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STRONGLY CONNECTED COMPONENTS OF A NETWORK IN PRETOPOLOGY

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Abstract: In this paper, we present properties of strongly connected components in the case of a network (which is defined as a family of pretopologies). The network can be analyse by the union or by the intersection or by the composition of the different pretopologies.

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1. Introduction

In Pretopology (see [1], [2], [3]), a network is defined as a family of pretopologies. Most often, it is studied by the union or by the intersection or by the composition of the different pretopologies constituting it (see [4]).

The aim of this paper is to give results concerning the relationship between the decomposition of a network studied by the union (or by the intersection or by the composition) of the pretopologies of which it is formed and the decomposition of each pretopological space constituting it.

We highlight algorithms for searching the strongly connected components of a network given the strongly connected components of each pretopological space of the network.

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2. Different Types of Pretopological Spaces (see [1], [2], [3])

Definition 1. Let X be a non empty set. P(X) denotes the family of subsets of X. We call pseudoclosure on X any mapping a from P(X) onto P(X) such as :

$$a(\emptyset) = \emptyset$$
$$\forall \mathrm{A} \subset \mathrm{X},\ \mathrm{A} \subset a(\mathrm{A})$$

(X, a) is then called pretopological space.

We can define 4 different types of pretopological spaces.

1- (X, a) is a V type pretopological space if and only if

$$\forall \mathrm{A} \subset \mathrm{X}, \forall \mathrm{B} \subset \mathrm{X},\ \mathrm{A} \subset \mathrm{B} \Rightarrow a(\mathrm{A}) \subset a(\mathrm{B}).$$

2- (X, a) is a V_D type pretopological space if and only if

$$\forall \mathrm{A} \subset \mathrm{X}, \forall \mathrm{B} \subset \mathrm{X},\ a(\mathrm{A} \cup \mathrm{B}) = a(\mathrm{A}) \cup a(\mathrm{B}).$$

3- (X, a) is a V_S type pretopological space if and only if

$$\forall \mathrm{A} \subset \mathrm{X},\ a(A) = \bigcup_{x \in A} a(\{x\}).$$

4- (X, a) a V_D type pretopological space, is a topological space if and only if

$$\forall \mathrm{A} \subset \mathrm{X},\ a(a(\mathrm{A})) = a(\mathrm{A}).$$

Property 2. *If (X, a) is a V_S space then (X, a) is a V_D space. If (X, a) is a V_D space then (X, a) is a V space.*

Example 3. Let X be a non empty set and R be a binary relationship defined on X.

The pretopology of descendants, noted a_d, is defined by :
$\forall$ A $\subset$ X, a_d(A) = { x$\in$ X/ R(x) $\cap$ A $\neq \emptyset$ } $\cup$ A with R(x)= { y $\in$ X/ x R y } .

The pretopology of ascendants, noted a_a, is defined by :
$\forall$ A $\subset$ X, a_a(A) = { x$\in$ X/ R^{-1}(x) $\cap$ A $\neq \emptyset$ } $\cup$ A with R^{-1}(x) = { y $\in$ X/ y R x}.

These pretopologies are V_S ones.

3. Different Pretopological Spaces Defined from a Space (X, a) and Closures (see [1], [2], [4])

Definition 4. Let (X, a) be a V pretopological space. Let A $\subset$ X. A is a closed subset if and only if a(A) = A.

We note $\forall$A$\subset$X, a^0(A) = A and $\forall$n, n$\geq$ 1, a^n(A) = $a(a^{n-1})$(A).

We name closure of A the subset of X, denoted F_a(A), which is the smallest closed subset which contains A.

Remark 5. F_a(A) is the intersection of all closed subsets which contain A. In the case where (X, a) is a "general" pretopological space (i.e. is not a V space, nor a V_D space, nor a V_S space, nor a topological space), the closure may not exist.

Proposition 6. *Let (X, a) be a V space. Let A $\subset$ X. If one of the two following conditions is fulfilled :*

- *X is a finite set*
- *a is of V_S type*

then $F_a(A) = \bigcup_{n\geq 0} a^n(A)$.

Remark 7. If a is of V type then a^n and F_a also are of V type. If a is of V_S type then a^n and F_a are also of V_S type.

Definition 8. Let (X, a) be a V pretopological space. Let A $\subset$ X. We define the induced pretopology on A by a, denoted a_A, by :
$\forall$C $\subset$ A, a_A(C) = a(C) $\cap$ A.
(A, a_A) (or more simply A) is said pretopological subspace of (X, a).
We note $(F_a)_A$ the closing obtained by restriction of closing F_a on A. $(F_a)_A$ is such as $\forall$ C$\subset$ A, $(F_a)_A$(C) = F_a(C) $\cap$A.

4. Strong Connectivity in (X, a) (see [1], [2], [3], [5], [6], [7], [8], [9])

Definition 9. Let (X, a) be a V pretopological space. Let A a non empty subset of X. Let B a non empty subset of X. There exists a path in (X, a) from B to A if and only if B $\subset$ F_a(A).

Definition 10. Let (X, a) be a V pretopological space.
(X, a) is strongly connected if and only if $\forall$C $\subset$ X, C $\neq$ $\emptyset$, F_a(C) = X.

Proposition 11 (see [1][5]). *Let (X, a) be a V pretopological space.*
(X, a) is strongly connected $\Leftrightarrow \forall x \in X$ *and* $\forall y \in X$*, there exists a path in (X, a) from* $\{ y \}$ *to* $\{ x \}$.

Definition 12. Let (X, a) be a V pretopological space. Let A $\subset$ X with A non empty.

A is a strongly connected subset of (X, a) if and only if A endowed with $(\mathrm{F}_a)_A$ is strongly connected.

A is a strongly connected component of (X, a) if and only if A is a strongly connected subset of (X, a) and $\forall$ B, A $\subset$ B $\subset$ X with A$\neq$ B, B is not a strongly connected subset of (X, a).

Proposition 13 (see [3]). *Let (X, a) be a V pretopological space. Let* $A \subset X$ *with A non empty.*

A is a strongly connected subset of (X, a) if and only if $\forall x \in A$ *and* $\forall y \in A$*, there exists a path in (X, a) from* $\{ y \}$ *to* $\{ x \}$.

5. Definition of a Network and Different Closures (see [4])

Definition 14. Let X a non empty set. Let I a countable family of indices. The family $\{$ (X, a_i), $i\in$I $\}$ of pretopological spaces is a network on X.

Definition 15. Let X a non empty set. For any pretopologies a_1 and a_2 defined on X, for any subset A of X, we define the three following mappings :

$(a_1 \cup a_2)(\mathrm{A}) = a_1(\mathrm{A}) \cup a_2(\mathrm{A})$ [union of pretopologies]

$(a_1 \cap a_2)(\mathrm{A}) = a_1(\mathrm{A}) \cap a_2(\mathrm{A})$ [intersection of pretopologies]

$(a_1 \odot a_2)(\mathrm{A}) = a_1(a_2(\mathrm{A}))$ [composition of pretopologies]

More generally, in a network $\{$ (X, a_i), $i\in$I $\}$ such as for any $i\in$I, a_i is of V type, we note F_{ai} the closure according to a_i, $\mathrm{F}_\cup$ (respectively $\mathrm{F}_\cap$) the closure according to $\bigcup_{i\in I} a_i$ (respectively $\bigcap_{i\in I} a_i$), $\mathrm{F}_{\cup F}$ (respectively $\mathrm{F}_{\cap F}$) the closure according to $\bigcup_{i\in I} F_{ai}$ (respectively $\bigcap_{i\in I} F_{ai}$).
We define the mapping, denoted $\prod_{i\in I} a_i$, from P(X) onto P(X) by :
$\forall$A$\subset$X, $\prod_{i\in I} a_i(A)= \{$ x$\in$X/ there exists $n\in$I such as x $\in a_n(a_{n-1}(\ldots (a_1(\mathrm{A}))\ldots)) \}$.
And we denote $\mathrm{F}_{\prod}$ the closure according to $\prod_{i\in I} a_i$ and $\mathrm{F}_{\prod F}$ the closure according to $\prod_{i\in I} F_{ai}$.

Proposition 16. *Let { (X, a_i), $i \in I$ } a network such as for any $i \in I$, a_i is of V type.*

i- $F_{\cup} = F_{\cup F} = F_{\prod F} = F_{\prod}$.

ii- $\forall A \subset X$, $F_{\cap}(A) \subset F_{\cap F}(A)$.

6. Strongly Connected Components in a Network

Definition 17 (see [1]). Let X a non empty set. Let a_1 and a_2 two pretopologies on X.

a_1 is thinner than a_2 if and only if $\forall$A $\subset$ X, a_1(A) $\subset$ a_2(A).

Remark 18 (see [1]). Let X a non empty set. Let a_1 and a_2 two *V* type pretopologies on X.

If a_1 is thinner than a_2 then F_{a1} is thinner than F_{a2}.

Proposition 19. *Let X a non empty set. Let a_1 and a_2 two V type pretopologies on X such as a_1 thinner than a_2. Let $A \subset X$ with A non empty.*

If A is a strongly connected subset of (X, a_1) then A is a strongly connected subset of (X, a_2).

Proof.

If A is a strongly connected subset of (X, a_1)

then $\forall$ C$\subset$ A, C $\neq \emptyset$, F_{a1}(C) $\cap$ A = A (by définition)

then $\forall$ C$\subset$ A, C $\neq \emptyset$, F_{a2}(C) $\cap$ A = A (Remark 18)

and then A is a strongly connected subset of (X, a_2).

Proposition 20. *Let { (X, a_i), $i \in I$ } a network such as for any $i \in I$, a_i is of V type. Let $A \subset X$ with A non empty.*

i- The following assertions are equivalent :

(1) A strongly connected subset of (X, $\bigcup_{i\in I} a_i$)

(2) A strongly connected subset of (X, $\bigcup_{i\in I} F_{ai}$)

(3) A strongly connected subset of (X, $\prod_{i\in I} a_i$)

(4) A strongly connected subset of (X, $\prod_{i\in I} F_{ai}$).

ii- If A is strongly connected subset of (X, $\bigcap_{i\in I} a_i$) then A is strongly connected subset of (X, $\bigcap_{i\in I} F_{ai}$).

Proof.

i- Let's show that (1) is equivalent to (2) :

A is a strongly connected subset of (X, $\bigcup_{i\in I} a_i$)

$\Leftrightarrow$ $\forall$ x $\in$ A and $\forall$ y $\in$ A, {y}$\subset$ $F_{\cup}$({x}) (Proposition 13 and Definition 9)

$\Leftrightarrow$ $\forall$ x $\in$ A and $\forall$ y $\in$ A, {y}$\subset$ $F_{\cup F}$({x}) (Proposition 16-i)

$\Leftrightarrow$ A is a strongly connected subset of (X, $\bigcup_{i\in I} F_{ai}$) (Proposition 13 and Definition 9).

Let's show that (1) is equivalent to (4) :

A is a strongly connected subset of (X, $\bigcup_{i\in I} a_i$)

$\Leftrightarrow$ $\forall$ x $\in$ A and $\forall$ y $\in$A, {y}$\subset$ $F_{\cup}$({x}) (Proposition 13 and Definition 9)

$\Leftrightarrow$ $\forall$ x $\in$ A and $\forall$ y $\in$ A, {y}$\subset$ $F_{\prod F}$({x}) (Proposition 16-i)

$\Leftrightarrow$ A is a strongly connected subset of (X, $\prod_{i\in I} F_{ai}$) (Proposition 13 and Definition 9) .

Let's show that (3) is equivalent to (4) :

A is a strongly connected subset of (X, $\prod_{i\in I} a_i$)

$\Leftrightarrow$ $\forall$ x $\in$ A and $\forall$ y $\in$ A, {y}$\subset$ $F_{\prod}$({x}) (Proposition 13 and Definition 9)

$\Leftrightarrow$ $\forall$ x $\in$ A and $\forall$ y $\in$ A, {y}$\subset$ $F_{\prod F}$({x}) (Proposition 16-i)

$\Leftrightarrow$ A is a strongly connected subset of (X, $\prod_{i\in I} F_{ai}$) (Proposition 13 and Definition 9).

ii- A is a strongly connected subset of (X, $\bigcap_{i\in I} a_i$)

$\Leftrightarrow$ $\forall$ x $\in$ A and $\forall$ y $\in$ A, {y}$\subset$ $F_{\cap}$({x}) (Proposition 13 and Definition 9)

Then $\forall$ x $\in$ A and $\forall$ y $\in$ A, {y}$\subset$ $F_{\cap F}$({x}) (Proposition 16-ii)

Then A is a strongly connected subset of (X, $\bigcap_{i\in I} F_{ai}$) (Proposition 13 and Definition 9).

Corollary 21. *Let { (X, a_i), i$\in$I } a network such as for any i$\in$I, a_i is of V type. Let A $\subset$ X with A non empty.*

The following assertions are equivalent :

(1) A strongly connected component of (X, $\bigcup_{i\in I} a_i$)

(2) A strongly connected component of (X, $\bigcup_{i\in I} F_{ai}$)

(3) A strongly connected component of (X, $\prod_{i\in I} a_i$)

(4) A strongly connected component of (X, $\prod_{i\in I} F_{ai}$).

Proof.

Obvious from Proposition 20-i and Definition 12.

Consequence. Decomposing (X, $\bigcup_{i\in I} a_i$) into strongly connected components is equivalent to decomposing (X, $\prod_{i\in I} a_i$) into strongly connected components. So we propose algorithms to look for strongly connected components in a network in the case of the union and in the case of the intersection of the different pretopologies.

Proposition 22. *Let $\{ (X, a_i), i\in I \}$ a network such as for any $i\in I$, a_i is of V type. Let $A \subset X$ with A non empty.*

i- If there exists $i\in I$ such as A strongly connected subset of (X, a_i) then A is strongly connected subset of $(X, \bigcup_{i\in I} a_i)$.

ii- If A is strongly connected subset of $(X, \bigcap_{i\in I} a_i)$ then for any $i\in I$, A is strongly connected subset of (X, a_i).

Proof.

i- $\forall\ i\in$I, a_i is thinner than $\bigcup_{i\in I} a_i$. We get the result according to Proposition 19.

ii- $\forall\ i\in$I, $\bigcap_{i\in I} a_i$ is thinner than a_i. We get the result according to Proposition 19.

Remark 23. The converses of i- and ii- are not true generally speaking.

Examples 24.

i- Let $\{$ (X, a_i), $i\in$I $\}$ a network with X= $\{$ a, b, c $\}$, I = $\{$ 1, 2 $\}$, a_1 and a_2 pretopologies of descendants defined respectively by the following graphs 1 and 2 :

x	R(x)
a	{b}
b	{a}
c	ø

Graph 1

x	R(x)
a	ø
b	{c}
c	{b}

Graph 2

X is strongly connected subset of (X, $\bigcup_{i\in I} a_i$) but X is not strongly connected subset of (X, a_1) and X is not strongly connected subset of (X, a_2).

ii- Let { (X, a_i), i∈I } a network with X= { a, b, c }, I = { 1, 2 }, a_1 and a_2 respectively pretopology of ascendants and pretopology of descendants defined by the following graph 3 :

x	R(x)
a	{b}
b	{c}
c	{a}

Graph 3

X is strongly connected subset of (X, a_1) and strongly connected subset of (X, a_2) but X is not strongly connected subset of (X, $\bigcap_{i\in I} a_i$).

Consequence. Given Proposition 22-ii, it does not seem possible to find a more judicious algorithm from the study of each a_i for the study of $\bigcap_{i\in I} a_i$. We will take into account only the case of $\bigcup_{i\in I} a_i$.

Proposition 25. *Let { (X, a_i), i∈I } a network such as for any i∈I, a_i is of V type. Let { C_k, k∈K } a family of subsets non empty of X such as :*

1- $\bigcup_{k\in K} C_k = X$

2- ∀ k∈K, there exists { A_j, j∈J } a family of subsets non empty of X such as:

2-1- $Ck = \bigcup_{j\in J} A_j$

2-2- ∀ j∈J, there exists i∈I, A_j strongly connected component of (X, a_i)

2-3- ∀ j∈J, ∀j'∈J, there exists a sequence $j_0 \ldots j_r$ of elements of J such as $j_0 = j$, $j_r = j'$ and ∀ l = 0,...,r-1, $A_{jl} \cap A_{jl+1} \neq \emptyset$

2-4- ∀ A'⊂ X, A' ∉ { A_j, j∈J }, if there exists i∈I such as A' strongly connected component of (X, a_i) then A' ∩ $C_k = \emptyset$.

We have :

i- ∀ k∈K, C_k strongly connected subset of (X, $\bigcup_{i\in I} a_i$).

ii- { C_k, k∈K } is a partition of X.

Proof.

i- ∀ j∈J, there exists i∈I, A_j strongly connected component of (X, a_i)

then ∀ j∈J, there exists i∈I, A_j strongly connected subset of (X, a_i) (Definition 12)

then ∀ j∈J, A_j is strongly connected subset of (X, $\bigcup_{i\in I} a_i$) (Proposition 22-i).

Morever, the union of two strongly connected subsets with a non empty intersection is a strongly connected subset (see [1]) hence the result.

ii- It is sufficient to show that $\forall\ k \in$K, $\forall\ k' \in$K with $k \neq k'$, $C_k \cap C_{k'} = \emptyset$ which is ensured by 2.

Remark 26. C_k is not strongly connected component of (X, $\bigcup_{i \in I} a_i$) generally speaking.

Example 27. Let { (X, a_i), $i \in$I } a network with X= { a, b, c, d, e, f, g }, I = { 1, 2 }, a_1 and a_2 prétopologies of descendants defined respectively by the following graphs 4 and 5 :

x	R(x)
a	{b}
b	{a}
c	{d}
d	{c}
e	{d}
f	ø
g	ø

Graph 4

x	R(x)
a	ø
b	{e}
c	{f}
d	ø
e	{b}
f	{g}
g	{ a,c }

Graph 5

Let A_1 = { a, b }. A_1 is strongly connected component of (X, a_1). Let A_2 = { e, b }. A_2 is strongly connected component of (X, a_2). Let C = $A_1 \cup A_2$. { A_1, A_2 } checks the conditions of the Proposition so C is strongly connected subset of (X, $\bigcup_{i \in I} a_i$) but C is not strongly connected component of (X, $\bigcup_{i \in I} a_i$). Indeed, X is strongly connected component of (X, $\bigcup_{i \in I} a_i$).

Proposition 28. *Let $\{ (X, a_i), i \in I \}$ a network such as for any $i \in I$, a_i is of V type. The same conditions apply as in Proposition 25.*

$\forall\ k \in K$, we define $K_k = \{ l \in K\ ,\ F_\cup(C_l) = F_\cup(C_k) \}$.

There exists $Q \subset K$ such as $\{ K_q,\ q \in Q \}$ which is a partition of K.

$\forall\ q \in Q$, we denote $F_q = \bigcup_{l \in K_q} C_l$.

We have :

i- $\{ F_q,\ q \in Q \}$ is a partition of X.

ii- $\{ F_q,\ q \in Q \}$ is the family of strongly connected components of (X, $\bigcup_{i \in I} a_i$).

Proof.

i- and ii- See Proposition 5 in [3] and Proposition 25.

7. Conclusion

Finally, if X is a finite set, we can give two algorithms to find strongly connected components of (X, $\bigcup_{i \in I} a_i$) (i.e. of (X, $\prod_{i \in I} a_i$)).

The first algorithm is to disregard the above results and therefore to consider $\bigcup_{i \in I} a_i$ as a pretopology. This first solution seems to be judicious when $\forall\ i \in$I, a_i is of V_S type. In this case, $\bigcup_{i \in I} a_i$ is also of V_S type and we can use previous results (in particular, see Proposition 8 in [3]). However, we can also use this algorithm when there exists i such as a_i is of V type. But, in this case, $\bigcup_{i \in I} a_i$ is also of V type and we need to use other more complex previous results (see Proposition 5 in [3]).

The following second algorithm uses the results developped in this paper (Propositions 25 and 28). In particular, it will be used when strongly connected components are known for each pretopologies of the network :

- For any a_i, find the strongly connected components of (X, a_i)

- Build $\{$ C$_k$, $k \in$K $\}$ by joining step by step all strongly connected components with a non empty intersection (Proposition 25)

- For any C$_k$, compute F$_\cup$(C$_k$) (which is equivalent to F$_{\prod F}$(C$_k$) according to Proposition 16-i)

- The union of all C$_k$ which have the same closure for $\bigcup_{i \in I} a_i$ (i.e. for $\prod_{i \in I} F_{ai}$) is a strongly connected components of $\bigcup_{i \in I} a_i$ (i.e. for $\prod_{i \in I} F_{ai}$).

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