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ON THE APPROXIMATION OF THE STEP FUNCTION BY A NEW MODIFIED LAPLACE CUMULATIVE DISTRIBUTION FUNCTION

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Abstract: In this paper the Hausdorff approximation of the Heaviside step function by a new Modified Laplace Cumulative Distribution Function is considered and precise upper and lower bounds for the one–sided Hausdorff distance are obtained. Numerical examples illustrating the obtained results are given too.

AMS Subject Classification: 68N30, 41A46
Key Words: Heaviside step function, Modified Laplace Cumulative Distribution Function (MLCDF), Hausdorff distance, upper and lower bounds

1. Introduction

The Laplace Cumulative Distribution Function (LCDF) is used for modeling in signal processing, various biological processes, finance and economics.

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Examples of events that may be modeled by LCDF include: credit risk and exotic options in financial engineering, insurance claims and structural changes in switching–regime model and Kalman filter [12]. For more results, see [3], [6], [7], [11], [13], [28], [47]–[49].

In this paper we study the Hausdorff approximation (see [46]) of the Heaviside step function by a new Modified Laplace Cumulative Distribution Function (MLCDF) introduced in [1].

For a special case of the considered MLCDF we find an expression for the error of the best approximation.

Furthermore - numerical examples illustrating applications in the Biology Modeling and to the Software Reliability Growth Theory of the obtained results are given.

2. Preliminaries

Definition 1. The Heaviside step function $h_0(t)$ is defined by

$$h_0(t) = \begin{cases} 0, & \text{for} \quad t < 0, \\ [0,1], & \text{for} \quad t = 0 \\ 1, & \text{for} \quad t > 0 \end{cases} \tag{1}$$

Definition 2. [46] The one–sided Hausdorff distance $\overrightarrow{\rho}(f,g)$ between two interval functions f, g on $\Omega \subseteq \mathbb{R}$ is the one–sided Hausdorff distance between their completed graphs $\mathcal{F}(f)$ and $\mathcal{F}(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\overrightarrow{\rho}(f,g) = \sup_{B\in\mathcal{F}(g)} \inf_{A\in\mathcal{F}(f)} ||A - B||,$$

where $|| \cdot ||$ is a norm in $\mathbb{R}^2$.

We recall that the completed graph of f is the closure of the graph of f as a subset of $\Omega \times \mathbb{R}$. If the graph of an interval function f equals $\mathcal{F}(f)$, then the f is called S-continuous.

Definition 3. The Laplace Cumulative Distribution function (LCDF) is defined by:

$$F(t;b) = \begin{cases} \frac{1}{2}e^{\frac{t}{b}}, & \text{for} \quad t \leq 0, \\ 1 - \frac{1}{2}e^{-\frac{t}{b}}, & \text{for} \quad t > 0, \end{cases} \tag{2}$$

where $b > 0$ is the scale parameter.

In [25] for the LCDF (2) are obtained the following estimates:

Theorem 4. *[25] The one–sided Hausdorff distance $d = d(b)$ between the Heaviside function and the LCDF (2) can be expressed in terms of the parameter b for any real $0 < b < 1.012$ as follows:*

$$\frac{1}{3\left(1+\frac{1}{2b}\right)} < d < \frac{\ln\left(3\left(1+\frac{1}{2b}\right)\right)}{3\left(1+\frac{1}{2b}\right)}. \tag{3}$$

Definition 5. [1] The Modified Laplace Cumulative Distribution function (MLCDF) is defined by:

$$f(t;\mu,b,\alpha) = \begin{cases} \left(\frac{1}{2}e^{\frac{t-\mu}{b}}\right)^{\alpha}, & \text{for} \quad t \leq \mu, \\ \left(1-\frac{1}{2}e^{-\frac{t-\mu}{b}}\right)^{\alpha}, & \text{for} \quad t > \mu, \end{cases} \tag{4}$$

where $\alpha > 0$; $b > 0$ is the scale parameter and μ is the location parameter.

3. Main Results

For the introduced in [1] MLCDF (4), without loss of generality, for $\mu = 0$ and $\alpha > 1$ we will consider the following **Special case**:

$$\tilde{f}(t;0,b,\alpha) = \begin{cases} \left(\frac{1}{2}e^{\frac{t}{b}}\right)^{\alpha}, & \text{for} \quad t \leq 0, \\ \left(1-\frac{1}{2}e^{-\frac{t}{b}}\right)^{\alpha}, & \text{for} \quad t \geq 0. \end{cases} \tag{5}$$

Now we will study the Hausdorff approximation [46] of the Heaviside function by MLCDF of the form (5) and will find an expression for the error of the best approximation.

The one–sided Hausdorff distance $d = d(b,\alpha)$ between the step function and the function $\tilde{f}(t;0,b,\alpha)$ satisfies the relation

$$\tilde{f}(d;0,b,\alpha) = \left(1-\frac{1}{2}e^{-\frac{d}{b}}\right)^{\alpha} = 1-d, \tag{6}$$

i.e.

$$e^{-\frac{d}{b}} = 2\left(1-(1-d)^{\frac{1}{\alpha}}\right). \tag{7}$$

The following theorem gives upper and lower bounds for $d(b,\alpha)$.

Theorem 6. *Let* $p=-1+\left(\frac{1}{2}\right)^{\alpha}$; $q=1+\frac{\alpha}{b}\left(\frac{1}{2}\right)^{\alpha}$, $\alpha>1$ *and* $q>\frac{1}{2}e^{2\left(1-\left(\frac{1}{2}\right)^{\alpha}\right)}$.

The one–sided Hausdorff distance $d=d(b,\alpha)$ *between the function* $h_0(t)$ *and the function* (5) *can be expressed in terms of the parameters* b *and* α *as follows:*

$$d_l = \frac{1}{2q} < d < \frac{\ln(2q)}{2q} = d_r. \tag{8}$$

Proof. Let us examine the function

$$L(d) = \left(1-\frac{1}{2}e^{-\frac{d}{b}}\right)^{\alpha} - 1 + d.$$

Evidently, $p<0$ and $q>0$. From $L'(d)>0$ we conclude that the function L is strictly monotonically increasing.

Consider the function

$$M(d) = p + qd.$$

From Taylor expansion we obtain $M(d)-L(d)=O(d^2)$. Hence $M(d)$ approximates $L(d)$ with $d\to 0$ as $O(d^2)$ (see Figure 1). In addition $M'(d)>0$.

Further, from the condition of the theorem we have

$$\begin{aligned} M(d_l) &= p+\frac{1}{2} = -\frac{1}{2}+\left(\frac{1}{2}\right)^{\alpha} < 0, \\ M(d_r) &= p+\frac{1}{2}\ln(2q) > 0. \end{aligned}$$

This completes the proof. □

The (MLCDF) of the form (5) for $b=0.26$, $\alpha=1.3$ is visualized on Figure 2. From the nonlinear equation (6) and inequalities (8) we have: $d=0.245288$, $d_l=0.164982$, $d_r=0.297284$.

The cumulative function (5) with $b=0.16$, $\alpha=1.1$; Hausdorff distance $d=0.178588$; $d_l=0.118841$, $d_r=0.253128$ is visualized on Figure 3.

The cumulative function (5) with $b=0.05$, $\alpha=1.04$; Hausdorff distance $d=0.0884703$; $d_l=0.0449818$, $d_r=0.139511$ is visualized on Figure 4.

Some computational examples using relations (6) and (8) are presented in Table 1.

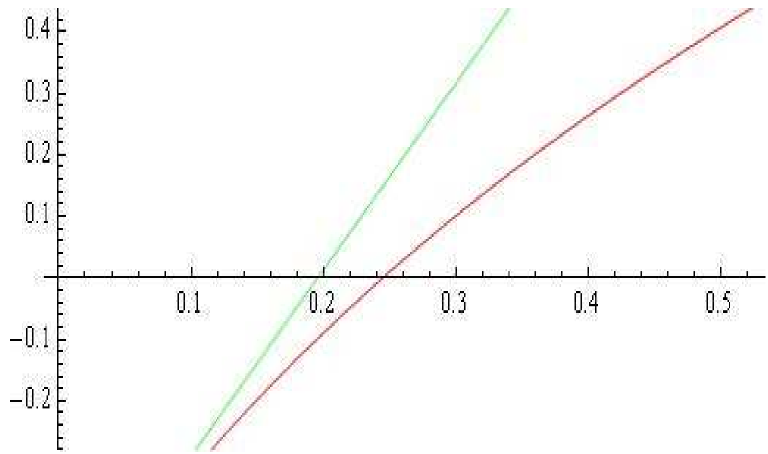

Figure 1: The functions $L(d)$ and $M(d)$ with $b = 0.26$ and $\alpha = 1.3$.

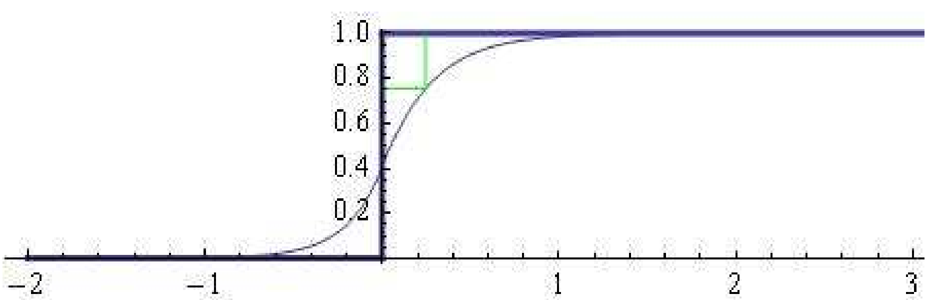

Figure 2: The cumulative function (5) with $b = 0.26$, $\alpha = 1.3$; Hausdorff distance $d = 0.245288$; $d_l = 0.164982$, $d_r = 0.297284$.

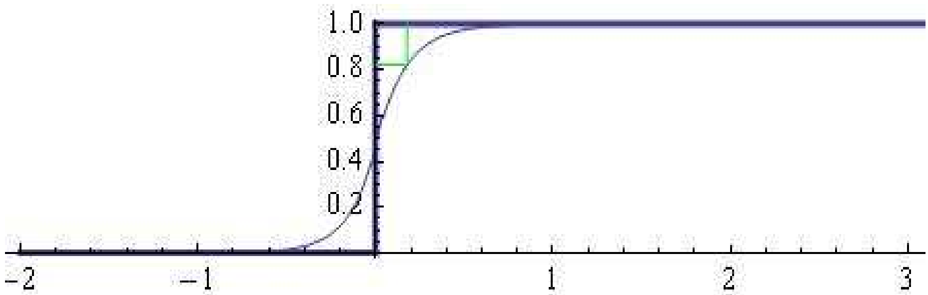

Figure 3: The cumulative function (5) with $b = 0.16$, $\alpha = 1.1$; Hausdorff distance $d = 0.178588$; $d_l = 0.118841$, $d_r = 0.253128$.

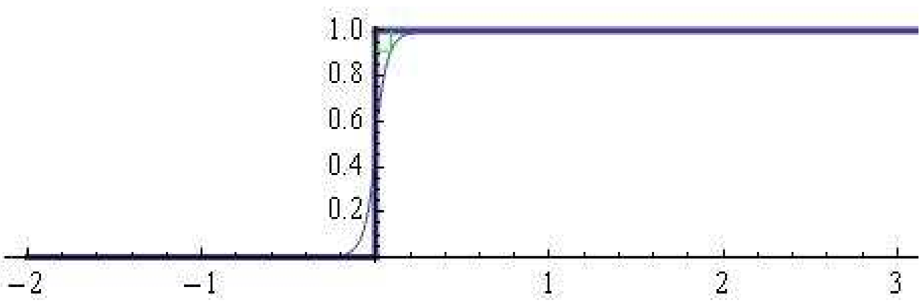

Figure 4: The cumulative function (5) with $b = 0.05$, $\alpha = 1.04$; Hausdorff distance $d = 0.0884703$; $d_l = 0.0449818$, $d_r = 0.139511$.

b	α	d_l	d	d_r
0.26	1.3	0.164982	0.245288	0.297284
0.16	1.1	0.118841	0.178588	0.253128
0.05	1.04	0.0449818	0.0884703	0.139511
0.04	1.01	0.0369336	0.0758281	0.121831
0.03	1.005	0.0282613	0.062519	0.100787
0.01	1.001	0.00980098	0.0286162	0.0453322

Table 1: Bounds for d computed by (6) and (8) for various b and α.

4. Numerical Examples

4.1. Application to the Modeling in Biology

We examine the following data for the growth of red abalone *Haliotis Rufescens* in Northern California. (The extended data for modeling the growth of red abalone is shown in Table 2. For more details see [45]).

The model

$$\tilde{f}(t) = \omega\left(1 - \frac{1}{2}e^{-\frac{t}{b}}\right)^{\alpha}$$

based on the data of Table 2 for the estimated parameters:

$$\omega = 199;\;\; b = 3.6724;\;\; \alpha = 5.20403$$

is plotted on Fig. 5.

For the predictive power criterion

$$PP = \sum_{i=1}^{n}\left(\frac{\tilde{f}(t_i) - y_i}{y_i}\right)^2$$

measures the distance of model actual data from the estimates against the actual data, we find $PP = 0.00121916$.

4.2. Application to the Software Reliability Growth Theory

We examine the following data for the software reliability growth. (The small on–line data entry software package test data, available since 1980 in Japan [33], is shown in Table 3. For more details see [43]).

Age	*Length(mm)*
1	16.1
2	33.9
3	54.3
4	76.2
5	97.8
6	117.1
7	133.3
8	146.5
9	157.2
10	166
11	173.3
12	179.6
13	185
14	189.9
15	194

Table 2: The extended data for modeling the growth of red abalone *Haliotis Rufescens* in Northern California [45]

The fitted model based on the data of Table 3 for the estimated parameters:

$$\omega = 76; \;\; b = 11.1584; \;\; \alpha = 6.06412$$

is plotted on Fig. 6.

5. Concluding Remarks

For given $\mu \neq 0$, based on the methodology proposed in the present paper, the reader may formulate the corresponding approximation problems on his/her own.

The estimates of the value of the best Hausdorff approximation of the Heaviside function and MLCDF obtained in this article can be used in practice as one possible additional criterion in "saturation" study of sigmoidal cumulative function.

The obtained estimates give more insight on the parameters in the strategy "Insurance responsibility" [14].

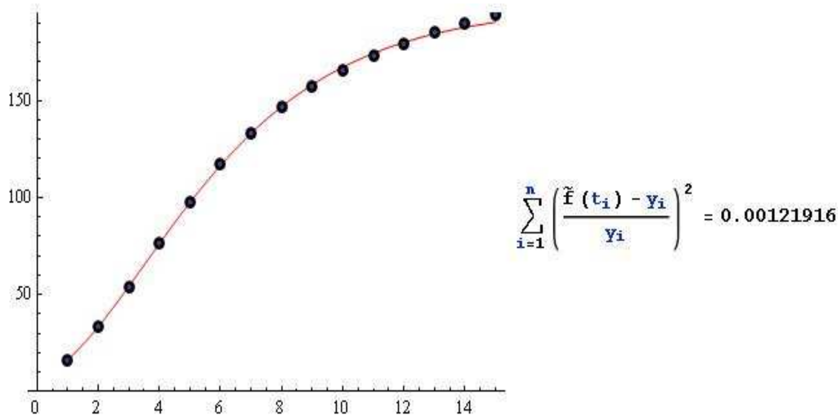

Figure 5: The model $\tilde{f}(t)$

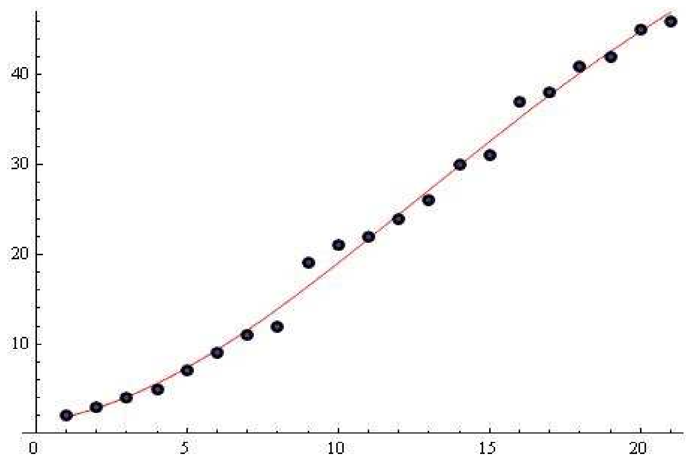

Figure 6: The fitted model.

$Testing\,time\,(day)$	$Failures$	$Cumulative\,failures$
1	2	2
2	1	3
3	1	4
4	1	5
5	2	7
6	2	9
7	2	11
8	1	12
9	7	19
10	2	21
11	1	22
12	2	24
13	2	26
14	4	30
15	1	31
16	6	37
17	1	38
18	3	41
19	1	42
20	3	45
21	1	46

Table 3: On–line IBM entry software package [33]

In the present paper we do not consider sigmoid functions generated as cumulative functions of other probabilistic distributions, such as the Skew–Laplace [4], α–Skew Laplace [8], and multimodal–Skew Laplace distribution [5].

For other results, see [2], [9], [10], [15]–[23], [29]–[32], [44].

We will explicitly note that in some cases the presented software reliability model provides better results than other much more sophisticated models.

The model (5) has a certain right of existence insofar as the theory of sigmoidal functions is well developed.

The results obtained in this paper can be used when controlling growth in Software Reliability Models, see [24], [26], [27], [34]–[42].

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