

ON $(1, 2)^*$ -PREGENERALIZED CLOSED SETS IN BITOPOLOGICAL SPACES

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Abstract: In this paper, we introduce the concept of $(1, 2)^*$ -pregeneralized closed (briefly, $(1, 2)^*$ -pg-closed) sets and study its basic properties. Furthermore, we define $(1, 2)^*$ -pg-closed functions and we obtain some important results of this class and investigate the relationships with other functions. Also, we define and investigate the notion of $(1, 2)^*$ -normal spaces.

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1. Introduction and Preliminaries

Levine [3] introduced the concept of generalized closed sets in topological spaces. After the work of Levine various authors [1, 2, 4] have contributed to the development of this theory. The purpose of this paper is to obtain some generalized results in bitopological spaces. In most of the occasions, our generalizations can be illustrated by suitable examples.

Throughout this paper (X, τ_1, τ_2) , (Y, σ_1, σ_2) and (Z, ν_1, ν_2) (or briefly, X , Y and Z) are bitopological spaces.

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Definition 1.1. ([5]) Let (X, τ_1, τ_2) be a bitopological space and A a subset of X . Then A is said to be $\tau_{1,2}$ -open if $A = S \cup T$, where $S \in \tau_1$ and $T \in \tau_2$. The complement of a $\tau_{1,2}$ -open set is said to be $\tau_{1,2}$ -closed.

Definition 1.2. ([5]) Let (X, τ_1, τ_2) be a bitopological space and A a subset of X . Then

1. The $\tau_{1,2}$ -closure of A , denoted by $\tau_{1,2}\text{-cl}(A)$, is defined by $\cap\{F : A \subset F \text{ and } F \text{ is } \tau_{1,2}\text{-closed}\}$.
2. The $\tau_{1,2}$ -interior of A , denoted by $\tau_{1,2}\text{-int}(A)$, is defined by $\cup\{U : U \subset A \text{ and } U \text{ is } \tau_{1,2}\text{-open}\}$.

Definition 1.3. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X . Then A is said to be

1. regular $(1, 2)^*$ -open (resp. regular $(1, 2)^*$ -closed) [8] if $A = \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(A))$ (resp. $A = \tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(A))$).
2. $(1, 2)^*$ -preclosed [9] if $\tau_{1,2}\text{-cl}(\tau_{1,2}\text{-int}(A)) \subseteq A$.
3. $(1, 2)^*$ - g -closed [8] if $\tau_{1,2}\text{-cl}(A) \subseteq U$ whenever $A \subset U$ and U is $\tau_{1,2}$ -open in X .
4. $(1, 2)^*$ - rg -closed [8] if $\tau_{1,2}\text{-cl}(A) \subset U$ whenever $A \subset U$ and U is regular $(1, 2)^*$ -open in X .

The complement of $(1, 2)^*$ -preclosed (resp. $(1, 2)^*$ - g -closed, $(1, 2)^*$ - rg -closed) set is called $(1, 2)^*$ -preopen (resp. $(1, 2)^*$ - g -open, $(1, 2)^*$ - rg -open).

The family of all $\tau_{1,2}$ -open (resp. $\tau_{1,2}$ -closed, $(1, 2)^*$ -preopen, $(1, 2)^*$ -preclosed, $(1, 2)^*$ - g -closed, $(1, 2)^*$ - rg -closed) sets of X are denoted by $\tau_{1,2}\text{-O}(X)$ (resp. $\tau_{1,2}\text{-C}(X)$, $(1, 2)^*\text{-PO}(X)$, $(1, 2)^*\text{-PC}(X)$, $(1, 2)^*\text{-GC}(X)$, $(1, 2)^*\text{-RGC}(X)$).

Definition 1.4. A mapping $f : X \rightarrow Y$ is called

1. $(1, 2)^*$ -continuous [6] if $f^{-1}(U)$ is $\tau_{1,2}$ -open in X for each $\sigma_{1,2}$ -open set U in Y .
2. $(1, 2)^*$ -closed [6] if for each $\tau_{1,2}$ -closed set F of X , $f(F)$ is $\sigma_{1,2}$ -closed in Y .
3. $(1, 2)^*$ - g -closed [8] if $f(F)$ is $(1, 2)^*$ - g -closed in Y for every $\tau_{1,2}$ -closed set F of X .
4. $(1, 2)^*$ - rg -closed [8] if $f(F)$ is $(1, 2)^*$ - rg -closed in Y for every $\tau_{1,2}$ -closed set F of X .

5. $(1, 2)^*$ -preclosed [9] if $f(F)$ is $(1, 2)^*$ -preclosed in Y for every $\tau_{1,2}$ -closed set F of X .

Lemma 1.5. For any subset A of X , $\tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(A) \setminus A) = \phi$.

Proof. The proof is obvious. □

Theorem 1.6. Let A be a subset of X . Then

1. Every regular $(1, 2)^*$ -closed set is $\tau_{1,2}$ -closed [8].
2. Every $\tau_{1,2}$ -closed set is $(1, 2)^*$ -preclosed [7].
3. Every regular $(1, 2)^*$ -closed set is $(1, 2)^*$ -preclosed [7].
4. Every $(1, 2)^*$ - g -closed set is $(1, 2)^*$ - rg -closed [6].

Theorem 1.7. ([8]) Let $f : X \rightarrow Y$ be a mapping. Then

1. Every $\tau_{1,2}$ -closed mapping is $(1, 2)^*$ - g -closed.
2. Every $(1, 2)^*$ - g -closed mapping is $(1, 2)^*$ - rg -closed.

2. $(1, 2)^*$ -pregeneralized-closed sets

Definition 2.1. A subset A of a bitopological space (X, τ_1, τ_2) is called $(1, 2)^*$ -pregeneralized closed (briefly, $(1, 2)^*$ - pg -closed) if $\tau_{1,2}\text{-cl}(A) \subset U$ whenever $A \subseteq U$ and U is an $(1, 2)^*$ -preopen in X .

The family of all $(1, 2)^*$ - pg -closed sets of X is denoted by $(1, 2)^*\text{-PGC}(X)$.

Definition 2.2. Let A be a subset of X . Then

1. The $\tau_{1,2}$ -closure p of A , denoted by $\tau_{1,2}\text{-cl}_p(A)$, is defined by $\cap\{F : A \subset F \text{ and } F \text{ is } (1, 2)^*\text{-preclosed}\}$.
2. The $\tau_{1,2}$ -interior p of A , denoted by $\tau_{1,2}\text{-int}_p(A)$, is defined by $\cup\{F : F \subset A \text{ and } F \text{ is } (1, 2)^*\text{-preopen}\}$.

Theorem 2.3. If A is $\tau_{1,2}$ -closed, then A is $(1, 2)^*$ - pg -closed.

Proof. It is obvious. □

Remark 2.4. The converse of Theorem 2.3 need not true, as shown by the following example.

Example 2.5. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$, $\tau_2 = \{\phi, \{b\}, X\}$. Then $\tau_{1,2}\text{-}O(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_{1,2}\text{-}C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$. Then $A = \{a, b\}$ is $(1, 2)^*$ -pg-closed but it is not $\tau_{1,2}$ -closed.

Theorem 2.6. Every $(1, 2)^*$ -pg-closed is $(1, 2)^*$ -g-closed.

Proof. It follows from Definitions 1.3 (iii) and Theorem 1.6 (ii). $\square$

Remark 2.7. The converse of Theorem 2.6 need not true, as shown by the following example.

Example 2.8. Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{b, c\}, X\}$. Then $\tau_{1,2}\text{-}O(X) = \{\phi, \{a\}, \{b, c\}, X\}$. Then $A = \{b\}$ is $(1, 2)^*$ -g-closed but it is not $(1, 2)^*$ -pg-closed.

Theorem 2.9. The union of two $(1, 2)^*$ -pg-closed sets are $(1, 2)^*$ -pg-closed.

Proof. Let $A \cup B \subseteq U$, then $A \subseteq U$ and $B \subseteq U$ where U is $(1, 2)^*$ -preopen set. As A and B are $(1, 2)^*$ -pg-closed, $\tau_{1,2}\text{-}cl(A) \subseteq U$ and $\tau_{1,2}\text{-}cl(B) \subseteq U$. Hence $\tau_{1,2}\text{-}cl(A \cup B) = \tau_{1,2}\text{-}cl(A) \cup \tau_{1,2}\text{-}cl(B) \subseteq U$. $\square$

Remark 2.10. The intersection of two $(1, 2)^*$ -pg-closed sets need not be $(1, 2)^*$ -pg-closed as shown by the following example.

Example 2.11. Refer Example 2.8. Clearly $(1, 2)^*\text{-}PGC(X) = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, then $A = \{a, b\}$ and $B = \{b, c\}$ are two $(1, 2)^*$ -pg-closed sets but $A \cap B$ is not $(1, 2)^*$ -pg-closed.

Theorem 2.12. An $(1, 2)^*$ -pg-closed set A is $\tau_{1,2}$ -closed in X if and only if $\tau_{1,2}\text{-}cl(A) \setminus A$ is regular $(1, 2)^*$ -closed set.

Proof. Suppose A is $\tau_{1,2}$ -closed. Then $\tau_{1,2}\text{-}cl(A) \setminus A = \phi$ and hence $\tau_{1,2}\text{-}cl(A) \setminus A$ is regular $(1, 2)^*$ -closed set.

Conversely suppose $\tau_{1,2}\text{-}cl(A) \setminus A$ is regular $(1, 2)^*$ -closed. Then $X \setminus [\tau_{1,2}\text{-}cl(A) \setminus A] = (X \setminus \tau_{1,2}\text{-}cl(A)) \cup A$ is regular $(1, 2)^*$ -open and hence $(1, 2)^*$ -preopen. Since $A \subset (X \setminus \tau_{1,2}\text{-}cl(A)) \cup A$ and A is $(1, 2)^*$ -pg-closed, $\tau_{1,2}\text{-}cl(A) \subset (X \setminus \tau_{1,2}\text{-}cl(A)) \cup A$ and $\tau_{1,2}\text{-}cl(A) \subset A$. Therefore $\tau_{1,2}\text{-}cl(A) = A$. Hence A is $\tau_{1,2}$ -closed set. $\square$

Theorem 2.13. If A is $(1, 2)^*$ -pg-closed and $A \subset B \subset \tau_{1,2}\text{-}cl(A)$, then B is $(1, 2)^*$ -pg-closed.

Proof. Let $B \subseteq U$ and U be $(1, 2)^*$ -preopen. Then $A \subseteq U$. Since A is $(1, 2)^*$ -pg-closed, we have $\tau_{1,2}\text{-}cl(B) \subseteq \tau_{1,2}\text{-}cl(A) \subseteq U$. Hence B is $(1, 2)^*$ -pg-closed. $\square$

Theorem 2.14. *If A is $(1, 2)^*$ -pg-closed set and $A \in (1, 2)^*$ -PO(X), then A is $\tau_{1,2}$ -closed.*

Proof. Since A is $(1, 2)^*$ -pg-closed and $A \in (1, 2)^*$ -PO(X), then $\tau_{1,2}\text{-cl}(A) \subseteq A$. Hence A is $\tau_{1,2}$ -closed. $\square$

Theorem 2.15. *For each $x \in X$, either $\{x\} \in (1, 2)^*$ -PC(X) or $X \setminus \{x\}$ is $(1, 2)^*$ -pg-closed.*

Proof. Suppose that $\{x\} \notin (1, 2)^*$ -PC(X). Since $X \setminus \{x\}$ is not $(1, 2)^*$ -preopen set, the space X itself is only $(1, 2)^*$ -preopen set containing $X \setminus \{x\}$. Therefore $\tau_{1,2}\text{-cl}(X \setminus \{x\}) \subset X$ holds and so, $X \setminus \{x\}$ is $(1, 2)^*$ -pg-closed. $\square$

Definition 2.16. A subset A in (X, τ_1, τ_2) is said to be $(1, 2)^*$ -pg-open if $X \setminus A$ is $(1, 2)^*$ -pg-closed.

Theorem 2.17. *The intersection of two $(1, 2)^*$ -pg-open sets is $(1, 2)^*$ -pg-open.*

Proof. The proof is obvious. $\square$

Theorem 2.18. *A subset A is $(1, 2)^*$ -pg-open if and only if $F \subset \tau_{1,2}\text{-int}(A)$ whenever $F \in (1, 2)^*$ -PC(X) and $F \subseteq A$.*

Proof. Let A be a $(1, 2)^*$ -pg-open set in X and $F \in (1, 2)^*$ -PC(X) such that $F \subset A$. Then $X \setminus A \subset X \setminus F$. Since $X \setminus F$ is a $(1, 2)^*$ -preopen set and $(1, 2)^*$ -pg-closedness of $X \setminus A$, $\tau_{1,2}\text{-cl}(X \setminus A) \subset X \setminus F$ implies $X \setminus \tau_{1,2}\text{-int}(A) \subset X \setminus F$ implies $F \subset \tau_{1,2}\text{-int}(A)$.

Conversely, let $F \in (1, 2)^*$ -PC(X) and $F \subset A$ implies $F \subset \tau_{1,2}\text{-int}(A)$. Let $X \setminus A \subset U$ when $U \in (1, 2)^*$ -PO(X). Since $X \setminus U \in (1, 2)^*$ -PC(X), by hypothesis, $X \setminus U \subset \tau_{1,2}\text{-int}(A)$. Therefore $X \setminus \tau_{1,2}\text{-int}(A) \subset U$ implies $\tau_{1,2}\text{-cl}(X \setminus A) \subseteq U$. Thus $X \setminus A$ is a $(1, 2)^*$ -pg-closed set in X . Hence A is a $(1, 2)^*$ -pg-open set in X . $\square$

Theorem 2.19. *If $\tau_{1,2}\text{-int}(A) \subseteq B \subset A$ and A is $(1, 2)^*$ -pg-open, then B is $(1, 2)^*$ -pg-open.*

Proof. Let $\tau_{1,2}\text{-int}(A) \subseteq B \subset A$ and A be $(1, 2)^*$ -pg-open. Then $X \setminus A \subset X \setminus B \subset \tau_{1,2}\text{-cl}(X \setminus A)$ and $X \setminus A$ is $(1, 2)^*$ -pg-closed. By using Theorem 2.13., B is $(1, 2)^*$ -pg-open. $\square$

Theorem 2.20. *A set A is $(1, 2)^*$ -pg-closed in X if and only if $\tau_{1,2}\text{-cl}(A) \setminus A$ is $(1, 2)^*$ -pg-open.*

Proof. Let A be $(1, 2)^*$ - pg -closed. Suppose that $F \subset \tau_{1,2}\text{-cl}(A) \setminus A$ and F is $(1, 2)^*$ -preclosed. Then $A \subset X \setminus F$ and $X \setminus F$ is $(1, 2)^*$ -preopen. Hence $\tau_{1,2}\text{-cl}(A) \subset X \setminus F$. Therefore, $F \subset [X \setminus \tau_{1,2}\text{-cl}(A)] \cap \tau_{1,2}\text{-cl}(A) = \emptyset$. For any $(1, 2)^*$ -preclosed set F such that $F \subset \tau_{1,2}\text{-cl}(A) \setminus A, \emptyset = F \subset \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(A) \setminus A)$ and hence by Theorem 2.18, $\tau_{1,2}\text{-cl}(A) \setminus A$ is $(1, 2)^*$ - pg -open.

Conversely, suppose $\tau_{1,2}\text{-cl}(A) \setminus A$ is $(1, 2)^*$ - pg -open set. Let $A \subset F$ where $F \in (1, 2)^*\text{-PO}(X)$. Then $X \setminus F \subset X \setminus A$ that is $\tau_{1,2}\text{-cl}(A) \cap (X \setminus F) \subset \tau_{1,2}\text{-cl}(A) \cap (X \setminus A)$. Thus $\tau_{1,2}\text{-cl}(A) \cap (X \setminus F)$ is a $(1, 2)^*$ -preclosed subset of $\tau_{1,2}\text{-cl}(A) \cap (X \setminus A) = \tau_{1,2}\text{-cl}(A) \setminus A$. Therefore by Theorem 2.18, $\tau_{1,2}\text{-cl}(A) \cap (X \setminus F) \subset \tau_{1,2}\text{-int}(\tau_{1,2}\text{-cl}(A) \setminus A) = \phi$, by Lemma 1.5. Hence $\tau_{1,2}\text{-cl}(A) \subset F$ implies A is a $(1, 2)^*$ - pg -closed set. $\square$

Theorem 2.21. *A subset A is $(1, 2)^*$ - pg -closed if and only if $\tau_{1,2}\text{-cl}_p(\{x\}) \cap A \neq \phi$ for every $x \in \tau_{1,2}\text{-cl}(A)$.*

Proof. Suppose that $\tau_{1,2}\text{-cl}_p(\{x\}) \cap A = \phi$ for some $x \in \tau_{1,2}\text{-cl}(A)$. Then $X \setminus (\tau_{1,2}\text{-cl}_p(\{x\})) \in (1, 2)^*\text{-PO}(X)$ such that $A \subset X \setminus (\tau_{1,2}\text{-cl}_p(\{x\}))$. Furthermore, $x \in \tau_{1,2}\text{-cl}(A) \setminus (X \setminus \tau_{1,2}\text{-cl}_p(\{x\}))$ and hence $\tau_{1,2}\text{-cl}(A) \not\subset (X \setminus \tau_{1,2}\text{-cl}_p(\{x\}))$. This shows that A is not $(1, 2)^*$ - pg -closed.

Now, suppose that A is not $(1, 2)^*$ - pg -closed. There exists $U \in (1, 2)^*\text{-PO}(X)$ such that $A \subset U$ and $\tau_{1,2}\text{-cl}(A) \setminus U \neq \phi$. There exists $x \in \tau_{1,2}\text{-cl}(A)$ such that $x \notin U$. Hence $\tau_{1,2}\text{-cl}_p(\{x\}) \cap U = \phi$. Therefore $\tau_{1,2}\text{-cl}_p(\{x\}) \cap A = \phi$ for some $x \in \tau_{1,2}\text{-cl}(A)$. $\square$

3. $(1, 2)^*$ - pg -closed mappings

We introduce new classes of generalized closed mappings as follows:

Definition 3.1. A mapping $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be $(1, 2)^*$ - pg -closed if $f(F)$ is a $(1, 2)^*$ - pg -closed set in Y for every $\tau_{1,2}$ -closed set F of X .

Remark 3.2. The composition of two $(1, 2)^*$ - pg -closed (resp. $(1, 2)^*$ - pg -open) mappings need not be $(1, 2)^*$ - pg -closed (resp. $(1, 2)^*$ - pg -open).

Example 3.3. Let $X = Y = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$, $\tau_2 = \{\phi, \{b\}, X\}$, $\sigma_1 = \{\phi, \{c\}, Y\}$ and $\sigma_2 = \{\phi, \{a, c\}, Y\}$. Then $\tau_{1,2}\text{-O}(X) = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, $\tau_{1,2}\text{-C}(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$, $\sigma_{1,2}\text{-O}(Y) = \{\phi, \{c\}, \{a, c\}, Y\}$, $\sigma_{1,2}\text{-C}(Y) = \{\phi, \{b\}, \{a, b\}, Y\}$ and $(1, 2)^*\text{-PGC}(Y) = P(Y)$. Let $Z = \{p, q, r\}$, $\nu_1 = \{\phi, \{p\}, Z\}$ and $\nu_2 = \{\phi, \{q, r\}, Z\}$. Then $\nu_{1,2}\text{-O}(Z) = \{\phi, \{p\}, \{q, r\}, Z\}$.

$Z\} = \nu_{1,2}\text{-}C(Z)$ and $(1, 2)^*\text{-}PGC(Z) = \{\phi, \{p\}, \{p, q\}, \{p, r\}, \{q, r\}, Z\}$. Define $f : X \rightarrow Y$ as $f(a) = c, f(b) = b, f(c) = a$ and define $g : Y \rightarrow Z$ as $g(a) = r, g(b) = p, g(c) = q$. Clearly f and g are $(1, 2)^*\text{-}pg\text{-closed}$ (resp. $(1, 2)^*\text{-}pg\text{-open}$). However, their composition is not $(1, 2)^*\text{-}pg\text{-closed}$ (resp. $(1, 2)^*\text{-}pg\text{-open}$). Since $g(f(\{c\})) = \{r\}$ is not a $(1, 2)^*\text{-}pg\text{-closed}$ set in Z and $g(f(\{a, b\})) = \{p, q\}$ is not a $(1, 2)^*\text{-}pg\text{-open}$ set in Z .

Remark 3.4. A bijective mapping is $(1, 2)^*\text{-}pg\text{-open}$ if and only if it is $(1, 2)^*\text{-}pg\text{-closed}$.

Theorem 3.5. Every $(1, 2)^*\text{-closed}$ mapping is $(1, 2)^*\text{-}pg\text{-closed}$.

Proof. It is obvious from Theorem 2.3. □

Remark 3.6. The converse of Theorem 3.5 need not be true as shown by the following example.

Example 3.7. Let $X = Y = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, X\}, \tau_2 = \{\phi, \{b\}, X\}, \sigma_1 = \{\phi, \{c\}, Y\}, \sigma_2 = \{\phi, \{a, c\}, Y\}$. Then $\tau_{1,2}\text{-}C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}, \sigma_{1,2}\text{-}C(Y) = \{\phi, \{b\}, \{a, b\}, Y\}$ and $(1, 2)^*\text{-}PGC(Y) = P(Y)$. Define $f : X \rightarrow Y$ be the identity mapping. Clearly f is a $(1, 2)^*\text{-}pg\text{-closed}$ mapping but it is not $(1, 2)^*\text{-closed}$, since $\{c\}$ is a $\tau_{1,2}\text{-closed}$ set but it is not $\sigma_{1,2}\text{-closed}$.

Theorem 3.8. Every $(1, 2)^*\text{-}pg\text{-closed}$ mapping is $(1, 2)^*\text{-}rg\text{-closed}$.

Proof. It is obvious from Theorems 1.6 and 2.6. □

Example 3.9. The converse of Theorem 3.8 may be not true.

Let $X = Y = \{a, b, c\}, \tau_1 = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}, \tau_2 = \{\phi, \{b, c\}, X\}, \sigma_1 = \{\phi, \{a\}, Y\}$ and $\sigma_2 = \{\phi, \{b, c\}, Y\}$. Then $\tau_{1,2}\text{-}O(X) = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}, \tau_{1,2}\text{-}C(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{b, c\}, X\}, \sigma_{1,2}\text{-}O(Y) = \sigma_{1,2}\text{-}C(Y) = \{\phi, \{a\}, \{b, c\}, Y\}, (1, 2)^*\text{-}RGC(Y) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, Y\}$ and $(1, 2)^*\text{-}PGC(Y) = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}, Y\}$. Define $f : X \rightarrow Y$ be the identity mapping. Clearly f is a $(1, 2)^*\text{-}rg\text{-closed}$ mapping but it is not $(1, 2)^*\text{-}pg\text{-closed}$, since $\{b\}$ is a $\tau_{1,2}\text{-closed}$ set in $X, f(\{b\}) = \{b\}$ is a $(1, 2)^*\text{-}rg\text{-closed}$ set but it is not a $(1, 2)^*\text{-}pg\text{-closed}$ set in Y .

Remark 3.10. For a mapping, the following implications hold.

$$\begin{array}{ccccc}
 (1, 2)^*\text{-closed} & \longrightarrow & (1, 2)^*\text{-}g\text{-closed} & \longrightarrow & (1, 2)^*\text{-}rg\text{-closed} \\
 & & \nwarrow & \uparrow & \nearrow \\
 & & (1, 2)^*\text{-}pg\text{-closed} & &
 \end{array}$$

Theorem 3.11. *If a mapping $f : X \rightarrow Y$ is $(1, 2)^*$ -closed and $g : Y \rightarrow Z$ is $(1, 2)^*$ -pg-closed, then $g \circ f : X \rightarrow Z$ is $(1, 2)^*$ -pg-closed.*

Proof. Let A be any $\tau_{1,2}$ -closed set of X . Since f is $(1, 2)^*$ -closed, $f(A)$ is a $\sigma_{1,2}$ -closed set in Y . Since g is $(1, 2)^*$ -pg-closed, $g(f(A))$ is $(1, 2)^*$ -pg-closed in Z . $\square$

Theorem 3.12. *A mapping $f : X \rightarrow Y$ is $(1, 2)^*$ -pg-closed if and only if for each subset S of Y and for each $\tau_{1,2}$ -open set U containing $f^{-1}(S)$ there is a $(1, 2)^*$ -pg-open set V of Y such that $S \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Proof. Suppose f is $(1, 2)^*$ -pg-closed. Let S be a subset of Y and U be an $\tau_{1,2}$ -open set of X such that $f^{-1}(S) \subset U$. Then $V = Y \setminus f(X \setminus U)$ is an $(1, 2)^*$ -pg-open set containing S such that $f^{-1}(V) \subset U$.

Clearly, suppose that F is a $\tau_{1,2}$ -closed set of X . Then $f^{-1}(Y \setminus f(F)) \subseteq X \setminus F$ and $X \setminus F$ is $\tau_{1,2}$ -open. By hypothesis, there is an $(1, 2)^*$ -pg-open set V of Y such that $Y \setminus f(F) \subset V$ and $f^{-1}(V) \subseteq X \setminus F$. Therefore, $F \subseteq X \setminus f^{-1}(V)$. Hence $Y \setminus V \subseteq f(F) \subseteq f(X \setminus f^{-1}(V)) \subset Y \setminus V$. Since $Y \setminus V$ is $(1, 2)^*$ -pg-closed, $f(F)$ is $(1, 2)^*$ -pg-closed in Y and thus f is $(1, 2)^*$ -pg-closed. $\square$

4. $(1, 2)^*$ -normal spaces

Definition 4.1. A bitopological space (X, τ_1, τ_2) is said to be $(1, 2)^*$ -normal if for disjoint $\tau_{1,2}$ -closed sets F_1 and F_2 , there exist $\tau_{1,2}$ -open sets U_1, U_2 such that $F_1 \subset U_1, F_2 \subset U_2$ and $U_1 \cap U_2 = \phi$.

Theorem 4.2. *For a bitopological space (X, τ_1, τ_2) , the following properties are equivalent:*

1. (X, τ_1, τ_2) is $(1, 2)^*$ -normal.
2. For any disjoint $\tau_{1,2}$ -closed sets F_1, F_2 , there exist $(1, 2)^*$ -pg-open sets V_1, V_2 such that $F_1 \subset V_1, F_2 \subset V_2$ and $V_1 \cap V_2 = \phi$.
3. For any $\tau_{1,2}$ -closed set F and any $\tau_{1,2}$ -open set U containing F , there exists an $(1, 2)^*$ -pg-open set V such that $F \subset V \subset \tau_{1,2}\text{-cl}(V) \subset U$.
4. For any $\tau_{1,2}$ -closed set F and any $\tau_{1,2}$ -open set U containing F , there exists a $\tau_{1,2}$ -open set G such that $F \subset G \subset \tau_{1,2}\text{-cl}(G) \subset U$.
5. For any disjoint $\tau_{1,2}$ -closed sets F_1, F_2 , there exists an $(1, 2)^*$ -pg-open set V such that $F_1 \subset V$ and $\tau_{1,2}\text{-cl}(V) \cap F_2 = \phi$.

6. For any disjoint $\tau_{1,2}$ -closed sets F_1, F_2 , there exists a $\tau_{1,2}$ -open set G such that $F_1 \subset G$ and $\tau_{1,2}\text{-cl}(G) \cap F_2 = \phi$.

Proof. (1) $\Rightarrow$ (2) : It follows from Theorem 2.3.

(2) $\Rightarrow$ (3) : Let F be a $\tau_{1,2}$ -closed set and U is a $\tau_{1,2}$ -open set such that $F \subset U$. Then $F, X \setminus U$ are disjoint $\tau_{1,2}$ -closed sets and by (2) there exist $(1, 2)^*$ - pg -open sets V_1 and V_2 such that $F \subset V_1, X \setminus U \subset V_2$ and $V_1 \cap V_2 = \phi$. Since V_2 is $(1, 2)^*$ - pg -open, by using Theorem 2.18, $X \setminus U \subset \tau_{1,2}\text{-int}(V_2)$ and $\tau_{1,2}\text{-int}(V_2)$ is $\tau_{1,2}$ -open. Hence, $\tau_{1,2}\text{-cl}(V_1) \cap \tau_{1,2}\text{-int}(V_2) = \phi$. Therefore, we obtain $F \subset V_1 \subset \tau_{1,2}\text{-cl}(V_1) \subset X \setminus \tau_{1,2}\text{-int}(V_2) \subset U$.

(3) $\Rightarrow$ (4) : Let F be a $\tau_{1,2}$ -closed set and U be a $\tau_{1,2}$ -open set such that $F \subset U$. Then by using (3), there exists an $(1, 2)^*$ - pg -open set V such that $F \subset V \subset \tau_{1,2}\text{-cl}(V) \subset U$. By using Theorem 2.18, $F \subset \tau_{1,2}\text{-int}(V)$. Put $G = \tau_{1,2}\text{-int}(V)$. Then $F \subset G \subset \tau_{1,2}\text{-cl}(G) \subset \tau_{1,2}\text{-cl}(V) \subset U$.

(4) $\Rightarrow$ (5) : Let F_1, F_2 be any disjoint $\tau_{1,2}$ -closed sets. Since $X \setminus F_2$ is a $\tau_{1,2}$ -open set such that $F_1 \subset X \setminus F_2$, by (4) there exists a $\tau_{1,2}$ -open set V such that $F_1 \subset V \subset \tau_{1,2}\text{-cl}(V) \subset X \setminus F_2$. By using Theorem 2.3, V is $(1, 2)^*$ - pg -open. Hence $F_1 \subset V$ and $\tau_{1,2}\text{-cl}(V) \cap F_2 = \phi$.

(5) $\Rightarrow$ (6) : Let F_1, F_2 be any disjoint $\tau_{1,2}$ -closed sets. Then there exists an $(1, 2)^*$ - pg -open set V such that $F_1 \subset V$ and $\tau_{1,2}\text{-cl}(V) \cap F_2 = \phi$. By using Theorem 2.18, $F_1 \subset \tau_{1,2}\text{-int}(V)$. Let $G = \tau_{1,2}\text{-int}(V)$, then we have G is $\tau_{1,2}$ -open, $F_1 \subset G$ and $\tau_{1,2}\text{-cl}(G) \cap F_2 = \phi$.

(6) $\Rightarrow$ (1) : Let F_1, F_2 be any disjoint $\tau_{1,2}$ -closed sets. Then by (6), there exists a $\tau_{1,2}$ -open set G such that $F_1 \subset G$ and $\tau_{1,2}\text{-cl}(G) \cap F_2 = \phi$. Now, put $U_1 = G$ and $U_2 = X \setminus \tau_{1,2}\text{-cl}(G)$. Then U_1 and U_2 are disjoint $\tau_{1,2}$ -open sets $F_1 \subset U_1$ and $F_2 \subset U_2$. This shows that (X, τ_1, τ_2) is $(1, 2)^*$ -normal. $\square$

Theorem 4.3. If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is a $(1, 2)^*$ -continuous and $(1, 2)^*$ - pg -closed surjection and (X, τ_1, τ_2) is $(1, 2)^*$ -normal, then (Y, σ_1, σ_2) is $(1, 2)^*$ -normal.

Corollary 4.4. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a $(1, 2)^*$ -continuous and $(1, 2)^*$ -closed bijection. Then (X, τ_1, τ_2) is $(1, 2)^*$ -normal if and only if (Y, σ_1, σ_2) is $(1, 2)^*$ -normal.

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