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OSCILLATION CRITERIA FOR SECOND ORDER DIFFERENCE EQUATIONS WITH SEVERAL NONLINEAR NEUTRAL TERMS

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Abstract: In this paper we establish some new sufficient conditions for the oscillation of all solutions of second order difference equations with several nonlinear neutral terms of the form

$$\Delta\left(a(n)\Delta\left(x(n)+\sum_{i=1}^{m}p_i(n)x^{\alpha_i}(n-k_i)\right)\right)+q(n)x^{\beta}(n+1-\ell)=0.$$

Using an elementary inequality we reduce the nonlinear neutral terms to linear neutral terms, four new oscillation criteria are obtained. Examples are included to illustrate the main results.

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Key Words: second order, difference equation, oscillation, nonlinear neutral terms

1. Introduction

In this paper, we study the oscillatory behavior of second order difference equa-

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tion with several nonlinear neutral terms of the form

$$\Delta(a(n)\Delta z(n)) + q(n)x^{\beta}(n+1-\ell) = 0, \quad n \geq n_0 \geq 0 \tag{1.1}$$

where $z(n) = x(n) + \sum_{i=1}^{m} p_i(n)x^{\alpha_i}(n-k_i)$, and m is a positive integer subject to the following conditions:

(H_1) $0 < \alpha_i \leq 1$ for $i = 1,\ 2,\ \ldots,\ m$ and β are ratios of odd positive integers;

(H_2) $\{a(n)\},\ \{p_i(n)\}$ and $\{q(n)\}$ are positive real sequences for $i = 1,\ 2,\ \ldots,\ m$ with

$$\sum_{n=n_0}^{\infty} \frac{1}{a(n)} < \infty; \tag{1.2}$$

(H_3) ℓ and k_i are positive integers for $i = 1,\ 2,\ \ldots,\ m$.

Let $\theta = \max\{\ell,\ k_1, k_2,\ \ldots,\ k_m\}$. By a solution of equation (1.1), we mean a real sequence $\{x(n)\}$ defined for $n \geq n_0 - \theta$ and satisfies equation (1.1) for all $n \geq n_0$. We consider only those solutions $\{x(n)\}$ of equation (1.1) which satisfy $\sup\{|x(n)| : n \geq N\} > 0$ for all $n \geq N$, and assume that the equation (1.1) possesses such solutions. A solution of equation (1.1) is called oscillatory if its terms are neither eventually positive nor eventually negative, and non oscillatory otherwise.

The problem of investigating the oscillatory properties of solutions of second order difference equations received a great interest in the past few decades, see for example [1, 2, 3, 4, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] for recent references. But few results are available in the literature dealing with the oscillation of second order difference equations with a nonlinear neutral term, see [1, 4, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], even though such equations has many applications, see [1, 2, 6].

In [9], Liu considered the equation of the form

$$\Delta\left(x(n) - p(n)x^{\alpha}(n-k)\right) + q(n)x^{\beta}(n-\ell) = 0,\ n \geq n_0$$

and studied its oscillatory behavior. In [13], Thandapani et al. investigated the oscillation of all solutions of the equation

$$\Delta\left(a(n)\Delta\left(x(n) - px^{\alpha}(n-k)\right)\right) + q(n)x^{\beta}(n+1-\ell) = 0,\ n \geq n_0.$$

A special case of the equation considered in [16] has the form

$$\Delta^2\left(x(n) + p(n)x^{\alpha}(n-k)\right) + q(n)x^{\beta}(n-\ell) = 0,\ n \geq n_0$$

and they too discussed the oscillatory behavior of solutions.

In [11, 15], the authors considered equation (1.1), and obtained criteria for the oscillation of all solutions provided $\sum_{n=n_0}^{\infty} \frac{1}{a(n)} = \infty$ or $\sum_{n=n_0}^{\infty} \frac{1}{a(n)} < \infty$ and $\alpha_i = 1$ for $i = 1, 2, \ldots, m$.

Therefore in this paper, we obtain sufficient conditions for the oscillation of all solutions of equation (1.1) when condition (1.2) is satisfied since the same problem investigated in [10] when $\sum_{n=n_0}^{\infty} \frac{1}{a(n)} = \infty$. Our method of proof makes use of an fundamental inequality and Riccati type transformation. The results obtained here are new and generalize those reported in [4, 8, 10, 11, 12, 14, 15, 16, 17]. Examples are provided to illustrate the main results.

2. Oscillation Results

In this section, we obtain sufficient conditions for the oscillation of all solutions of equation (1.1). Due to the form of the equation (1.1), we only need to give proofs for the case of eventually positive solutions since the proof for eventually negative solutions would be similar.

We begin with the following lemma extracted from [7].

Lemma 2.1. *If a and b are nonnegative, then*

$$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \quad for \quad 0 < \alpha \leq 1,$$

when equality holds if and only if $a = b$.

Lemma 2.2. *Let $\{x(n)\}$ be an eventually positive solution of equation* (1.1)*. Then the corresponding sequence $\{z(n)\}$ satisfies one of the following two cases for all sufficiently large n:*

$$(1) z(n) > 0, \ \Delta z(n) > 0, \ \Delta(a(n)\Delta z(n)) \leq 0,$$

$$(2) z(n) > 0, \ \Delta z(n) < 0, \ \Delta(a(n)\Delta z(n)) \leq 0.$$

Proof. Let $\{x(n)\}$ be a positive solution of equation (1.1) for all $n \geq n_1 \geq n_0$. It is easy to see that $z(n) > 0$ for $n \geq n_1$ and that equation (1.1) can be written as

$$\Delta(a(n)\Delta z(n)) = -q(n) x^{\beta}(n+1-\ell) < 0.$$

From this, we see that $a(n)\Delta z(n)$ is decreasing and so either $\Delta z(n) > 0$ or $\Delta z(n) < 0$ for all $n \geq n_1$. This completes the proof.

In the following, for convenience we denote

$$A(n) = \sum_{s=n}^{\infty} \frac{1}{a(s)}, \quad B(n) = \sum_{s=n_0}^{n-1} \frac{1}{a(s)},$$

$$P(n) = \sum_{i=1}^{m} \left(\alpha_i + \frac{1}{A(n)}(1-\alpha_i) \right) p_i(n),$$

and

$$Q(n) = \sum_{i=1}^{m} \left(\alpha_i + \frac{1}{A^2(n)}(1-\alpha_i) \right) p_i(n) \left(\frac{A(n-k_i)}{A(n)} \right)^{\alpha_i}.$$

□

Lemma 2.3. *Let $\{x(n)\}$ be an eventually positive solution of equation (1.1), and suppose Case (1) of Lemma 2.2 holds. Then there is an integer $N \geq n_0$ such that*

$$x(n) \geq (1 - P(n))z(n) \tag{2.1}$$

for all $n \geq N$.

Proof. From (H_2) and the definition of $z(n)$, we have $z(n) \geq x(n)$ for all $n \geq n_1 \geq n_0$. By using Lemma 2.1 with $b = 1$ we have

$$\begin{aligned} x(n) &\geq z(n) - \sum_{i=1}^{m} p_i(n) z^{\alpha_i}(n-k_i) \\ &\geq z(n) - \sum_{i=1}^{m} p_i(n) z^{\alpha_i}(n) \\ &\geq z(n) - \sum_{i=1}^{m} p_i(n) \left(\alpha_i z(n) + (1-\alpha_i) \right) \end{aligned} \tag{2.2}$$

for all $n \geq n_1$. Since $z(n)$ is positive and increasing and $A(n)$ is positive, decreasing and tending to zero, there is an integer $N \geq n_1$ such that

$$z(n) \geq A(n), \quad n \geq N. \tag{2.3}$$

From (2.2) and (2.3), we obtain

$$x(n) \geq (1 - P(n))z(n).$$

This completes the proof. □

Lemma 2.4. *Let $\{x(n)\}$ be an eventually positive solution of equation* (1.1) *and suppose Case (2) of Lemma 2.2 holds. Then there is an integer $N \geq n_0$ such that*

$$x(n) \geq (1 - Q(n))z(n) \tag{2.4}$$

for all $n \geq N$.

Proof. From (H_2) and the definition of $z(n)$, we have $z(n) \geq x(n)$ for all $n \geq n_1 \geq n_0$. Now

$$z(j) - z(n) = \sum_{s=n}^{j-1} \frac{a(s)\Delta(s)}{a(s)} \leq a(n)\Delta z(n) \sum_{s=n}^{j-1} \frac{1}{a(s)}$$

where we have used $a(n)\Delta z(n)$ is decreasing. As $j \to \infty$, we obtain

$$0 \leq z(n) + A(n)a(n)\Delta z(n), \quad n \geq n_1,$$

and hence

$$\Delta\left(\frac{z(n)}{A(n)}\right) > 0 \quad for \quad n \geq n_1. \tag{2.5}$$

From (2.5), it follows that $\frac{z(n)}{A(n)}$ is increasing and $A(n)$ is decreasing and tending to zero, we have $\frac{z(n)}{A(n)} \geq A(n)$ for all $n \geq N \geq n_1$. From Lemma 2.1 with $b = 1$, we have

$$\begin{aligned} x(n) &\geq z(n) - \sum_{i=1}^{m} p_i(n) z^{\alpha_i}(n - k_i) \\ &= z(n) - \sum_{i=1}^{m} p_i(n) \left(\frac{z(n-k_i)}{A(n-k_i)}\right)^{\alpha_i} A^{\alpha_i}(n-k_i) \\ &\geq z(n) - \sum_{i=1}^{m} p_i(n) \left(\frac{A(n-k_i)}{A(n)}\right)^{\alpha_i} z^{\alpha_i}(n) \\ &\geq z(n) - \sum_{i=1}^{m} p_i(n) \left(\frac{A(n-k_i)}{A(n)}\right)^{\alpha_i} (\alpha_i z(n) + (1-\alpha_i)) \end{aligned} \tag{2.6}$$

where we have used $\frac{z(n)}{A(n)}$ is increasing. Using $z(n) \geq A^2(n)$ in (2.6), we obtain

$$x(n) \geq (1 - Q(n))z(n), \quad n \geq N,$$

and this completes the proof. □

Let $\beta \geq 1$, condition (1.2) and $\max\{P(n),\ Q(n)\} < 1$ be hold. If there is a positive real sequence $\{\mu(n)\}$ such that

$$\lim_{n\to\infty} \sup \sum_{s=N}^{n} \Big[\mu(s)q(s)(1-P(s+1-\ell))^{\beta}A^{\beta-1}(s+1-\ell) - \frac{a(s+1-\ell)(\Delta\mu(s))^2}{4\mu(s)}\Big] = \infty \quad (2.7)$$

and

$$\lim_{n\to\infty} \sup \sum_{s=N}^{n-1} \Big[A(s+1)q(s)(1-Q(s+1-\ell))^{\beta}A^{2\beta-2}(s+1-\ell) - \frac{1}{4a(s)A(s+1)}\Big] = \infty \quad (2.8)$$

then every solution of equation (1.1) is oscillatory.
Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1.1), say $x(n) > 0$, $x(n-k_i) > 0$ and $x(n+1-\ell) > 0$ for $n \geq n_1 \geq n_0$ for $i = 1,\ 2,\ \ldots,\ m$. It is easy to see that $z(n) > 0$ for all $n \geq n_1$ and choose an integer $N \geq n_1$ so that both cases of Lemma 2.2 holds for all $n \geq N$. We shall show that in each case we are led to a contradiction.
Case (1) From equation (1.1) and (2.1), we have

$$\Delta(a(n)\Delta z(n)) + q(n)(1-P(n+1-\ell))^{\beta}z^{\beta}(n+1-\ell) \leq 0 \quad (2.9)$$

for all $n \geq N$. Since $\beta \geq 1$, we have from (2.3) and (2.9) that

$$\Delta(a(n)\Delta z(n)) + q(n)(1-P(n+1-\ell))^{\beta}A^{\beta-1}(n+1-\ell)z(n+1-\ell) \leq 0 \quad (2.10)$$

for all $n \geq N$. Define

$$w(n) = \mu(n)\frac{a(n)\Delta z(n)}{z(n-\ell)}, \quad n \geq N.$$

Then $w(n) > 0$ for $n \geq N$, and

$$\Delta w(n) = \mu(n)\frac{\Delta(a(n)\Delta z(n))}{z(n+1-\ell)} + \frac{\Delta\mu(n)}{\mu(n+1)}w(n+1) - \mu(n)\frac{a(n+1)\Delta z(n+1)}{z(n+1-\ell)z(n-\ell)}\Delta z(n-\ell), \quad n \geq N. \quad (2.11)$$

From equation (1.1) and (2.11), we obtain

$$\Delta w(n) \leq -\mu(n)q(n)(1-P(n+1-\ell))^{\beta}A^{\beta-1}(n+1-\ell)+\frac{\Delta\mu(n)}{\mu(n+1)}w(n+1)$$

$$-\frac{\mu(n)}{\mu^2(n+1)}\frac{w^2(n+1)}{a(n+1-\ell)}, \quad n \geq N \tag{2.12}$$

where we have used $a(n)\Delta z(n)$ is decreasing and $z(n)$ is increasing. Now, using completing the square in (2.12), we have

$$\Delta w(n) \leq -\mu(n)q(n)(1-P(n+1-\ell))^{\beta}A^{\beta-1}(n+1-\ell)+\frac{a(n+1-\ell)(\Delta\mu(n))^2}{4\mu(n)}$$

for $n \geq N$. Summing the last inequality from N to n yields

$$\sum_{s=N}^{n}\left[\mu(s)q(s)(1-P(s+1-\ell))^{\beta}A^{\beta}(s+1-\ell)-\frac{a(s+1-\ell)(\Delta\mu(s))^2}{4\mu(s)}\right] \leq w(N)$$

which on taking sup as $n \to \infty$, we obtain a contradiction with (2.7).
Case (2) From equation (1.1) and (2.4) we have

$$\Delta(a(n)\Delta z(n)) + q(n)(1-Q(n+1-\ell))^{\beta}z^{\beta}(n+1-\ell) \leq 0, \quad n \geq N. \tag{2.13}$$

Since $\beta \geq 1$ and $\frac{z(n)}{A(n)} \geq A(n)$, the above inequality implies that

$$\Delta(a(n)\Delta z(n))+q(n)(1-Q(n+1-\ell))^{\beta}A^{2\beta-2}(n+1-\ell)z(n+1-\ell) \leq 0 \tag{2.14}$$

for $n \geq N$. Define

$$u(n) = \frac{a(n)\Delta z(n)}{z(n)}, \ n \geq N. \tag{2.15}$$

Then $u(n) < 0$ for all $n \geq N$. Since $a(n)\Delta z(n)$ is decreasing we have

$$\Delta z(s) \leq \frac{a(n)\Delta z(n)}{a(s)}, \quad s \geq n.$$

Hence

$$z(j) - z(n) \leq a(n)\Delta z(n)\sum_{s=n}^{j-1}\frac{1}{a(s)}$$

which as $j \to \infty$ yields

$$\frac{a(n)\Delta z(n)}{z(n)}A(n) \geq -1$$

or

$$u(n)A(n) \geq -1, \;\; n \geq N. \tag{2.16}$$

From (2.15), we have

$$\Delta u(n) = \frac{\Delta(a(n)\Delta z(n))}{z(n+1)} - \frac{a(n)\Delta z(n)}{z(n)z(n+1)}\Delta z(n)$$

$$\leq -q(n)(1-Q(n+1-\ell))^{\beta}A^{2\beta-2}(n+1-\ell) - \frac{u^2(n)}{a(n)}, \;\; n \geq N,$$

where we have used $a(n)\Delta z(n)$ is negative decreasing and $z(n)$ is positive and decreasing. Multiplying the last inequality by $A(n+1)$ and then applying summation by parts formula, we obtain

$$A(n)u(n) - A(N)u(N) + \sum_{s=N}^{n-1} A(s+1)q(s)(1-Q(s+1-\ell))^{\beta}A^{2\beta-2}(s+1-\ell)$$

$$+ \sum_{s=N}^{n-1}\left(\frac{u(s)}{a(s)} + \frac{A(s+1)u^2(s)}{a(s)}\right) \leq 0$$

which on using completing the square yields

$$\sum_{s=N}^{n-1}\left(A(s+1)q(s)(1-Q(s+1-\ell))^{\beta}A^{\beta-1}(s+1-\ell) - \frac{1}{4a(s)A(s+1)}\right)$$
$$\leq 1 + A(N)u(N)$$

where we have used (2.16). Taking sup as $n \to \infty$ in the last inequality, we obtain a contradiction with (2.8). This completes the proof.

If $\beta = 1$, then from Theorem 2.1 one can immediately obtain the following oscillation result for the equation (1.1).

Theorem 2.2 *Let $\beta = 1$, condition (1.2) and $\max\{P(n),\, Q(n)\} < 1$ be hold. If there is a positive real sequence $\{\mu(n)\}$ such that*

$$\lim_{n\to\infty}\sup\sum_{s=N}^{n}\left[\mu(s)q(s)(1-P(s+1-\ell)) - \frac{a(s+1-\ell)(\Delta\mu(s))^2}{4\mu(s)}\right] = \infty \tag{2.17}$$

and

$$\lim_{n\to\infty}\sup\sum_{s=N}^{n-1}\left[A(s+1)q(s)(1-Q(s+1-\ell)) - \frac{1}{4a(s)A(s+1)}\right] = \infty \tag{2.18}$$

then every solution of equation (1.1) is oscillatory.

In the following, we present an oscillation result for the equation (1.1) in the case $0 < \beta < 1$.

Theorem 2.3 *Let* $0 < \beta < 1$, *condition* (1.2) *and* $\max\{P(n),\ Q(n)\} < 1$ *be hold. If there is a positive real sequence* $\{\mu(n)\}$ *such that*

$$\lim_{n\to\infty}\sup\sum_{s=N}^{n}\left[\mu(s)q(s)\frac{(1-P(s+1-\ell))^{\beta}B^{\beta-1}(s+1-\ell)}{K^{1-\beta}} - \frac{a(s+1-\ell)(\Delta\mu(s))^2}{4\mu(s)}\right] = \infty \quad (2.19)$$

and

$$\lim_{n\to\infty}\sup\sum_{s=N}^{n-1}\left[A(s+1)q(s)\frac{(1-Q(s+1-\ell))^{\beta}}{M^{1-\beta}} - \frac{1}{4a(s)A(s+1)}\right] = \infty \quad (2.20)$$

for every constants $K > 0$ and $M > 0$, then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1.1), say $x(n) > 0$, $x(n-k_i) > 0$ and $x(n+1-\ell) > 0$ for $n \ge n_1$ and $i = 1, \ldots, m$. Proceeding as in proof of Theorem 2.1, we see that two cases of Lemma 2.2 hold.

Case (1) Proceeding as in the proof of Case (1) of Theorem 2.1 we have (2.9) holds. Now

$$z(n) - z(n_1) = \sum_{s=n_1}^{n-1}\frac{a(s)\Delta z(s)}{a(s)}$$

or

$$z(n) \ge a(n)\Delta z(n)B(n) \quad (2.21)$$

and hence

$$\Delta\left(\frac{z(n)}{B(n)}\right) \le 0 \quad for \quad n \ge n_1.$$

From (2.9), we have

$$\Delta(a(n)\Delta z(n)) + q(n)(1-P(n+1-\ell))^{\beta}B^{\beta-1}(n+1-\ell)\left(\frac{z(n+1-\ell)}{B(n+1-\ell)}\right)^{\beta-1} z(n+1-\ell) \le 0 \quad (2.22)$$

for all $n \geq n_1$. Since $\frac{z(n)}{B(n)}$ is decreasing there is a constant $K > 0$ such that

$$\frac{z(n)}{B(n)} \leq K \quad for \quad n \geq n_1. \tag{2.23}$$

In view of (2.23), and $\beta < 1$ the inequality (2.22) becomes

$$\Delta(a(n)\Delta z(n)) + q(n)\frac{(1 - P(n+1-\ell))^{\beta} B^{\beta-1}(n+1-\ell)}{K^{1-\beta}} z(n+1-\ell) \leq 0, \quad n \geq N.$$

The remaining part of the proof is similar to that of Case (1) of Theorem 2.1 and hence the details are omitted.

Case(2) In this case $z(n)$ is decreasing and therefore there is a constant $M > 0$ such that $z(n) \leq M$ for all $n \geq N$. Now from (2.13) and $\beta < 1$, we have

$$\Delta(a(n)\Delta z(n)) + M^{\beta-1} q(n)(1 - Q(n+1-\ell))^{\beta} z(n+1-\ell) \leq 0, \quad n \geq N.$$

The rest of the proof is similar to that of Case (2) of Theorem 2.1 and hence details are omitted. This completes the proof of the theorem.

In our next result, we use comparison method to prove our results for the case $\beta \in (0, \infty)$.

Theorem 2.4 *Let condition* (1.2) *and* $\max\{P(n), Q(n)\} < 1$ *be hold. If* $\ell \geq 2$ *and for some integer* $k \geq 1$, *the first order delay difference inequalities*

$$\Delta w(n) + q(n)(1 - P(n+1-\ell))^{\beta} B^{\beta}(n+1-\ell) w^{\beta}(n+1-\ell) \leq 0, \quad n \geq N \tag{2.24}$$

and

$$\Delta u(n) - q(n)(1 - Q(n+1-\ell))^{\beta} A^{\beta}(n+k) u^{k}(n+k) \geq 0, \quad n \geq N \tag{2.25}$$

have no positive solutions, then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1.1), say $x(n) > 0$, $x(n-k_i) > 0$ and $x(n+1-\ell) > 0$ for $n \geq n_1$ and $i = 1, 2, \ldots, m$. Processing as in the proof of Theorem 2.1, we see that two cases of Lemma 2.2 hold.

Case(1) From (2.9) and (2.26), we see that

$$\Delta(a(n)\Delta z(n)) + q(n)(1 - P(n+1-\ell))^{\beta} B^{\beta}(n+1-\ell)(a(n+1-\ell)\Delta z(n+1-\ell))^{\beta} \leq 0$$

for $n \geq N$. Set $w(n) = a(n)\Delta z(n)$. Then $w(n) > 0$ and from the last inequality, we have

$$\Delta w(n) + q(n)(1 - P(n+1-\ell))^{\beta} B^{\beta}(n+1-\ell) w^{\beta}(n+1-\ell) \leq 0.$$

Therefore $\{w(n)\}$ is a positive solution of the inequality (2.24) which is a contradiction.
Case (2) Since $\Delta z(n) < 0$, the inequality (2.13) reduces to

$$\Delta(-a(n)\Delta z(n)) - q(n)(1 - Q(n + 1 - \ell))^{\beta} z^{\beta}(n + 1 - \ell) \geq 0.$$

Note that $z(n)$ is decreasing and therefore $z(n + 1 - \ell) \geq z(n + k)$ and

$$\Delta(-a(n)\Delta z(n)) - q(n)(1 - Q(n + 1 - \ell))^{\beta} z^{\beta}(n + k) \geq 0. \tag{2.26}$$

Setting $u(n) = -a(n)\Delta z(n)$ and using (2.16) in (2.26), we see that $\{u(n)\}$ is a positive solution of the inequality (2.25). This contradicts our assumption and the proof is complete.

If $\beta = 1$, then we can obtain the following corollary by using the results in Theorem 7.6.1 of [5] and Theorem 2.4.
Corollary 2.1 *Let all conditions of Theorem 2.4 hold with $\beta = 1$. If*

$$\liminf_{n\to\infty} \sum_{s=n+1-\ell}^{n-1} q(s)(1 - P(s + 1 - \ell))B(s + 1 - \ell) > \left(\frac{\ell - 1}{\ell}\right)^{\ell}, \tag{2.27}$$

and

$$\liminf_{n\to\infty} \sum_{s=n}^{n+k} q(s)(1 - Q(s + 1 - \ell))A(s + k) > \left(\frac{k + 1}{k}\right)^{k+1} \tag{2.28}$$

then every solution of equation (1.1) is oscillatory.
Proof. As in [[5], Theorem 7.6.1], condition (2.27) ensures that the difference inequality (2.24) has no positive solutions. On the otherhand, it follows from [[5], Theorem 7.6.1] that condition (2.28) guarantees that the difference inequality (2.28) has no positive solutions. Application of Theorem 2.4 completes the proof.

2.1. 3.Examples

In this section, we present some examples to illustrate the main results.
Example 3.1 *Consider a second order difference equation with nonlinear neutral terms*

$$\Delta\left(n(n+1)\Delta\left(x(n) + \frac{1}{n^3}x^{1/3}(n-1) + \frac{1}{n^3}x^{1/5}(n-2)\right)\right) + n^{\gamma}x^3(n) = 0 \tag{2.29}$$

for $n \geq 3$. Here $a(n) = n(n+1)$, $p_1(n) = \frac{1}{n^3}$, $p_2(n) = \frac{1}{n^3}$, $k_1 = 1$, $k_2 = 2$, $\ell = 1$, $q(n) = n^\gamma$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{1}{5}$ and $\beta = 3$. Simple calculation shows that $A(n) = \frac{1}{n} \to 0$ as $n \to \infty$, and

$$1 - P(n) = \frac{15n^3 - 22n - 8}{15n^3} > 0 \quad for \quad n \geq 3$$

and

$$1 - Q(n) = \frac{15n^3 - 5(1+2n^2)\left(\frac{n}{n-1}\right)^{1/3} - 3(1+4n^2)\left(\frac{n}{n-2}\right)^{1/5}}{15n^3} > 0 \quad for \quad n \geq 3.$$

By taking $\mu(n) = 1$, we see conditions (2.7) and (2.8) are satisfied provided $\gamma \geq 4$. Therefore by Theorem 2.1, every solution of equation (2.29) is oscillatory provided $\gamma \geq 4$.

Example 3.2 *Consider a second order difference equation with nonlinear neutral terms*

$$\Delta\left(2^{n+1}\Delta\left(x(n) + \frac{1}{8^n}x^{3/5}(n-1) + \frac{1}{8^n}x^{1/5}(n-2)\right)\right) + 2^{\gamma n}x(n-1) = 0 \tag{2.30}$$

for $n \geq 3$. Here $a(n) = 2^{n+1}$, $p_1(n) = \frac{1}{8^n}$, $p_2(n) = \frac{1}{8^n}$, $k_1 = 1$, $k_2 = 2$, $\ell = 2$, $q(n) = 2^{\gamma n}$, $\alpha_1 = \frac{3}{5}$, $\alpha_2 = \frac{1}{5}$and $\beta = 1$. Simple calculation shows that $A(n) = \frac{1}{2^n} \to 0$ as $n \to \infty$, and

$$1 - P(n) = \frac{8^n - 6 \times 2^n - 4}{5 \times 8^n} > 0 \quad for \quad n \geq 3$$

and

$$1 - Q(n) = \frac{1}{5}\left[5 - \frac{1}{8^n}(3 + 2^{2n+1})8^{1/5} + (1 + 4^{n+1})4^{1/5}\right] > 0 \quad for \quad n \geq 3.$$

By taking $\mu(n) = 1$, we see conditions (**??**) and (2.18) are satisfied provided $\gamma \geq 1$. Therefore by Theorem 2.2, every solution of equation (2.30) is oscillatory provided $\gamma \geq 1$.

Example 3.3 *Consider a second order difference equation with nonlinear neutral terms*

$$\Delta\left(n(n+1)\Delta\left(x(n) + \frac{1}{n^3}x^{1/3}(n-1) + \frac{1}{n^3}x^{3/5}(n-2)\right)\right) + n^\gamma x^{1/3}(n) = 0 \tag{2.31}$$

for $n \geq 3$. Here $a(n) = n(n+1)$, $p_1(n) = \frac{1}{n^3}$, $p_2(n) = \frac{1}{n^3}$, $k_1 = 1$, $k_2 = 2$, $\ell = 1$, $q(n) = n^\gamma$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{3}{5}$ and $\beta = \frac{1}{3}$. Simple calculation shows that $A(n) = \frac{1}{n} \to 0$ as $n \to \infty$, $B(n) = \frac{n-3}{3n}$, and

$$1 - P(n) = \frac{15n^3 - 16n - 14}{15n^3} > 0 \quad for \quad n \geq 3$$

and

$$1 - Q(n) = \frac{15n^3 - 5(1+2n^2)\left(\frac{n}{n-1}\right)^{1/3} - 3(3+2n^2)\left(\frac{n}{n-2}\right)^{3/5}}{15n^3} > 0 \quad for \quad n \geq 3.$$

By taking $\mu(n) = 1$, we see conditions (2.19) and (2.20) are satisfied provided $\gamma \geq 1$. Therefore by Theorem 2.3, every solution of equation (2.31) is oscillatory provided $\gamma \geq 1$.

We conclude this paper with the following remark.

Remark 3.1 *The results obtained in this paper are new and complement to that of in [4, 8, 10, 11, 12, 14, 15, 16, 17]. The results established in [4, 8, 10, 11, 12, 14, 15, 16, 17] cannot be applied to equations* (2.29) *to* (2.31) *since neutral terms contain more than one nonlinear neutral terms. Thus the results presented in this paper are applicable to several classes of neutral type difference equations.*

References

[1] R.P.Agarwal, M.Bohner, S.R.Grace and D.O'Regan, Discrete Oscillation Theory, *Hindawi Pub. Corp., New York,* 2005.

[2] R.P.Agarwal, S.R.Grace and D.O'Regan, Oscillation Theory for Difference and Functional Differential Equations, *Kluwer, The Netherlands,* 2000.

[3] Y.Bolat, Oscillation criteria for second order functional difference equation with neutral terms, *Demonstr. Math.,* 41(2008), 615-625.

[4] C.Dharuman, J.R.Graef, E.Thandapani and K.S.Vidhyaa, Oscillation of second order difference equation with a sub-linear neutral term, *J. Math. Appl.,* 40 (2017), 1-10.

[5] I.Gyori and G.Ladas, Oscillation of Theory of Delay Differential Equations with Applications, *Clarendan Press, Oxford,* 1991.

[6] J.K.Hale, Theory of Functional Differential Equations, *Springer, New York,* 1987.

[7] G.H.Hardy, J.E.Littlewood and G.Polya, Inequalities, *Cambridge University Press, London,* 1934.

[8] B.Kamaraj and R.Vasuki, Some oscillation results for second order difference equation with a sublinear neutral term, *Asia Pac. J. Math.,* 4(2017), 100-109.

[9] X.Liu, Oscillation of solutions of neutral difference equations with nonlinear term, *Comput. Math. Appl.,* 52 (2006), 439-448.

[10] S.Mehar Banu, S.Nalini and Sandra Pinelas, Oscillation of second order difference equations with several sublinear neutral term, *Far East J. Math. Sci.,* (to appear).

[11] S.Selvarangam, S.A.Rupadevi and E.Thandapani, Oscillation theorems for second order quasilinear difference equations with linear neutral terms, *Transylv. J. Math. Mech.,* 9(2017), 63-72 .

[12] E.Thandapani and K.Mahalingam, Necessary and sufficient conditions for oscillation of second order neutral difference equations, *Tamkang J. Math.,* 34 (2003), 137-145.

[13] E.Thandapani, Z.Liu, R.Arul and P.S.Raja, Oscillation and asymptotic behavior of second order difference equations with nonlinear neutral term, *Appl. Math. E-Notes,* 4 (2004), 59-67.

[14] E.Thandapani, S.Pandian and R.K.Balasubramanian, Oscillation of solutions of nonlinear neutral difference equations with nonlinear neutral term, *Far East J. Appl. Math.,* 15 (2004), 47-62.

[15] E.Thandapani, D.Seghar and S.Selvarangam, Oscillation of second order quasilinear difference equations with sevaral neutral terms, *Transylv. J. Math. Mech.,* 7(2015), 67-74.

[16] M.K.Yildiz and H.Ogunmez, Oscillation results of higher order nonlinear neutral delay difference equations with a nonlinear neutral term, *Hacettepe J. Math. Stat.,* 43(2004), 809-814.

[17] Z.Zhang, J.Chen and C.Zhang, Oscillation of solutions for second order nonlinear difference equations with nonlinear neutral term, *Comput. Math. Appl.,* 41(2001), 1487-1494.