2} \left( \frac{\partial^2 L}{\partial \alpha^2} - \frac{1}{\alpha} \frac{\partial L}{\partial \alpha} \right). \tag{3}
\]
For a Finsler space with \((\alpha, \beta)\)-metric, we have,
\[
\rho_{-1}\beta + \rho_{-2}\alpha^2 = 0.
\] (4)

With respect to these notations, the fundamental metric tensor \(g_{ij}\) of a Finsler space with \((\alpha, \beta)\)-metric is given by [10]:
\[
g_{ij}(x, y) = \rho a_{ij}(x) + \rho_0 b_i(x) b_j(x) + \rho_{-1}\{b_i(x)y_j + b_j(x)y_i\} + \rho_{-2}y_iy_j.
\] (5)

The metric tensor \(g_{ij}\) of a Lagrange space with \((\alpha, \beta)\)-metric can be arranged into the form:
\[
g_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j) + \frac{1}{\rho_{-2}}(\rho_0\rho_{-2} - \beta^2) b_ib_j.
\] (6)

From (6), we can see that \(g_{ij}\) is the result of two Finsler deformations
\[
a_{ij} \mapsto h_{ij} = \rho a_{ij} + \frac{1}{\rho_{-2}}(\rho_{-1}b_i + \rho_{-2}y_i)(\rho_{-1}b_j + \rho_{-2}y_j)
\] and
\[
h_{ij} \mapsto g_{ij} = h_{ij} + \frac{1}{\rho_{-2}}(\rho_0\rho_{-2} - \beta^2) b_ib_j.
\] (7) (8)

The nonholonomic Finsler frame that corresponds to the first deformation (7) is, according to the theorem (7.9.1) in [9], given by
\[
X^i_j = \sqrt{\rho}\delta^i_j - \frac{1}{B^2}\left(\sqrt{\rho} \pm \sqrt{\frac{\rho + \frac{B^2}{\rho_{-2}}}{\rho_{-2}}}\right)(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b_j + \rho_{-2}y_j),
\] (9)
where \(B^2 = a_{ij}(\rho_{-1}b^i + \rho_{-2}y^i)(\rho_{-1}b^j + \rho_{-2}y^j) = \rho_{-1}b^2 + \beta\rho_{-1}\rho_{-2}\). The metric tensors \(a_{ij}\) and \(h_{ij}\) are related by
\[
h_{ij} = X^k_i X^l_j a_{kl}.
\] (10)

According to the theorem (7.9.1) in [9], the nonholonomic Finsler frame that corresponds to the second deformation (8), is given by
\[
Y^i_j = \delta^i_j - \frac{1}{C^2}\left(1 \pm \sqrt{1 + \frac{\rho_{-2}C^2}{\rho_0\rho_{-2} - \beta^2}}\right)b^ib_j,
\] (11)
where,
\[
C^2 = h_{ij} b^i b^j = \rho b^2 + \frac{1}{\rho_{-2}}(\rho_{-1}b^2 + \rho_{-2}\beta)^2.
\] (12)
The metric tensors $h_{ij}$ and $g_{ij}$ are related by the formula:

$$g_{mn} = Y_m^i Y_n^j h_{ij}.$$  \hfill (13)

From (10) and (13), we have that $V^k_m = X^k_i Y_m^i$, with $X^k_i$ given by (9) and $Y_m^i$ given by (11), is a nonholonomic Finsler frame of the Finsler space with $(\alpha, \beta)$-metric.

3. The Condition for Space to be Positive Definite

We consider the Finsler space $(M, F)$, where $F = \alpha^2(\alpha^2 + \epsilon \beta^2)^{1/4}$, $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i(x)y^i$. Since the case of either $\beta = 0$ or $\epsilon = 0$ implies that $F = \alpha$, therefore, we mainly consider the case of $\beta \neq 0$ and positive $\epsilon$ throughout the paper. Putting $s = \beta/\alpha$, we get

$$F = \alpha \phi(s)$$ \hfill (14)

Thus, we have,

$$\phi(s) = (1 + \epsilon s^2)^{1/4}.$$ \hfill (15)

We know that the fundamental tensor $g_{ij}$ is defines by $g_{ij} := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$. We check whether the matrix $(g_{ij})$ of a Finsler space $(M, F)$ is positive definite or not.

We have a tool to make a judgment for the positive definiteness of the matrix $(g_{ij})$. It is the following Chern and Shen’s lemma:

\textbf{Lemma 3.1.}  \quad $F = \alpha \phi(\beta/\alpha)$ is a Minkowski norm for any Riemannian metric $\alpha$ and 1- form $\beta$ with $\|\beta\|_{\alpha} < b_0$ if and only if $\phi = \phi(s)$ satisfies the following conditions:

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0,$$ \hfill (16)

where $s$ and $b$ are arbitrary numbers with $|s| < b < b_0$.

The first condition in (16) is trivially satisfied and we have,

$$\Phi'(s) = \frac{\epsilon s}{2(1 + \epsilon s^2)^{3/4}} \quad \text{and} \quad \Phi''(s) = \frac{\epsilon (2 - \epsilon s^2)}{4(1 + \epsilon s^2)^{7/4}}.$$  

Now,

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s)$$
Therefore, it follows that the inequality \(3\epsilon^2 s^4 + \epsilon(4 - \epsilon b^2)s^2 + 2\epsilon b^2 + 4 > 0\) holds for any \(s\) such that \(s^2 \leq b^2\) if and only if

\[\epsilon b^2 \leq 4\]  

or

\[\epsilon b^2 > 4 \text{ and } \epsilon^2 b^4 - 32\epsilon b^2 - 32 < 0\]  

So we get

**Theorem 2.** The metric \((g_{ij})\) of the Finsler space \((M, F)\), where \(F = \alpha(1 + \epsilon s^2)^{1/4}\), is positive definite if and only if \(\beta\) satisfies the inequality given in equations (17) or (18).

### 4. Nonholonomic Frames for Finsler Space with a Special \((\alpha, \beta)\)-Quartic Metric

Let us consider a Finsler space \(F^n = (M, F)\) with special \((\alpha, \beta)\)-metric \(F = \alpha(1 + \epsilon s^2)^{1/4}\), where \(s = \beta/\alpha\), then the Finsler invariants (3), are given by:

\[
\rho = \frac{2 + \epsilon s^2}{2\sqrt{(1 + \epsilon s^2)^3}}, \quad \rho_0 = \frac{\epsilon}{2\sqrt{(1 + \epsilon s^2)^3}}, \quad \rho_{-1} = \frac{\epsilon^2 s^3}{2\alpha\sqrt{(1 + \epsilon s^2)^3}}, \\
\rho_{-2} = \frac{-\epsilon^2 s^4}{2\alpha^2\sqrt{(1 + \epsilon s^2)^3}}, \quad B^2 = \frac{\epsilon^2 s^3}{4\alpha(1 + \epsilon s^2)^3} \left\{2b^2\sqrt{(1 + \epsilon s^2)^3} - \epsilon^2 s^5\right\}. \tag{19}
\]

Using equation (19) in equations (9), (11) and (12), we get:

\[
X^i_j = \sqrt{\frac{2 + \epsilon^2}{2\sqrt{2} + \epsilon^2}} \left[\delta^i_j - \frac{1}{B^2} \left\{1 \pm D(b^i - \frac{s}{\alpha} y^i)(b_j - \frac{s}{\alpha} y_j)\right\}\right], \tag{20}
\]

\[
Y^i_j = \delta^i_j - \frac{1}{C^2} \left[1 \pm \sqrt{1 + \frac{C^2 2\sqrt{(1 + \epsilon s^2)^3}}{1 + 2\epsilon s^2}} \right] b^i b_j, \tag{21}
\]

where

\[
D = \sqrt{1 - \frac{\alpha \left\{2b^2\sqrt{(1 + \epsilon s^2)^3} - \epsilon^2 s^5\right\}}{s(1 + \epsilon s^2)^3(2 + \epsilon s^2)}}.
\]
and
\[
C^2 = \frac{(1 + \epsilon s^2)(2 + \epsilon s^2) - s^2 \epsilon^2 (b^2 - s^2)^2}{2\sqrt{(1 + \epsilon s^2)^3}}.
\]

Hence, we have the following:

**Theorem 3.** Let \( F^n = (M, F) \) be a Finsler space with \((\alpha, \beta)\)-metric \( F = \{\alpha^2(\alpha^2 + \epsilon \beta^2)\}^{1/4} \), then its nonholonomic frame is \( V^k_m = X^i_k Y^i_m \), where \( X^k_i \) and \( Y^i_m \) are given by (20) and (21) respectively.

When \( \epsilon = 1 \) then, we have,
\[
L = F^2 = \alpha^2(1 + s^2)^{1/2},
\]
\[
\rho = \frac{2 + s^2}{2\sqrt{(1 + s^2)^3}}, \rho_0 = \frac{1}{2\sqrt{(1 + s^2)^3}}, \rho_1 = \frac{s^3}{2\alpha\sqrt{(1 + s^2)^3}}, \rho_2 = \frac{-s^4}{2\alpha^2\sqrt{(1 + s^2)^3}}, B^2 = \frac{s^3}{4\alpha(1 + s^2)^6} \left\{ 2b^2 \sqrt{(1 + s^2)^3} - s^5 \right\}.
\]

Using equation (22) in equations (9), (11) and (12), we get:
\[
X^i_j = \sqrt{\frac{3}{2}} \left\{ \delta^i_j - \frac{1}{B^2} \left\{ 1 \pm D(b^i - \frac{s}{\alpha} y^i)(b_j - \frac{s}{\alpha} y_j) \right\} \right\},
\]
\[
Y^i_j = \delta^i_j - \frac{1}{C^2} \left[ 1 \pm \sqrt{1 + \frac{2C^2 \sqrt{(1 + s^2)^3}}{1 + 2s^2}} \right] b^i b_j,
\]
where,
\[
D = \sqrt{1 - \frac{\alpha \left\{ 2b^2 \sqrt{(1 + s^2)^3} - s^5 \right\}}{s(1 + s^2)^3(2 + s^2)}},
\]
and
\[
C^2 = \frac{(1 + s^2)(2 + s^2) - s^2 (b^2 - s^2)^2}{2\sqrt{(1 + s^2)^3}}.
\]

Hence, we have the following:

**Corollary 4.1.** Let \( F^n = (M, F) \) be a Finsler space with \((\alpha, \beta)\)-metric \( F = \{\alpha^2(\alpha^2 + \beta^2)\}^{1/4} \), then its nonholonomic frame is \( V^k_m = X^k_i Y^i_m \), where \( X^k_i \) and \( Y^i_m \) are given by (23) and (24) respectively.
References


