

CURVATURE BASED OFFSETS TO ELLIPTIC CYLINDERS

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Abstract: Elliptic cylinders form an important class of regular surfaces with many applications. In this paper, we study two associated rational surfaces of a given elliptic cylinder. One of these surfaces is the unique focal surface of the elliptic cylinder. The other one is the generalized focal surface which is obtained by the reciprocal mean curvature of the elliptic cylinder. Such generalized focal surfaces are classified with respect to their self-intersecting and self-contact lines.

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1. Introduction

An oriented regular surface in the Euclidean 3-space $\mathbb{E}^3$ is a significant object studied in differential geometry. There are two main approaches for obtaining a new surface from a given regular surface. The set of points placed at the same distance $d > 0$ from a given regular surface S is called a parallel (or offset) surface S_d of S , or shortly, S_d is called an offset to S . If the surface S has nonzero principle curvatures κ_1 and κ_2 , there exist two focal surfaces S^{f1} and S^{f2} whose

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points are placed at the oriented distances $\frac{1}{\kappa_1}$ and $\frac{1}{\kappa_2}$, respectively, from the corresponding points of S . Parallel (offset) surfaces and focal surfaces are closely related to the Euclidean motions. In fact, any Euclidean motion transforms the pair formed by a regular surface and its offset of a distance d into another pair of surfaces that are also a regular surface and its offset of the same distance d . Since the principle curvatures are invariant under the Euclidean motion, the pair formed by a regular surface S and its focal surface S^{f1} (resp. S^{f2}) is mapped by an arbitrary Euclidean motion $\Psi : \mathbb{E}^3 \rightarrow \mathbb{E}^3$ into another pair of a regular surface $S_\Psi = \Psi(S)$ and its focal surfaces S_Ψ^{f1} (resp. S_Ψ^{f2}). In [4], the first author introduced the so-called similarity offset to a surface with the following property: If $\tilde{S}$ is a similarity offset to a regular surface S and if $\Phi : \mathbb{E}^3 \rightarrow \mathbb{E}^3$ is a direct similarity, then the surface $\Phi(\tilde{S})$ is a similarity offset to the regular surface $\Phi(S)$. For a surface with nonzero Gaussian curvature K and nonzero mean curvature H , such a kind of an offsets obtained by variable distance function $\frac{H}{K}$ or $\frac{1}{H}$ is called a generalized focal surface (see [4] for details). It is well known that the cylinders in $\mathbb{E}^3$ have zero Gaussian curvature everywhere and nonzero mean curvature at non planar points. Equivalently, exactly one of principle curvatures of a cylinder, say κ_1 , is identically zero, and the other principle curvature κ_2 is nonzero at non planar points. This means that every cylinder possesses exactly one focal surface and exactly one generalized focal surface with variable distance function $\frac{1}{H}$. Elliptic cylinders form an important class of surfaces with many applications (see for instance [1] and [2]). This class is closely related to the direct similarities of $\mathbb{E}^3$ because every two elliptic cylinders with the same aspect ratio are similarity equivalent. The aim of this paper is to examine focal and generalized focal surfaces of an arbitrary elliptic cylinder. Both standard trigonometric and rational parametrizations of elliptic cylinders are discussed. The paper is organized as follows. The next two sections give a short description of different offsets to a regular surface. Section 4 is devoted to the focal surfaces of elliptic cylinders. Main properties of generalized focal surfaces of elliptic cylinders are obtained in Section 5. In particular, a classification theorem for these surfaces is proved. The paper completes with illustrative examples and concluding remarks.

2. Parallel and Focal Surfaces of Regular Surfaces

Let $S : D \rightarrow \mathbb{E}^3$ be a regular surface of class C^2 defined on a domain $D \subseteq \mathbb{R}^2$ with a parametrization

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))^T, \quad (u, v) \in D. \quad (1)$$

Then the tangent vectors to the parametric lines $\mathbf{r}_u = \frac{\partial}{\partial u}\mathbf{r}(u, v)$ and $\mathbf{r}_v = \frac{\partial}{\partial v}\mathbf{r}(u, v)$ determine the unit normal vector field

$$\mathbf{n}(u, v) = \frac{1}{\|\mathbf{r}_u \times \mathbf{r}_v\|} (\mathbf{r}_u \times \mathbf{r}_v)$$

Under above assumptions, the parallel (or offset) surface S_d of distance d possesses a parametrization

$$\mathbf{r}_d(u, v) = \mathbf{r}(u, v) + d\mathbf{n}(u, v) = \mathbf{r}(u, v) + \frac{d}{\|\mathbf{r}_u \times \mathbf{r}_v\|} (\mathbf{r}_u \times \mathbf{r}_v). \quad (2)$$

If the principle curvatures κ_1 and κ_2 of the surface S are nonzero everywhere, there exist two focal surfaces S^{f1} and S^{f2} given by

$$\begin{aligned} \mathbf{r}^{fi}(u, v) &= \mathbf{r}(u, v) + \frac{1}{\kappa_i} \mathbf{n}(u, v) \\ &= \mathbf{r}(u, v) + \frac{1}{\kappa_i \|\mathbf{r}_u \times \mathbf{r}_v\|} (\mathbf{r}_u \times \mathbf{r}_v), \quad i = 1, 2. \end{aligned} \quad (3)$$

Properties of parallel and focal surfaces and some relations between surfaces (1) and (2) are described in [5] and [8]. Recall that any Euclidean motion of the Euclidean 3-space $\mathbb{E}^3$ is an affine transformation of $\mathbb{E}^3$ that preserves the distances and the orientation of surfaces. This implies that any Euclidean motion transforms the pair of a regular surface and its offset of distance $d > 0$ into another pair of a regular surface and its offset of distance $d > 0$. Since principle curvatures are invariant under Euclidean motions, the image of a regular surface and the image of one of its focal surface under an arbitrary Euclidean motion form also a pair of a regular surface and one of its focal surfaces.

3. Generalized Offsets to Regular Surfaces

One natural generalization of the parallel surfaces can be obtained by replacing in the definition the distance with a variable distance function. More exactly, this generalization called a generalized offset to a regular surface is represented by

$$\mathbf{r}_g(u, v) = \mathbf{r}(u, v) + f(u, v)\mathbf{n}(u, v), \quad (4)$$

where $\mathbf{n}(u, v)$ is the unit normal vector to the surface and $f(u, v)$ is a variable distance function (both depending on the point of the original surface). In

1995, Hagen, Hahmann and Schreiber [6] considered the case in which $f(u, v)$ is a certain function of the principal curvatures κ_1 and κ_2 . They called such a generalized offset "a generalized focal surface". Other type of a generalized offset to a surface is introduced and studied by Moon in [7].

The Euclidean space $\mathbb{E}^3$ can be considered as an affine space with an associate vector space $\mathbb{R}^3$. This means that the points of $\mathbb{E}^3$ can be identified with their position vectors. Let us recall that the transformation $\Phi : \mathbb{E}^3 \rightarrow \mathbb{E}^3$ is called an orientation preserving similarity transformation, or shortly a direct similarity, if Φ is an orientation preserving affine transformation and if there exists a positive constant ϱ such that for every two points $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{E}^3$ and $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{E}^3$ it is fulfilled $\|\Phi(\mathbf{x}) - \Phi(\mathbf{y})\| = \varrho \|\mathbf{x} - \mathbf{y}\|$.

Two important types of generalized focal surfaces related to direct similarities have been introduced in [4]. Let S be a regular surface of class C^2 with a parametrization (1). Suppose that S has a nonzero Gaussian curvature function K and a nonzero mean curvature function H . Then, there exist a generalized focal surface $\tilde{S}$ given by

$$\tilde{\mathbf{r}}(u, v) = \mathbf{r}(u, v) + \frac{1}{H}\mathbf{n}(u, v) \quad (5)$$

and a generalized focal surface $\overline{S}$ given by

$$\overline{\mathbf{r}}(u, v) = \mathbf{r}(u, v) + \frac{H}{K}\mathbf{n}(u, v). \quad (6)$$

Main properties of these generalized focal surfaces are proved in [4]:

A. The set of all pairs $(S, \tilde{S})$ is invariant under an arbitrary orientation preserving similarity transformation $\Phi : \mathbb{E}^3 \rightarrow \mathbb{E}^3$. This mean that if $S_0 = \Phi(S)$ is given by a vector-valued function $\mathbf{r}_0(u, v)$ and if $\tilde{S}_0 = \Phi(\tilde{S})$ is given by a vector-valued function $\tilde{\mathbf{r}}_0(u, v)$, then

$$\tilde{\mathbf{r}}_0(u, v) = \mathbf{r}_0(u, v) + \frac{1}{H_0}\mathbf{n}_0(u, v),$$

where H_0 is a mean curvature of S_0 and $\mathbf{n}_0(u, v)$ is a unit normal vector field of S_0 .

B. The set of all pairs $(S, \overline{S})$ is invariant under an arbitrary orientation preserving similarity transformation $\Phi : \mathbb{E}^3 \rightarrow \mathbb{E}^3$.

C. If the original surface S is rational, then generalized focal surfaces $\tilde{S}$ and $\overline{S}$ also are rational.

Definition 1. The focal surfaces (3) and generalized focal surfaces (5) and (6) are called curvature based offsets to a regular surface.

There are exactly two curvature based offsets to an elliptic cylinder. One of these offsets is the unique focal surface obtained by the unique nonzero principle curvature of the elliptic cylinder. The other one is the generalized focal surface given by (5). Both offsets will be studied in the next sections.

4. Focal Surfaces of Elliptic Cylinders

We start with a short description of two known parametrizations of an elliptic cylinder. Without loss of generality we may assume that any elliptic cylinder S is an implicitly defined surface given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a > 0$ and $b > 0$ are constants. This surface S is a ruled surface which has **a standard parametrization**

$$\mathbf{r}(u, v) = (a \cos u, b \sin u, v) = \boldsymbol{\alpha}(u) + v\mathbf{q}, \quad (7)$$

where $(u, v) \in [0, 2\pi] \times \mathbb{R}$, $\boldsymbol{\alpha}(u) = (a \cos u, b \sin u, 0)$ is a parametrization of the ellipse $\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the coordinate xy -plane, and $\mathbf{q} = (0, 0, 1)$ is a fixed unit 3-dimensional vector perpendicular to xy -plane(cf. [5]). It is easy to see that the tangent vectors are $\mathbf{r}_u = \boldsymbol{\alpha}'(u) = (-a \sin u, b \cos u, 0)$ and $\mathbf{r}_v = \mathbf{q}$. Thus, the elliptic cylinder is a regular surface at any point since the vector cross product $\mathbf{r}_u \times \mathbf{r}_v = \boldsymbol{\alpha}' \times \mathbf{q} = (b \cos u, a \sin u, 0)$ is nonzero everywhere. Consequently, the unit normal vector of S is

$$\begin{aligned} \mathbf{n}(u, v) &= \frac{1}{\|\mathbf{r}_u \times \mathbf{r}_v\|} (\mathbf{r}_u \times \mathbf{r}_v) \\ &= \left(\frac{b \cos u}{\sqrt{a^2 \sin^2 u + b^2 \cos^2 u}}, \frac{a \sin u}{\sqrt{a^2 \sin^2 u + b^2 \cos^2 u}}, 0 \right). \end{aligned} \quad (8)$$

The elliptic cylinder possesses also **a rational parametrization**

$$\mathbf{r}_1(w, v) = \left(a \frac{1-w^2}{1+w^2}, b \frac{2w}{1+w^2}, v \right) = \boldsymbol{\beta}(w) + v\mathbf{q}, \quad (9)$$

where $(w, v) \in \mathbb{R}^2$ and $\boldsymbol{\beta}(w) = \left(a \frac{1-w^2}{1+w^2}, b \frac{2w}{1+w^2}, 0 \right)$ is a rational parametrization of the same ellipse $\mathcal{E}$ in the xy -plane. Then, the unit normal vector of the

elliptic cylinder S with respect to the rational parametrization (9) can be written as

$$\begin{aligned} \mathbf{n}_1(w, v) &= \\ &= \frac{1}{\|\mathbf{r}_{1w}(w, v) \times \mathbf{r}_{1v}(w, v)\|} (\mathbf{r}_{1w}(w, v) \times \mathbf{r}_{1v}(w, v)) \\ &= \left(\frac{b(1-w^2)}{\sqrt{4a^2w^2 + b^2(1-w^2)^2}}, \frac{2aw}{\sqrt{4a^2w^2 + b^2(1-w^2)^2}}, 0 \right). \end{aligned} \quad (10)$$

Let us recall some basic computation for elliptic cylinders. Suppose that the elliptic cylinder is given by the standard parametrization (7). Then, for the coefficients of the first fundamental form, we have $E = \boldsymbol{\alpha}' \cdot \boldsymbol{\alpha}' = a^2 \sin^2 u + b^2 \cos^2 u$, $F = \boldsymbol{\alpha}' \cdot \mathbf{q} = 0$ and $G = \mathbf{q} \cdot \mathbf{q} = 1$. Since $\mathbf{r}_{uu} = \boldsymbol{\alpha}'' = (-a \cos u, -b \sin u, 0)$, $\mathbf{r}_{uv} = 0$, $\mathbf{r}_{vv} = 0$, the coefficients of the second fundamental form are $L = \mathbf{n} \cdot \mathbf{r}_{uu} = \frac{-ab}{\sqrt{a^2 \sin^2 u + b^2 \cos^2 u}}$, $M = \mathbf{n} \cdot \mathbf{r}_{uv} = 0$ and $N = \mathbf{n} \cdot \mathbf{r}_{vv} = 0$. It is well-known that the Gaussian curvature of any cylinder (in particular, the Gaussian curvature of any elliptic cylinder) vanishes everywhere. Furthermore, the mean curvature of the elliptic cylinder (7) can be expressed as

$$H(u, v) = \frac{L}{2E} = \frac{-ab}{2(a^2 \sin^2 u + b^2 \cos^2 u)^{3/2}} < 0. \quad (11)$$

Similarly, the coefficients of the first fundamental form regarding to the rational parametrization (9) are

$$\begin{aligned} E_1(w, v) &= \boldsymbol{\beta}' \cdot \boldsymbol{\beta}' = \frac{4(4a^2w^2 + b^2(1-w^2)^2)}{(1+w^2)^4} \\ F_1(w, v) &= \boldsymbol{\beta}' \cdot \mathbf{q} = 0 \\ G_1(w, v) &= \mathbf{q} \cdot \mathbf{q} = 1. \end{aligned}$$

By direct computations we obtain the coefficients of the second fundamental form in this case

$$\begin{aligned} L_1(w, v) &= \mathbf{n}_1(w, v) \cdot \mathbf{r}_{1ww}(w, v) \\ &= -\frac{4ab}{(1+w^2)\sqrt{4a^2w^2 + b^2(1-w^2)^2}} \\ M_1(w, v) &= \mathbf{n}_1(w, v) \cdot \mathbf{r}_{1wv}(w, v) = 0 \\ N_1(w, v) &= \mathbf{n}_1(w, v) \cdot \mathbf{r}_{1vv}(w, v) = 0. \end{aligned}$$

Hence, the mean curvature of the elliptic cylinder in terms of rational parametrization is

$$H_1(w, v) = \frac{L_1(w, v)}{2E_1(w, v)} = -\frac{ab(1+w^2)^3}{2\left(4a^2w^2 + b^2(1-w^2)^2\right)^{3/2}} < 0. \quad (12)$$

Every elliptic cylinder S in the Euclidean 3-space can be considered as a right cylinder over an ellipse. Its Gaussian curvature K is identically zero, but its mean curvature H is nonzero everywhere and does not change the sign. This implies that the one principle curvature (e.g. κ_1) of S is identically zero and the other principle curvature $\kappa_2 = 2H$ is nonzero everywhere. Consequently, every elliptic cylinder has exactly one focal surface. Now, we will study the relationship between these two surfaces.

Proposition 2. *Let S be the elliptic cylinder defined by (7) or (9), and let S^f be its unique focal surface. Then:*

- (a) *The surface S^f is a right cylinder over the evolute of the ellipse $\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.*
- (b) *The surface S^f possesses a rational parametrization.*

Proof. (a) Assume that S is given by standard parametrization (7). From (8) and (11) it follows that the parametrization of S^f

$$\mathbf{r}^f(u, v) = \mathbf{r}(u, v) + \frac{1}{\kappa_2} \mathbf{n}(u, v) = \mathbf{r}(u, v) + \frac{1}{2H(u, v)} \mathbf{n}(u, v)$$

can be written in the form

$$\mathbf{r}^f(u, v) = \left(\frac{(a^2 - b^2) \cos^3 u}{a}, -\frac{(a^2 - b^2) \sin^3 u}{b}, v \right) \quad (13)$$

$(u, v) \in [0, 2\pi] \times \mathbb{R}.$

Since the plane curve parameterized by

$$\left(\frac{(a^2 - b^2) \cos^3 u}{a}, -\frac{(a^2 - b^2) \sin^3 u}{b}, 0 \right), \quad u \in [0, 2\pi]$$

is the evolute of the ellipse $\mathcal{E}$ (see for instance [3] and [9]), S^f is a right cylinder over this evolute.

(b) From rational representations concerning the elliptic cylinder (9), (10), (12) and

$$\kappa_2(w, v) = 2H_1(w, v) = -\frac{ab(1+w^2)^3}{\left(4a^2w^2 + b^2(1-w^2)^2\right)^{3/2}},$$

it follows that the unique focal surface S^f (given by (3)) of the elliptic cylinder has the following rational parametrization

$$\begin{aligned} \mathbf{r}_1^f(w, v) &= \left(\frac{(a^2 - b^2)(1 - w^2)^3}{a(1 + w^2)^3}, -\frac{8(a^2 - b^2)w^3}{b(1 + w^2)^3}, v \right), \\ (w, v) &\in \mathbb{R}^2. \end{aligned} \quad (14)$$

□

We can also examine the position of the focal surface relative to the original elliptic cylinder.

Proposition 3. *Let S be an elliptic cylinder defined by (7) with $a \geq b > 0$. Then, its focal surface S^f given by (13) is placed inside S , i.e. every interior point of S^f is an interior point of S , if and only if the aspect ratio $\frac{a}{b}$ is less than or equal to $\sqrt{2}$.*

Proof. The cusps of horizontal sections of S^f lying on xz -plane form two vertical lines $\left(\pm \frac{a^2 - b^2}{a}, 0, v\right)$ which are placed inside the cylinder S . The cusps of horizontal sections of S^f lying on yz -plane form other two vertical lines $\left(0, \mp \frac{a^2 - b^2}{b}, , v\right)$. The last lines are placed inside S or lie on S whenever $\left|\frac{a^2 - b^2}{b}\right| \leq b$, or equivalently, $a^2 \leq 2b^2$. This completes the proof. □

We may summarize the considerations in this section as follows. The unique focal surface of any elliptic cylinder is a right astroidal cylinder with four cuspidal edges parallel to z -axis.

5. Generalized Focal Surfaces of Elliptic Cylinders

First, we describe the common properties of a unique generalized focal surface (5) of an elliptic cylinder.

Theorem 4. *Let S be the elliptic cylinder defined by (7) or (9), and let $\tilde{S}$ be its generalized focal surface given by (5). Then,*

1. The surface $\tilde{S}$ is a right cylinder over the plane curve $\mathcal{G}$ defined by

$$\begin{aligned}\boldsymbol{\gamma}(u) &= (\tilde{x}(u), \tilde{y}(u), 0), \text{ where } u \in [0, 2\pi], \\ \tilde{x}(u) &= \frac{\cos u}{a} \left((a^2 - 2b^2) \cos^2 u - a^2 \sin^2 u \right) \\ \tilde{y}(u) &= -\frac{\sin u}{b} \left((2a^2 - b^2) \sin^2 u + b^2 \cos^2 u \right).\end{aligned}\quad (15)$$

2. The surface $\tilde{S}$ possesses a rational parametrization.

Proof. 1. Using (7), (8) and (11) we get the parametric equation (5) of the generalized focal surface $\tilde{S}$ with respect to the standard parametrization

$$\tilde{\mathbf{r}}(u, v) = (\tilde{x}(u), \tilde{y}(u), v), \text{ where } (u, v) \in [0, 2\pi] \times \mathbb{R}. \quad (16)$$

Hence, $\tilde{S}$ is a right cylinder over the curve $\boldsymbol{\gamma}(u) = (\tilde{x}(u), \tilde{y}(u), 0)$ in the coordinate xy -plane.

2. Substituting (9), (10) and (12) in (5) we obtain a rational parametrization of $\tilde{S}$

$$\begin{aligned}\tilde{\mathbf{r}}_1(w, v) &= (\tilde{x}_1(w), \tilde{y}_1(w), v), \text{ where } (w, v) \in \mathbb{R}^2, \\ \tilde{x}_1(w) &= \frac{1-w^2}{a(1+w^2)^3} \left((a^2 - 2b^2) (1 - w^2)^2 - a^2 (2w)^2 \right) \\ \tilde{y}_1(w) &= \frac{2w}{b(1+w^2)^3} \left(-b^2 (1 - w^2)^2 + (b^2 - 2a^2) (2w)^2 \right).\end{aligned}\quad (17)$$

□

Second, the considered generalized focal surfaces can be classified according to aspect ratios of the original elliptic cylinders.

Theorem 5. *Let S be an elliptic cylinder defined by (7) with $a \geq b > 0$ and let $\tilde{S}$ be its generalized focal surface given by (16). Then:*

- (a) *The self-intersection set of $\tilde{S}$ consists of two lines parallel to the z -axis if the aspect ratio $\frac{a}{b}$ is greater than $\sqrt{2}$.*
- (b) *The cylindrical surface $\tilde{S}$ contains the z -axis and any point on z -axis is self-contact if the aspect ratio $\frac{a}{b}$ is equal to $\sqrt{2}$.*
- (c) *The cylindrical surface $\tilde{S}$ has no self-intersection and self-contact points if the aspect ratio $\frac{a}{b}$ is less than $\sqrt{2}$.*

Proof. According to Theorem 5, it is sufficient to examine the regularity and the double points of the plane curve $\mathcal{G}$ determined by (15). The tangent vector of this curve is

$$\begin{aligned} \frac{d}{du}\boldsymbol{\gamma}(u) &= \left(\frac{d}{du}\tilde{x}(u), \frac{d}{du}\tilde{y}(u) \cdot 0 \right), \text{ where} \\ \frac{d}{du}\tilde{x}(u) &= \frac{\sin u}{a} \left((6b^2 - 5a^2) \cos^2 u + a^2 \sin^2 u \right) \\ \frac{d}{du}\tilde{y}(u) &= -\frac{\cos u}{b} \left((6a^2 - 5b^2) \sin^2 u + b^2 \cos^2 u \right). \end{aligned} \quad (18)$$

Since $\frac{d}{du}\boldsymbol{\gamma}(u)$ is nonzero vector for any u , $\boldsymbol{\gamma}(u)$ is a regular plane curve. Obviously, the points $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(2\pi)$ and $\boldsymbol{\gamma}(\pi)$ lie on the x -axis, and the points $\boldsymbol{\gamma}(\frac{\pi}{2})$ and $\boldsymbol{\gamma}(\frac{3\pi}{2})$ lie on the y -axis. It is easy to see that both x -axis and y -axis are axes of symmetry for the curve $\mathcal{G}$. Since $((2a^2 - b^2) \sin^2 u_1 + b^2 \cos^2 u_1) > 0$ for any parameter value u , the double points of $\mathcal{G}$, if they exist, must lie on the y -axis.

(a) Suppose that $\frac{a}{b} > \sqrt{2}$. There exists a unique $u_1 \in (0, \frac{\pi}{2})$ such that $\tan u_1 = \frac{\sqrt{a^2 - 2b^2}}{a}$. This implies $((a^2 - 2b^2) \cos^2 u_1 - a^2 \sin^2 u_1) = 0$. From (15) it follows that $\tilde{x}(u_1) = 0$ and

$$\tilde{y}(u_1) = -\frac{\sin u_1}{b} \left((2a^2 - b^2) \sin^2 u_1 + b^2 \cos^2 u_1 \right) < 0.$$

Choose $u_2 = \pi - u_1$. Obviously, $u_2 \in (\frac{\pi}{2}, \pi)$, $\sin u_2 = \sin u_1$ and $\cos u_2 = -\cos u_1$. By (15) we get

$$\tilde{x}(u_2) = \tilde{x}(u_1) = 0, \quad \tilde{y}(u_2) = \tilde{y}(u_1) < 0.$$

Consequently, the points $\boldsymbol{\gamma}(u_1)$ and $\boldsymbol{\gamma}(u_2)$ coincide. From (18) and $a > b\sqrt{2}$, we conclude that

$$\frac{d}{du}\tilde{x}(u_1) = \frac{d}{du}\tilde{x}(u_2) \text{ and } \frac{d}{du}\tilde{y}(u_1) = -\frac{d}{du}\tilde{y}(u_2) \neq 0.$$

This means that the nonzero tangent vectors $\frac{d}{du}\boldsymbol{\gamma}(u_1)$ and $\frac{d}{du}\boldsymbol{\gamma}(u_2)$ at the points $\boldsymbol{\gamma}(u_1) = \boldsymbol{\gamma}(u_2)$ are not parallel. Hence, the point $\boldsymbol{\gamma}(u_1) = \boldsymbol{\gamma}(u_2)$ is an ordinary double point of a regular plane curve $\mathcal{G}$. By (16) the points $\tilde{\mathbf{r}}(u_1, v)$ and $\tilde{\mathbf{r}}(u_2, v)$ on the surface $\tilde{S}$ coincide for any $v \in \mathbb{R}$. Furthermore, the normal vectors of $\tilde{S}$ at these points $\frac{d}{du}\boldsymbol{\gamma}(u_1) \times \mathbf{q}$ and $\frac{d}{du}\boldsymbol{\gamma}(u_2) \times \mathbf{q}$ are not parallel. Thus, any point on the line $(0, \tilde{y}(u_1), v)$ $v \in \mathbb{R}$ is a self-intersecting point of $\tilde{S}$. In the same way, if $u_3 = \pi + u_1 \in (\pi, \frac{3\pi}{2})$ and $u_4 = 2\pi - u_1 \in (\frac{3\pi}{2}, 2\pi)$, then $\boldsymbol{\gamma}(u_3) = \boldsymbol{\gamma}(u_4)$ and the

nonzero tangent vectors $\frac{d}{du}\boldsymbol{\gamma}(u_3)$, $\frac{d}{du}\boldsymbol{\gamma}(u_4)$ are not parallel. Since $\boldsymbol{\gamma}(u_3) \neq \boldsymbol{\gamma}(u_1)$, the point $\boldsymbol{\gamma}(u_3) = \boldsymbol{\gamma}(u_4)$ is a second ordinary double point of $\mathcal{G}$. Consequently, every point on the line $(0, \tilde{y}(u_3), v)$ $v \in \mathbb{R}$ is a self-intersecting point of $\tilde{S}$.

(b) Suppose that $\frac{a}{b} = \sqrt{2}$. Then, the parametrization of $\mathcal{G}$ becomes

$$\boldsymbol{\gamma}(u) = (-a \sin^2 u \cos u, -b \sin u (2 \sin^2 u + 1), 0), \quad u \in [0, 2\pi].$$

Hence, the points $\boldsymbol{\gamma}(0) = (0, 0, 0)$ and $\boldsymbol{\gamma}(\pi) = (0, 0, 0)$ coincide. By (18), the nonzero tangent vectors $\frac{d}{du}\boldsymbol{\gamma}(0) = (0, -b, 0)$ and $\frac{d}{du}\boldsymbol{\gamma}(\pi) = (0, b, 0)$ are parallel but pointing in opposite directions. This means that the origin $(0, 0, 0)$ is a self-contact point of $\mathcal{G}$. For any $v \in \mathbb{R}$ the points $\tilde{\mathbf{r}}(0, v)$ and $\tilde{\mathbf{r}}(\pi, v)$ on $\tilde{S}$ coincide, and the normal vectors at these points $\frac{d}{du}\boldsymbol{\gamma}(0) \times \mathbf{q}$ and $\frac{d}{du}\boldsymbol{\gamma}(\pi) \times \mathbf{q}$ are parallel but having opposite directions. In other words, any point on the z -axis is a self-contact point of $\tilde{S}$.

(c) If $\frac{a}{b} < \sqrt{2}$, then $((a^2 - 2b^2) \cos^2 u - a^2 \sin^2 u) > 0$ and $((2a^2 - b^2) \sin^2 u + b^2 \cos^2 u) > 0$ for any u . From (15) and (16) it follows that the curve $\mathcal{G}$ and the surface $\tilde{S}$ have no self-intersecting and self-contacts points. $\square$

6. Examples

Right circular cylinder. This particular case of an elliptic cylinder has parametrizations (7) or (9) with an additional condition $a = b$. From (13) or (14) it follows that the focal surface of right circular cylinder degenerates into the z -axis. Similarly, by (16) or (17), the generalized focal surface of the right circular cylinder coincides with the original cylinder. More precisely, the generalized focal surface can be obtained from the original right circular cylinder by the symmetry with respect to z -axis.

Elliptic cylinder implicitly defined by $\frac{x^2}{9} + \frac{y^2}{4} = 1$. In this case the standard parametrization (7) is determined by replacing a and b with 3 and 2, respectively. Then by (13), the focal surface is given by

$$\mathbf{r}^f(u, v) = \left(\frac{5 \cos^3 u}{3}, -\frac{5 \sin^3 u}{2}, v \right).$$

Furthermore, using (16) we can write the parametrization of the generalized focal surface as follows:

$$\tilde{\mathbf{r}}(u, v) = \begin{pmatrix} \frac{\cos u}{3} (\cos^2 u - 9 \sin^2 u) \\ -\frac{\sin u}{2} (4 \cos^2 u + 14 \sin^2 u) \\ v \end{pmatrix}^T.$$

Since $\frac{3}{2} > \sqrt{2}$, this cylindrical surface has two self-intersecting lines. The original elliptic cylinder and its generalized focal surface are plotted in Figure 1.

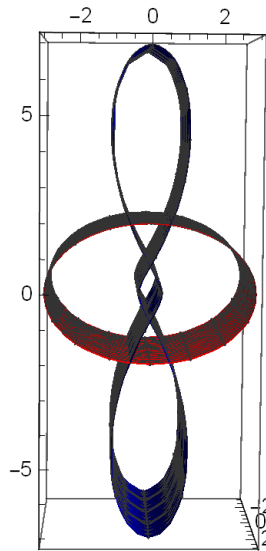


Figure 1: The elliptic cylinder (in red) and its generalized focal surface (in blue) with two self-intersecting lines.

Elliptic cylinder defined by $\frac{x^2}{16} + \frac{y^2}{9} = 1$. In this case $a = 4$, $b = 3$ and $\frac{a}{b} = \frac{4}{3} < \sqrt{2}$. According to Proposition 3, the focal surface given by

$$\mathbf{r}^f(u, v) = \left(\frac{7 \cos^3 u}{4}, -\frac{7 \sin^3 u}{3}, v \right).$$

is placed inside the original elliptic cylinder. Theorem 5 now shows that the generalized focal surface has no self-intersecting and self-contact points. By

(16), its parametrization can be written as

$$\tilde{\mathbf{r}}(u, v) = \begin{pmatrix} \frac{\cos u}{4} (-2 \cos^2 u - 16 \sin^2 u) \\ -\frac{\sin u}{3} (9 \cos^2 u + 23 \sin^2 u) \\ v \end{pmatrix}^T.$$

The original elliptic cylinder, the focal surface and the generalized focal surface are plotted in Figure 2.

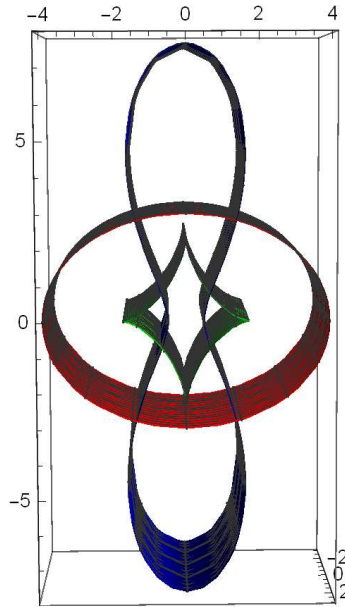


Figure 2: The focal surface (in green) and the generalized focal surface (in blue) of the original elliptic cylinder (in red).

7. Conclusion

In this paper we examine two rational surfaces which are associated to a given elliptic cylinder. The first surface is the unique focal surface of the elliptic cylinder. We show that this surface is a right astroidal cylinder with four cuspidal edges. The second associated surface is a generalized offset to the elliptic cylinder which is obtained by the reciprocal mean curvature as a variable distance function. We prove that such a kind of a generalized offset is a right

regular cylinder. We also classify these generalized offsets with respect to their self-intersecting and self-contact lines.

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