ON $(1, 2)^\ast$-PREGENERALIZED CLOSED SETS
IN BITOPOLITICAL SPACES

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Abstract: In this paper, we introduce the concept of $(1, 2)^\ast$-pregeneralized closed (briefly, $(1, 2)^\ast$-pg-closed) sets and study its basic properties. Furthermore, we define $(1, 2)^\ast$-pg-closed functions and we obtain some important results of this class and investigate the relationships with other functions. Also, we define and investigate the notion of $(1, 2)^\ast$-normal spaces.

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1. Introduction and Preliminaries

Levine [3] introduced the concept of generalized closed sets in topological spaces. After the work of Levine various authors [1, 2, 4] have contributed to the development of this theory. The purpose of this paper is to obtain some generalized results in bitopological spaces. In most of the occasions, our generalizations can be illustrated by suitable examples.

Throughout this paper $(X, \tau_1, \tau_2), (Y, \sigma_1, \sigma_2)$ and $(Z, \nu_1, \nu_2)$ (or briefly, $X, Y$ and $Z$) are bitopological spaces.
Definition 1.1. ([5]) Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(A\) a subset of \(X\). Then \(A\) is said to be \(\tau_{1,2}\)-open if \(A = S \cup T\), where \(S \in \tau_1\) and \(T \in \tau_2\). The complement of a \(\tau_{1,2}\)-open set is said to be \(\tau_{1,2}\)-closed.

Definition 1.2. ([5]) Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(A\) a subset of \(X\). Then

1. The \(\tau_{1,2}\)-closure of \(A\), denoted by \(\tau_{1,2}-cl(A)\), is defined by \(\cap\{F : A \subset F\) and \(F\) is \(\tau_{1,2}\)-closed\}. 

2. The \(\tau_{1,2}\)-interior of \(A\), denoted by \(\tau_{1,2}-int(A)\), is defined by \(\cup\{U : U \subset A\) and \(U\) is \(\tau_{1,2}\)-open \}. 

Definition 1.3. Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(A\) a subset of \(X\). Then \(A\) is said to be

1. regular \((1, 2)^*-open\) (resp. regular \((1, 2)^*-closed\)) [8] if \(A = \tau_{1,2}-int(\tau_{1,2}-cl(A))\) (resp. \(A = \tau_{1,2}-cl(\tau_{1,2}-int(A))\)).

2. \((1, 2)^*-preclosed\) [9] if \(\tau_{1,2}-cl(\tau_{1,2}-int(A)) \subset A\).

3. \((1, 2)^*-g-closed\) [8] if \(\tau_{1,2}-cl(A) \subset U\) whenever \(A \subset U\) and \(U\) is \(\tau_{1,2}\)-open in \(X\).

4. \((1, 2)^*-rg-closed\) [8] if \(\tau_{1,2}-cl(A) \subset U\) whenever \(A \subset U\) and \(U\) is regular \((1, 2)^*-open\) in \(X\).

The complement of \((1, 2)^*-preclosed\) (resp. \((1, 2)^*-g-closed, (1, 2)^*-rg-closed\)) set is called \((1, 2)^*-preopen\) (resp. \((1, 2)^*-g-open, (1, 2)^*-rg-open\)).

The family of all \(\tau_{1,2}\)-open (resp. \(\tau_{1,2}\)-closed, \(\tau_{1,2}\)-preopen, \(\tau_{1,2}\)-preclosed, \(\tau_{1,2}\)-\(g\)-closed, \(\tau_{1,2}\)-\(rg\)-closed) sets of \(X\) are denoted by \(\tau_{1,2}-O(X)\) (resp. \(\tau_{1,2}-C(X), (1, 2)^*-PO(X), (1, 2)^*-PC(X), (1, 2)^*-GC(X), (1, 2)^*-RGC(X)\)).

Definition 1.4. A mapping \(f : X \to Y\) is called

1. \((1, 2)^*-continuous\) [6] if \(f^{-1}(U)\) is \(\tau_{1,2}\)-open in \(X\) for each \(\sigma_{1,2}\)-open set \(U\) in \(Y\).

2. \((1, 2)^*-closed\) [6] if for each \(\tau_{1,2}\)-closed set \(F\) of \(X\), \(f(F)\) is \(\sigma_{1,2}\)-closed in \(Y\).

3. \((1, 2)^*-g-closed\) [8] if \(f(F)\) is \((1, 2)^*-g\)-closed in \(Y\) for every \(\tau_{1,2}\)-closed set \(F\) of \(X\).

4. \((1, 2)^*-rg-closed\) [8] if \(f(F)\) is \((1, 2)^*-rg\)-closed in \(Y\) for every \(\tau_{1,2}\)-closed set \(F\) of \(X\).
5. $(1, 2)^*\text{-preclosed}$ [9] if $f(F)$ is $(1, 2)^*\text{-preclosed}$ in $Y$ for every $\tau_{1,2}\text{-closed}$ set $F$ of $X$.

**Lemma 1.5.** For any subset $A$ of $X$, $\tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(A) \setminus A) = \phi$.

**Proof.** The proof is obvious. \hfill \Box

**Theorem 1.6.** Let $A$ be a subset of $X$. Then
1. Every regular $(1, 2)^*\text{-closed}$ set is $\tau_{1,2}\text{-closed}$ [8].
2. Every $\tau_{1,2}\text{-closed}$ set is $(1, 2)^*\text{-preclosed}$ [7].
3. Every regular $(1, 2)^*\text{-closed}$ set is $(1, 2)^*\text{-preclosed}$ [7].
4. Every $(1, 2)^*\text{-g-closed}$ set is $(1, 2)^*\text{-rg-closed}$ [6].

**Theorem 1.7.** ([8]) Let $f : X \to Y$ be a mapping. Then
1. Every $\tau_{1,2}\text{-closed}$ mapping is $(1, 2)^*\text{-g-closed}$. 
2. Every $(1, 2)^*\text{-g-closed}$ mapping is $(1, 2)^*\text{-rg-closed}$. 

## 2. $(1, 2)^*\text{-pregeneralized-closed sets}$

**Definition 2.1.** A subset $A$ of a bitopological space $(X, \tau_1, \tau_2)$ is called $(1, 2)^*\text{-pregeneralized closed}$ (briefly, $(1, 2)^*\text{-pg-closed}$) if $\tau_{1,2}\text{-cl}(A) \subset U$ whenever $A \subseteq U$ and $U$ is an $(1, 2)^*\text{-preopen}$ in $X$.

The family of all $(1, 2)^*\text{-pg-closed}$ sets of $X$ is denoted by $(1, 2)^*\text{-PGC}(X)$.

**Definition 2.2.** Let $A$ be a subset of $X$. Then
1. The $\tau_{1,2}\text{-closure}$ $p$ of $A$, denoted by $\tau_{1,2}\text{-cl}_p(A)$, is defined by $\bigcap\{F : A \subset F$ and $F$ is $(1, 2)^*\text{-preclosed}\}$.
2. The $\tau_{1,2}\text{-interior}$ $p$ of $A$, denoted by $\tau_{1,2}\text{-int}_p(A)$, is defined by $\bigcup\{F : F \subset A$ and $F$ is $(1, 2)^*\text{-preopen}\}$.

**Theorem 2.3.** If $A$ is $\tau_{1,2}\text{-closed}$, then $A$ is $(1, 2)^*\text{-pg-closed}$. 

**Proof.** It is obvious. \hfill \Box

**Remark 2.4.** The converse of Theorem 2.3 need not true, as shown by the following example.
Example 2.5. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$, $\tau_2 = \{\phi, \{b\}, X\}$. Then $\tau_{1,2}$-$O(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$, $\tau_{1,2}$-$C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$. Then $A = \{a, b\}$ is $(1,2)*$-pg-closed but it is not $\tau_{1,2}$-closed.

**Theorem 2.6.** Every $(1,2)*$-pg-closed is $(1,2)*$-g-closed.

**Proof.** It follows from Definitions 1.3 (iii) and Theorem 1.6 (ii).

**Remark 2.7.** The converse of Theorem 2.6 need not true, as shown by the following example.

Example 2.8. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{b, c\}, X\}$. Then $\tau_{1,2}$-$O(X) = \{\phi, \{a\}, \{b, c\}, X\}$. Then $A = \{b\}$ is $(1,2)*$-g-closed but it is not $(1,2)*$-pg-closed.

**Theorem 2.9.** The union of two $(1,2)*$-pg-closed sets are $(1,2)*$-pg-closed.

**Proof.** Let $A \cup B \subseteq U$, then $A \subseteq U$ and $B \subseteq U$ where $U$ is $(1,2)*$-preopen set. As $A$ and $B$ are $(1,2)*$-pg-closed, $\tau_{1,2}$-$cl(A) \subseteq U$ and $\tau_{1,2}$-$cl(B) \subseteq U$. Hence $\tau_{1,2}$-$cl(A \cup B) = \tau_{1,2}$-$cl(A) \cup \tau_{1,2}$-$cl(B) \subseteq U$.

**Remark 2.10.** The intersection of two $(1,2)*$-pg-closed sets need not be $(1,2)*$-pg-closed as shown by the following example.

Example 2.11. Refer Example 2.8. Clearly $(1,2)*$-PGC$(X) = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, then $A = \{a, b\}$ and $B = \{b, c\}$ are two $(1,2)*$-pg-closed sets but $A \cap B$ is not $(1,2)*$-pg-closed.

**Theorem 2.12.** An $(1,2)*$-pg-closed set $A$ is $\tau_{1,2}$-closed in $X$ if and only if $\tau_{1,2}$-$cl(A) \setminus A$ is regular $(1,2)*$-closed set.

**Proof.** Suppose $A$ is $\tau_{1,2}$-closed. Then $\tau_{1,2}$-$cl(A) \setminus A = \phi$ and hence $\tau_{1,2}$-$cl(A) \setminus A$ is regular $(1,2)*$-closed set.

Conversely suppose $\tau_{1,2}$-$cl(A) \setminus A$ is regular $(1,2)*$-closed. Then $X \setminus [\tau_{1,2}$-$cl(A) \setminus A] = (X \setminus \tau_{1,2}$-$cl(A)) \cup A$ is regular $(1,2)*$-open and hence $(1,2)*$-preopen. Since $A \subseteq (X \setminus \tau_{1,2}$-$cl(A)) \cup A$ and $A$ is $(1,2)*$-pg-closed, $\tau_{1,2}$-$cl(A) \subseteq (X \setminus \tau_{1,2}$-$cl(A)) \cup A$ and $\tau_{1,2}$-$cl(A) \subseteq A$. Therefore $\tau_{1,2}$-$cl(A) = A$. Hence $A$ is $\tau_{1,2}$-closed set.

**Theorem 2.13.** If $A$ is $(1,2)*$-pg-closed and $A \subseteq B \subseteq \tau_{1,2}$-$cl(A)$, then $B$ is $(1,2)*$-pg-closed.

**Proof.** Let $B \subseteq U$ and $U$ be $(1,2)*$-preopen. Then $A \subseteq U$. Since $A$ is $(1,2)*$-pg-closed, we have $\tau_{1,2}$-$cl(B) \subseteq \tau_{1,2}$-$cl(A) \subseteq U$. Hence $B$ is $(1,2)*$-pg-closed.
**Theorem 2.14.** If $A$ is $(1, 2)^*$-pg-closed set and $A \in (1, 2)^*$-$PO(X)$, then $A$ is $\tau_{1,2}$-closed.

**Proof.** Since $A$ is $(1, 2)^*$-pg-closed and $A \in (1, 2)^*$-$PO(X)$, then $\tau_{1,2}$-$cl(A) \subseteq A$. Hence $A$ is $\tau_{1,2}$-closed. \hfill \Box

**Theorem 2.15.** For each $x \in X$, either $\{x\} \in (1, 2)^*$-$PC(X)$ or $X \setminus \{x\}$ is $(1, 2)^*$-pg-closed.

**Proof.** Suppose that $\{x\} \notin (1, 2)^*$-$PC(X)$. Since $X \setminus \{x\}$ is not $(1, 2)^*$-preopen set, the space $X$ itself is only $(1, 2)^*$-preopen set containing $X \setminus \{x\}$. Therefore $\tau_{1,2}$-$cl(X \setminus \{x\}) \subseteq X$ holds and so, $X \setminus \{x\}$ is $(1, 2)^*$-pg-closed. \hfill \Box

**Definition 2.16.** A subset $A$ in $(X, \tau_1, \tau_2)$ is said to be $(1, 2)^*$-pg-open if $X \setminus A$ is $(1, 2)^*$-pg-closed.

**Theorem 2.17.** The intersection of two $(1, 2)^*$-pg-open sets is $(1, 2)^*$-pg-open.

**Proof.** The proof is obvious. \hfill \Box

**Theorem 2.18.** A subset $A$ is $(1, 2)^*$-pg-open if and only if $F \subseteq \tau_{1,2}$-$int(A)$ whenever $F \in (1, 2)^*$-$PC(X)$ and $F \subseteq A$.

**Proof.** Let $A$ be a $(1, 2)^*$-pg-open set in $X$ and $F \in (1, 2)^*$-$PC(X)$ such that $F \subseteq A$. Then $X \setminus A \subset X \setminus F$. Since $X \setminus F$ is a $(1, 2)^*$-preopen set and $(1, 2)^*$-pg-closedness of $X \setminus A$, $\tau_{1,2}$-$cl(X \setminus A) \subseteq X \setminus F$ implies $X \setminus \tau_{1,2}$-$int(A) \subseteq X \setminus F$ implies $F \subseteq \tau_{1,2}$-$int(A)$.

Conversely, let $F \in (1, 2)^*$-$PC(X)$ and $F \subseteq A$ implies $F \subseteq \tau_{1,2}$-$int(A)$. Let $X \setminus A \subset U$ when $U \in (1, 2)^*$-$PO(X)$. Since $X \setminus U \in (1, 2)^*$-$PC(X)$, by hypothesis, $X \setminus U \subset \tau_{1,2}$-$int(A)$. Therefore $X \setminus \tau_{1,2}$-$int(A) \subset U$ implies $\tau_{1,2}$-$cl(X \setminus A) \subseteq U$. Thus $X \setminus A$ is a $(1, 2)^*$-pg-closed set in $X$. Hence $A$ is a $(1, 2)^*$-pg-open set in $X$. \hfill \Box

**Theorem 2.19.** If $\tau_{1,2}$-$int(A) \subseteq B \subset A$ and $A$ is $(1, 2)^*$-pg-open, then $B$ is $(1, 2)^*$-pg-open.

**Proof.** Let $\tau_{1,2}$-$int(A) \subseteq B \subset A$ and $A$ be $(1, 2)^*$-pg-open. Then $X \setminus A \subset X \setminus B \subset \tau_{1,2}$-$cl(X \setminus A)$ and $X \setminus A$ is $(1, 2)^*$-pg-closed. By using Theorem 2.13., $B$ is $(1, 2)^*$-pg-open. \hfill \Box

**Theorem 2.20.** A set $A$ is $(1, 2)^*$-pg-closed in $X$ if and only if $\tau_{1,2}$-$cl(A) \setminus A$ is $(1, 2)^*$-pg-open.
Proof. Let $A$ be $(1,2)^*\text{-}pg$-closed. Suppose that $F \subset \tau_{1,2}\text{-}cl(A) \setminus A$ and $F$ is $(1,2)^*\text{-}preclosed$. Then $A \subset X \setminus F$ and $X \setminus F$ is $(1,2)^*\text{-}preopen$. Hence $\tau_{1,2}\text{-}cl(A) \subset X \setminus F$. Therefore, $F \subset [X \setminus \tau_{1,2}\text{-}cl(A)] \cap \tau_{1,2}\text{-}cl(A) = \emptyset$. For any $(1,2)^*\text{-}preclosed$ set $F$ such that $F \subset \tau_{1,2}\text{-}cl(A) \setminus A$, $\emptyset = F \subset \tau_{1,2}\text{-}int(\tau_{1,2}\text{-}cl(A) \setminus A)$ and hence by Theorem 2.18, $\tau_{1,2}\text{-}cl(A) \setminus A$ is $(1,2)^*\text{-}pg$-open.

Conversely, suppose $\tau_{1,2}\text{-}cl(A) \setminus A$ is $(1,2)^*\text{-}pg$-open set. Let $A \subset F$ where $F \in (1,2)^*\text{-}PO(X)$. Then $X \setminus F \subset X \setminus A$ that is $\tau_{1,2}\text{-}cl(A) \cap (X \setminus F) \subset \tau_{1,2}\text{-}cl(A) \cap (X \setminus A)$. Thus $\tau_{1,2}\text{-}cl(A) \cap (X \setminus F)$ is a $(1,2)^*\text{-}preclosed$ subset of $\tau_{1,2}\text{-}cl(A) \cap (X \setminus A) = \tau_{1,2}\text{-}cl(A)|A$. Therefore by Theorem 2.18, $\tau_{1,2}\text{-}cl(A) \cap (X \setminus F)$ is $\tau_{1,2}\text{-}int(\tau_{1,2}\text{-}cl(A) \setminus A) = \emptyset$, by Lemma 1.5. Hence $\tau_{1,2}\text{-}cl(A) \subset F$ implies $A$ is a $(1,2)^*\text{-}pg$-closed set.

**Theorem 2.21.** A subset $A$ is $(1,2)^*\text{-}pg$-closed if and only if $\tau_{1,2}\text{-}cl_p(\{x\}) \cap A \neq \emptyset$ for every $x \in \tau_{1,2}\text{-}cl(A)$.

Proof. Suppose that $\tau_{1,2}\text{-}cl_p(\{x\}) \cap A = \emptyset$ for some $x \in \tau_{1,2}\text{-}cl(A)$. Then $X \setminus (\tau_{1,2}\text{-}cl_p(\{x\})) \in (1,2)^*\text{-}PO(X)$ such that $A \subset X \setminus (\tau_{1,2}\text{-}cl_p(\{x\}))$. Furthermore, $x \in \tau_{1,2}\text{-}cl(A) \setminus (X \setminus \tau_{1,2}\text{-}cl_p(\{x\}))$ and hence $\tau_{1,2}\text{-}cl(A) \not\subset (X \setminus \tau_{1,2}\text{-}cl_p(\{x\}))$. This shows that $A$ is not $(1,2)^*\text{-}pg$-closed.

Now, suppose that $A$ is not $(1,2)^*\text{-}pg$-closed. There exists $U \in (1,2)^*\text{-}PO(X)$ such that $A \subset U$ and $\tau_{1,2}\text{-}cl(A) \setminus U \neq \emptyset$. There exists $x \in \tau_{1,2}\text{-}cl(A)$ such that $x \notin U$. Hence $\tau_{1,2}\text{-}cl_p(\{x\}) \cap U = \emptyset$. Therefore $\tau_{1,2}\text{-}cl_p(\{x\}) \cap A = \emptyset$ for some $x \in \tau_{1,2}\text{-}cl(A)$.

3. $(1,2)^*\text{-}pg$-closed mappings

We introduce new classes of generalized closed mappings as follows:

**Definition 3.1.** A mapping $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be $(1,2)^*\text{-}pg$-closed if $f(F)$ is a $(1,2)^*\text{-}pg$-closed set in $Y$ for every $\tau_{1,2}$-closed set $F$ of $X$.

**Remark 3.2.** The composition of two $(1,2)^*\text{-}pg$-closed (resp. $(1,2)^*\text{-}pg$-open) mappings need not be $(1,2)^*\text{-}pg$-closed (resp. $(1,2)^*\text{-}pg$-open).

**Example 3.3.** Let $X = Y = \{a,b,c\}$, $\tau_1 = \{\phi, \{a\}, X\}$, $\tau_2 = \{\phi, \{b\}, X\}$, $\sigma_1 = \{\phi, \{c\}, Y\}$ and $\sigma_2 = \{\phi, \{a\}, Y\}$. Then $\tau_{1,2}\text{-}O(X) = \{\phi, \{a\}, \{b\}, \{a,b\}, X\}$, $\tau_{1,2}\text{-}C(X) = \{\phi, \{c\}, \{a,c\}, \{b,c\}, X\}$, $\sigma_{1,2}\text{-}O(Y) = \{\phi, \{c\}, \{a,c\}, Y\}$, $\sigma_{1,2}\text{-}C(Y) = \{\phi, \{b\}, \{a,b\}, Y\}$ and $(1,2)^*\text{-}PGC(Y) = P(Y)$. Let $Z = \{p, q, r\}$, $\nu_1 = \{\phi, \{p\}, Z\}$ and $\nu_2 = \{\phi, \{q, r\}, Z\}$. Then $\nu_{1,2}\text{-}O(Z) = \{\phi, \{p\}, \{q, r\}\}$,
be the identity mapping. Clearly \( f(1) \) is not a \( (1,2)^* \)-closed set but it is not a \( (1,2)^* \)-pg-open. However, their composition is not \( (1,2)^* \)-pg-closed (resp. \( (1,2)^* \)-pg-open). Since \( g(f(\{c\})) = \{r\} \) is not a \( (1,2)^* \)-pg-closed set in \( Z \) and \( g(f(\{a,b\})) = \{p,q\} \) is not a \( (1,2)^* \)-pg-open set in \( Z \).

**Remark 3.4.** A bijective mapping is \( (1,2)^* \)-pg-open if and only if it is \( (1,2)^* \)-pg-closed.

**Theorem 3.5.** Every \( (1,2)^* \)-closed mapping is \( (1,2)^* \)-pg-closed.

**Proof.** It is obvious from Theorem 2.3. \( \square \)

**Remark 3.6.** The converse of Theorem 3.5 need not be true as shown by the following example.

**Example 3.7.** Let \( X = Y = \{a,b,c\}, \tau_1 = \{a\}, X \), \( \tau_2 = \{b\}, X \), \( \sigma_1 = \{\phi\}, Y \), \( \sigma_2 = \{\phi\}, Y \). Then \( \tau_{1,2}(X) = \{\phi\}, X \), \( \sigma_{1,2}(Y) = \{\phi\}, Y \) and \( (1,2)^* \)-PGC \( Y = P(Y) \). Define \( f : X \to Y \) be the identity mapping. Clearly \( f \) is a \( (1,2)^* \)-pg-closed mapping but it is not \( (1,2)^* \)-closed, since \( \{c\} \) is \( \tau_{1,2} \)-closed set but it is not \( \sigma_{1,2} \)-closed.

**Theorem 3.8.** Every \( (1,2)^* \)-pg-closed mapping is \( (1,2)^* \)-rg-closed.

**Proof.** It is obvious from Theorems 1.6 and 2.6. \( \square \)

**Example 3.9.** The converse of Theorem 3.8 may be not true.

Let \( X = Y = \{a,b,c\}, \tau_1 = \{\phi\}, X \), \( \tau_2 = \{a\}, X \), \( \sigma_1 = \{\phi\}, Y \) and \( \sigma_2 = \{\phi\}, Y \). Then \( \tau_{1,2}(X) = \{\phi\}, X \), \( \sigma_{1,2}(Y) = \{\phi\}, Y \) and \( (1,2)^* \)-PGC \( Y = P(Y) \). Define \( f : X \to Y \) be the identity mapping. Clearly \( f \) is a \( (1,2)^* \)-pg-closed mapping but it is not \( (1,2)^* \)-pg-closed, since \( \{b\} \) is \( \tau_{1,2} \)-closed set in \( X \), \( f(\{b\}) = \{b\} \) is a \( (1,2)^* \)-rg-closed set but it is not a \( (1,2)^* \)-pg-closed set in \( Y \).

**Remark 3.10.** For a mapping, the following implications hold.

\[
(1,2)^* \text{-closed} \quad \rightarrow \quad (1,2)^* \text{-g-closed} \quad \rightarrow \quad (1,2)^* \text{-rg-closed}
\]

\[
(1,2)^* \text{-pg-closed}
\]
**Theorem 3.11.** If a mapping \( f : X \to Y \) is \((1,2)^*\)-closed and \( g : Y \to Z \) is \((1,2)^*\)-pg-closed, then \( g \circ f : X \to Z \) is \((1,2)^*\)-pg-closed.

**Proof.** Let \( A \) be any \( \tau_{1,2} \)-closed set of \( X \). Since \( f \) is \((1,2)^*\)-closed, \( f(A) \) is a \( \sigma_{1,2} \)-closed set in \( Y \). Since \( g \) is \((1,2)^*\)-pg-closed, \( g(f(A)) \) is \((1,2)^*\)-pg-closed in \( Z \).

**Theorem 3.12.** A mapping \( f : X \to Y \) is \((1,2)^*\)-pg-closed if and only if for each subset \( S \) of \( Y \) and for each \( \tau_{1,2} \)-open set \( U \) containing \( f^{-1}(S) \) there is a \((1,2)^*\)-pg-open set \( V \) of \( Y \) such that \( S \subseteq V \) and \( f^{-1}(V) \subseteq U \).

**Proof.** Suppose \( f \) is \((1,2)^*\)-pg-closed. Let \( S \) be a subset of \( Y \) and \( U \) be an \( \tau_{1,2} \)-open set of \( X \) such that \( f^{-1}(S) \subseteq U \). Then \( V = Y \setminus f(X \setminus U) \) is an \((1,2)^*\)-pg-open set containing \( S \) such that \( f^{-1}(V) \subseteq U \).

Clearly, suppose that \( F \) is a \( \tau_{1,2} \)-closed set of \( X \). Then \( f^{-1}(Y \setminus f(F)) \subseteq X \setminus F \) and \( X \setminus F \) is \( \tau_{1,2} \)-open. By hypothesis, there is an \((1,2)^*\)-pg-open set \( V \) of \( Y \) such that \( Y \setminus f(F) \subseteq V \) and \( f^{-1}(V) \subseteq X \setminus F \). Therefore, \( F \subseteq X \setminus f^{-1}(V) \). Hence \( Y \setminus V \subseteq f(F) \subseteq f(X \setminus f^{-1}(V)) \subseteq Y \setminus V \). Since \( Y \setminus V \) is \((1,2)^*\)-pg-closed, \( f(F) \) is \((1,2)^*\)-pg-closed in \( Y \) and thus \( f \) is \((1,2)^*\)-pg-closed.

### 4. \((1,2)^*\)-normal spaces

**Definition 4.1.** A bitopological space \((X, \tau_1, \tau_2)\) is said to be \((1,2)^*\)-normal if for disjoint \( \tau_{1,2} \)-closed sets \( F_1 \) and \( F_2 \), there exist \( \tau_{1,2} \)-open sets \( U_1, U_2 \) such that \( F_1 \subseteq U_1, F_2 \subseteq U_2 \) and \( U_1 \cap U_2 = \emptyset \).

**Theorem 4.2.** For a bitopological space \((X, \tau_1, \tau_2)\), the following properties are equivalent:

1. \((X, \tau_1, \tau_2)\) is \((1,2)^*\)-normal.

2. For any disjoint \( \tau_{1,2} \)-closed sets \( F_1, F_2 \), there exist \((1,2)^*\)-pg-open sets \( V_1, V_2 \) such that \( F_1 \subseteq V_1, F_2 \subseteq V_2 \) and \( V_1 \cap V_2 = \emptyset \).

3. For any \( \tau_{1,2} \)-closed set \( F \) and any \( \tau_{1,2} \)-open set \( U \) containing \( F \), there exists an \((1,2)^*\)-pg-open set \( V \) such that \( F \subseteq V \subseteq \tau_{1,2}\text{-cl}(V) \subseteq U \).

4. For any \( \tau_{1,2} \)-closed set \( F \) and any \( \tau_{1,2} \)-open set \( U \) containing \( F \), there exists a \( \tau_{1,2} \)-open set \( G \) such that \( F \subseteq G \subseteq \tau_{1,2}\text{-cl}(G) \subseteq U \).

5. For any disjoint \( \tau_{1,2} \)-closed sets \( F_1, F_2 \), there exists an \((1,2)^*\)-pg-open set \( V \) such that \( F_1 \subseteq V \) and \( \tau_{1,2}\text{-cl}(V) \cap F_2 = \emptyset \).
6. For any disjoint $\tau_{1,2}$-closed sets $F_1, F_2$, there exists a $\tau_{1,2}$-open set $G$ such that $F_1 \subset G$ and $\tau_{1,2} \text{-cl}(G) \cap F_2 = \phi$.

Proof. (1) $\Rightarrow$ (2) : It follows from Theorem 2.3.

(2) $\Rightarrow$ (3) : Let $F$ be a $\tau_{1,2}$-closed set and $U$ is a $\tau_{1,2}$-open set such that $F \subset U$. Then $F, X \setminus U$ are disjoint $\tau_{1,2}$-closed sets and by (2) there exist $(1, 2)^*\text{-pg}$-open sets $V_1$ and $V_2$ such that $F \subset V_1, X \setminus U \subset V_2$ and $V_1 \cap V_2 = \phi$. Since $V_2$ is $(1, 2)^*\text{-pg}$-open, by using Theorem 2.18, $X \setminus U \subset \tau_{1,2}\text{-int}(V_2)$ and $\tau_{1,2}\text{-int}(V_2)$ is $\tau_{1,2}$-open. Hence, $\tau_{1,2}\text{-cl}(V_1) \cap \tau_{1,2}\text{-int}(V_2) = \phi$. Therefore, we obtain $F \subset V_1 \subset \tau_{1,2}\text{-cl}(V_1) \subset X \setminus \tau_{1,2}\text{-int}(V_2) \subset U$.

(3) $\Rightarrow$ (4) : Let $F$ be a $\tau_{1,2}$-closed set and $U$ be a $\tau_{1,2}$-open set such that $F \subset U$. Then by using (3), there exists an $(1, 2)^*\text{-pg}$-open set $V$ such that $F \subset V \subset \tau_{1,2}\text{-cl}(V) \subset U$. By using Theorem 2.18, $F \subset \tau_{1,2}\text{-int}(V)$. Put $G = \tau_{1,2}\text{-int}(V)$. Then $F \subset G \subset \tau_{1,2}\text{-cl}(G) \subset \tau_{1,2}\text{-cl}(V) \subset U$.

(4) $\Rightarrow$ (5) : Let $F_1, F_2$ be any disjoint $\tau_{1,2}$-closed sets. Since $X \setminus F_2$ is a $\tau_{1,2}$-open set such that $F_1 \subset X \setminus F_2$, by (4) there exists a $\tau_{1,2}$-open set $V$ such that $F_1 \subset V \subset \tau_{1,2}\text{-cl}(V) \subset X \setminus F_2$. By using Theorem 2.3, $V$ is $(1, 2)^*\text{-pg}$-open. Hence $F_1 \subset V$ and $\tau_{1,2}\text{-cl}(V) \cap F_2 = \phi$.

(5) $\Rightarrow$ (6) : Let $F_1, F_2$ be any disjoint $\tau_{1,2}$-closed sets. Then there exists an $(1, 2)^*\text{-pg}$-open set $V$ such that $F_1 \subset V$ and $\tau_{1,2}\text{-cl}(V) \cap F_2 = \phi$. By using Theorem 2.18, $F_1 \subset \tau_{1,2}\text{-int}(V)$. Let $G = \tau_{1,2}\text{-int}(V)$, then we have $G$ is $\tau_{1,2}$-open, $F_1 \subset G$ and $\tau_{1,2}\text{-cl}(G) \cap F_2 = \phi$.

(6) $\Rightarrow$ (1) : Let $F_1, F_2$ be any disjoint $\tau_{1,2}$-closed sets. Then by (6), there exists a $\tau_{1,2}$-open set $G$ such that $F_1 \subset G$ and $\tau_{1,2}\text{-cl}(G) \cap F_2 = \phi$. Now, put $U_1 = G$ and $U_2 = X \setminus \tau_{1,2}\text{-cl}(G)$. Then $U_1$ and $U_2$ are disjoint $\tau_{1,2}$-open sets $F_1 \subset U_1$ and $F_2 \subset U_2$. This shows that $(X, \tau_1, \tau_2)$ is $(1, 2)^*\text{-normal}$. \hfill $\square$

Theorem 4.3. If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is a $(1, 2)^*\text{-continuous}$ and $(1, 2)^*\text{-pg}$-closed surjection and $(X, \tau_1, \tau_2)$ is $(1, 2)^*\text{-normal}$, then $(Y, \sigma_1, \sigma_2)$ is $(1, 2)^*\text{-normal}$.

Corollary 4.4. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a $(1, 2)^*\text{-continuous}$ and $(1, 2)^*\text{-closed bijection}$. Then $(X, \tau_1, \tau_2)$ is $(1, 2)^*\text{-normal}$ if and only if $(Y, \sigma_1, \sigma_2)$ is $(1, 2)^*\text{-normal}$.

References


