International Journal of Pure and Applied Mathematics
Volume 120 No. 3 2018, 447-459
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v120i3.13

ijpam.eu

ON COMMON FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN INTUITIONISTIC FUZZY METRIC SPACE

Rajinder Sharma[1 §], Deepti Thakur[2]

[1]Sohar College of Applied Sciences
Mathematics Section, OMAN

[2]Sohar College of Applied Sciences
Mathematics Section, OMAN

Abstract: This study deals with some common fixed point theorems for multi-valued mappings in intuitionistic fuzzy metric space by relaxing the condition of continuous mapping and replacing the completeness of the space with a set of an alternative conditions. We improve some earlier results.

AMS Subject Classification: 47H10, 54H25

Key Words: common fixed point, multi-valued mappings, weak compatible maps, intuitionistic fuzzy metric space

1. Introduction and Preliminaries

There have been a number of generalizations after the notion of fuzzy sets given by Zadeh [25]. Atanassov [1] introduced the concept of intuitionistic fuzzy sets and thereby attracted other researchers involved in the field of non linear analysis to explore further . Working in the same line, Park [18] used the concept of intuitionistic fuzzy sets to generalize fuzzy metric space due to George and Veeramani (see [8], [9]) to intuitionistic fuzzy metric spaces coinciding with con-

Received: November 13, 2016
Revised: January 2, 2019
Published: January 3, 2018

§Correspondence author

tinuous t norms and continuous t-conorms. Thereafter , Jungck and Rhoades [13] came out with the more generalized form of compatibility called weak compatibility and established common fixed point theorems for the same.

Kubiaczyk and Sharma (see [15],[16]) defined multivalued mappings in fuzzy metric spaces and studied common fixed point theorems for it. Sharma et al. [21] obtained common fixed point for multivalued mappings in intutionistic fuzzy metric spaces. For more details, we refer to (see [2], [3],[4], [5], [6], [7], [10],[11], [12], [14], [17], [18], [19], [20], [22], [23], [24]). In this paper, we established a common fixed point theorem for multi-valued mappings in intuitionistic fuzzy metric space by relaxing the condition of continuity and replacing the completeness of the space with an alternative set of conditions.

Definition 1. [24] A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is continuous t-norm if $*$ is satisfying the following conditions:

(i) $*$ is commutative and associative,

(ii) $*$ is continuous,

(iii) $a * 1 = a, \forall a \in [0,1]$,

(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d, \forall a,b,c,d \in [0,1]$.

Definition 2. [24] A binary operation $\diamond : [0,1]\times[0,1] \to [0,1]$ is continuous t-norm if $\diamond$ is satisfying the following conditions:

(i) $\diamond$ is commutative and associative,

(ii) $\diamond$ is continuous,

(iii) $a \diamond 0 = a, \forall a \in [0,1]$,

(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d, \forall a,b,c,d \in [0,1]$.

Definition 3. [2] A 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0,\infty)$ satisfying the following conditions ($\forall x,y,z \in X$ and $t,s > 0$,):

(i) $M(x,y,t) + N(x,y,t) \leq 1$,

(ii) $M(x,y,0) = 0$,

(iii) $M(x,y,t) = 1, \forall t > 0$ if and only if $x = y$,

(iv) $M(x,y,t) = M(y,x,t)$,

(v) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$,

(vi) $M(x,y,.) : [0,\infty) \to [0,1]$ is left continuous,

(vii) $lim_{t\to\infty} M(x,y,t) = 1, \forall x,y \in X$,

(viii) $N(x,y,0) = 1$,

(ix) $N(x,y,t) = 0, \forall t > 0$ if and only if $x = y$,

(x) $N(x,y,t) = N(y,x,t)$,

(xi) $N(x,y,t) \diamond N(y,z,s) \geq N(x,z,t+s)$,

(xii) $N(x,y,.) : [0,\infty) \to [0,1]$ is right continuous.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 4. *Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \diamond)$ such that t-norm $*$ and t-conorm $\diamond$ are associated , i.e., $x \diamond y = 1 - ((1-x) * (1-y)), \forall x, y \in X$.*

Example 5. *[18] Let (X, d) be a metric space. Define t-norm $a * b = min\{a, b\}$ and t-conorm $a \diamond b = max\{a, b\}$ and $\forall x, y \in X$ and $t > 0$,*

$$M(x,y,t) = \frac{t}{t + d(x,y)}, \quad N(x,y,t) = \frac{d(x,y)}{t + d(x,y)}.$$

Then $(X, M, N, *, \diamond)$ is an intuitionistic fuzzy metric space. We call this intuitionistic fuzzy metric (M, N) induced by the metric d, the standard intuitionistic fuzzy metric.

Example 6. *[18] Let $X = N$. Define $a * b = max\{0, a + b - 1\}$ and $a \diamond b = a + b - ab, \forall a, b \in [0,1]$ and let M and N be the fuzzy sets on $X^2 \times (0,\infty)$ as follows :*

$$M(x,y,t) = \begin{cases} \frac{x}{y}, & \text{if } x \leq y, \\ \frac{y}{x}, & \text{if} y \leq x, \end{cases}$$

and

$$N(x,y,t) = \begin{cases} \frac{y-x}{y}, & \text{if } x \leq y, \\ \frac{x-y}{x}, & \text{if} y \leq x, \end{cases}$$

$\forall x, y \in X$ and $t > 0$. Then $(X, M, N, *, \diamond)$ is an intuitionistic fuzzy metric space.

Remark 7. *Note that, in the above example, t-norm * and t-conorm $\diamond$ are not associated and there exists no metric d on X satisfying*

$$M(x,y,t) = \frac{t}{t + d(x,y)}, \quad N(x,y,t) = \frac{d(x,y)}{t + d(x,y)},$$

where $M(x,y,t)$ and $N(x,y,t)$ are as defined in above example. Also note the above functions (M, N) is not an intuitionistic fuzzy metric with the $t-$norm $*$ and $t-$ conorm $\diamond$ defined as $a * b = min\{a, b\}$ and $a \diamond b = max\{a, b\}$.

Lemma 8. *[2] In an intuitionistic fuzzy metric space X, $M(x,y,.)$ is non-decreasing and $N(x,y,.)$ is non-increasing, $\forall x,y \in X$.*

Definition 9. [2] Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space. Then
(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (denoted by $lim_{n\to\infty}x_n = x)\, if, \forall t > 0,\ lim_{n\to\infty}M(x_n,x,t) = 1,\ lim_{n\to\infty}N(x_n,x,t) = 0.$
(ii) A sequence $\{x_n\}$ in X is said to be a cauchy sequence if $\forall\, t > 0$ and $p > 0$, $lim_{n\to\infty}M(x_{n+p},x_n,t) = 1,\ lim_{n\to\infty}N(x_{n+p},x_n,t) = 0.$

Remark 10. *Since $*$ and $\diamond$ are continuous, the limit is uniquely determined from (v) and (xi), respectively.*

Definition 11. [1] An intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ is said to be complete if and only if every cauchy sequence in X is convergent. It is called compact if every sequence contains a convergent subsequence.

Lemma 12. *[21] Let $(X,M,N,*,\diamond)$ be an intuitionistic fuzzy metric space and $\{y_n\}$ be sequence in X. If there exists a number $k \in (0,1)$ such that:*

(i) $M(y_{n+2},y_{n+1},kt) \geq M(y_{n+1},y_n,t)$,

(ii) $N(y_{n+2},y_{n+1},kt) \leq N(y_{n+1},y_n,t), \forall t > 0$ and $n = 1,2,...$, then $\{y_n\}$ is a cauchy sequence in X.

Proof. By simple induction with the condition (i) and with the help of Alaca et. al. [2], we have $\forall t > 0$ and $n = 1,2,...$,
$(iii)\ M(y_{n+1},y_{n+2},t) \geq M(y_1,y_2,\frac{t}{k^n}), N(y_{n+1},y_{n+2},t) \leq N(y_1,y_2,\frac{t}{k^n})$
Thus by (iii) and definition (3) [(v) and (xi)], for any positive integer p and real number $t > 0$, we have

$$\begin{aligned} M(y_{n+p},y_{n+1},t) &\geq M(y_n,y_{n+1},\frac{t}{p}) * ...p - times... * M(y_{n+p-1},y_{n+p},\frac{t}{p}) \\ &\geq M(y_1,y_2,\frac{t}{pk^{n-1}}) * ...p - times... * M(y_1,y_2,\frac{t}{pk^{n+p-1}}) \end{aligned}$$

and

$$\begin{aligned} N(y_{n+p},y_{n+1},t) &\leq N(y_n,y_{n+1},\frac{t}{p}) \diamond ...p - times... \diamond N(y_{n+p-1},y_{n+p},\frac{t}{p}) \\ &\leq N(y_1,y_2,\frac{t}{pk^{n-1}}) \diamond ...p - times... \diamond N(y_1,y_2,\frac{t}{pk^{n+p-1}}). \end{aligned}$$

Therefore by Definition (3) [(vii) and (xiii)], we have

$$\lim_{n\to\infty} M(y_n,y_{n+p},t) \geq 1 * ...p - times... * 1 \geq 1$$

and

$$\lim_{n\to\infty} N(y_n, y_{n+p}, t) \leq 0 \diamond ...p - times... \diamond 0 \leq 0,$$

which implies that $\{y_n\}$ is a cauchy sequence in X. This completes the proof. The following Lemma establishes a relationship between x and y by virtue of Kubiaczyk and Sharma [15]. □

Lemma 13. *[21] Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and $\forall x, y \in X, t > 0$ and if for a number $k \in (0, 1), M(x, y, kt) \geq M(x, y, t)$ and $N(x, y, kt) \leq N(x, y, t)$, then $x = y$*

Proof. Since $M(x, y, kt) \geq M(x, y, t)$ and $N(x, y, t) \leq N(x, y, t)$, then using results of Kubiaczyk and Sharma[15], we have

$$M(x, y, kt) \geq M(x, y, \frac{t}{k}) \text{ and } N(x, y, t) \leq N(x, y, \frac{t}{k}).$$

By repeated application of above inequalities , we have
$M(x, y, t) \geq M(x, y, \frac{t}{k})$,

$$M(x, y, kt) \geq M(x, y, \frac{t}{k^2}) \geq ... \geq M(x, y, \frac{t}{k^n}) \geq ...,$$

and

$$N(x, y, kt) \leq N(x, y, \frac{t}{k^2}) \leq ... \leq N(x, y, \frac{t}{k^n}) \leq ..., n \in N$$

which tend to 1 and 0, respectively as $n \to \infty$. Thus $M(x, y, t) = 1$ and $N(x, y, t) = 0$ for all $t > 0$ and we get $x = y$. □

Definition 14. [1] Let A and B be maps from an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ into itself. The maps A and B are said to be compatible if $\forall t \geq 0$, $\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1$ and $\lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 15. [15] Two self maps A and B on a set X are said to be weakly compatible if they commute at coincidence points; i.e., if $Au = Bu$ for some $u \in X$, then $ABu = BAu$.

Definition 16. [21] Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy metric space and consider $I : X \to X$ and $T : X \to CB(X)$. A point $z \in X$ is called a coincidence point of I and T if and only if $Iz \in Tz$. Denote $CB(X)$, the set of all non-empty bounded and closed subsets of X. We have
$M^{\nabla}(B, y, t) = max\{M(b, y, t); b \in B\}$,

$N^{\Delta}(B,y,t) = min\{N(b,y,t); b \in B\}$ and
$M_{\bigtriangledown}(A,B,t) \geq min\{minM^{\bigtriangledown}(a,B,t); a \in A, minM^{\bigtriangledown}(A,b,t); b \in B\}$
$N_{\Delta}(A,B,t) \leq max\{maxN^{\Delta}(a,B,t); a \in A, minN^{\Delta}A,b,t); b \in B\}$ for all $A,B \in X$ and $t > 0$.

Remark 17. *(i) In [11],[12] and [13] we can find the equivalent formulations of definitions of compatible maps, compatible maps of type* (α) *and compatible maps of type* (β)*. Such maps are independent of each other and more general than commuting and weakly commuting maps ([10], [20]).*

(ii) Compatible or compatible of type (α) *or compatible of type* (β) *maps are weakly compatible but converse need not true.*

Alaca, Turkoglu and Yildiz [2] established the following.

Theorem 18. *(Intuitionistic fuzzy Banach contraction theorem). Let* $(X,M,N,*,\diamond)$ *be a complete intuitionistic fuzzy metric space. Let* $T : X \to X$ *be a mapping satisfying*

$$M(Tx,Ty,kt) \geq M(x,y,t) \text{ and } N(Tx,Ty,kt) \leq N(x,y,t),$$

for all $x,y \in X, 0 < k < 1$*. Then* T *has a unique fixed point.*

Theorem 19. *(Intuitionistic fuzzy Edelstein contraction theorem). Let* $(X,M,N,*,\diamond)$ *be a compact space. Let* $T : X \to X$ *be a mapping satisfying*

$$M(Tx,Ty,.) > M(x,y,.) \text{ and } N(Tx,Ty,.) < N(x,y,.)$$

for all $x \neq y$*, i.e.* $M(Tx,Ty,.) \geq M(x,y,.)$ *and* $M(Tx,Ty,.) \neq M(x,y,.)$ *and* $N(Tx,Ty,.) \leq N(x,y,.)$ *and* $N(Tx,Ty,.) \neq N(x,y,.)$*, for all* $x \neq y$*. Then* T *has a unique fixed point.*

Kubiaczyk and Sharma [15] proved the following for fuzzy metric space.

Theorem 20. *Let* $(X,M,)$ *be a complete fuzzy metric space with* $t*t \geq t$ *for all* $t \in [0,1]$*. Let* $P,Q : X \to CB(X)$ *be continuous and there exists mappings* $S,T : X \to X$ *satisfying:*

(i) $SP = PS, QT = TQ$,

(ii) $P(X) \subseteq S(X), Q(X) \subseteq T(X)$,

(iii) there exists a number $q \in (0,1)$ *such that*

$$M_{\bigtriangledown}(Px,Qy,qt) \geq min\{M^{\bigtriangledown}(Sx,Tx,t), M^{\bigtriangledown}(Px,Sx,t), M^{\bigtriangledown}(Qy,Ty,t), M^{\bigtriangledown}(Px,Ty,(2-\alpha)t), M^{\bigtriangledown}(Qy,Sx,t)\},$$

for all $x, y \in X, \alpha \in (0, 2), t > 0$. *Then* P, Q, S *and* T *have a common coincidence point, i.e.* $Sz \in Pz$ *and* $Tz \in Qz$.

Sharma, Servet and Rathore [21] proved the follwing results for intuitionistic fuzzy metric spaces.

Theorem 21. *Let* $(X, M, N, *, \diamond)$ *be a complete intuitionistic fuzzy metric space with continuous* $t-$*norm* $*$ *and continuous* $t-$*conorm* $\diamond$ *defined by* $t*t \geq t$ *and* $(1-t)\diamond(1-t) \leq (1-t)$, *for all* $t \in [0, 1]$. *Let* $P : X \to C(X)$, *such that:*
(i) $M(Px, Py, kt) \geq M(x, y, t)$
and
(ii) $N(Px, Py, kt) \leq N(x, y, t)$, *for all* $x, y \in X$ *and* $0 < k < 1$. *Then* P *has a fixed point. This means that there exists a point u such that* $u \in Pu$.

Theorem 22. *Let* $(X, M, N, *, \diamond)$ *be a complete intuitionistic fuzzy metric space with continuous* $t-$ *norm* $*$ *and continuous* $t-$ *conorm* $\diamond$ *defined by* $t*t \geq t$ *and* $(1-t)\diamond(1-t) \leq (1-t)$, *for all* $t \in [0, 1]$. *Let* $T_n : X \to CB(X)(n \in N)$ *and continuous mapping* $I : X \to X$ *be such that* $T_n(X) \subset I(X)$ *where* I *commute with* T_n *for every* $n \in N$ *and*
(i) there exists $q \in (0, 1)$, *such that*

$$M_{\bigtriangledown}(T_ix, T_jy, qt) \geq min\{M(Ix, Iy, t), M^{\bigtriangledown}(Ix, T_ix, t), M^{\bigtriangledown}(Iy, T_jy, t), M^{\bigtriangledown}(Ix, T_jy, (2-\alpha)t), M^{\bigtriangledown}(Iy, T_ix, t)\}$$

and

$$N_{\Delta}(T_ix, T_jy, qt) \leq max\{N(Ix, Iy, t), N^{\Delta}(Ix, T_ix, t), N^{\Delta}(Iy, T_jy, t), N^{\Delta}(Ix, T_jy, (2-\alpha)t), N^{\Delta}(Iy, T_ix, t)\}$$

for all $x, y \in X, \alpha \in (0, 2)$ *and* $t > 0$ *for every* $i, , j \in N(i \neq j)$.*Then there exists a common coincidence point of* T_n *and* I , *i.e. there exists a point* $z \in X$ *such that* $Iz \in \cap T_nz$, $n \in N$.

2. Main Results

We improve Theorem (22) by dropping the condition of continuity and replacing the completeness of the space with a set of an alternative conditions .

Theorem 23. *Let* $(X, M, N, *, \diamond)$ *be an intuitionistic fuzzy metric space with continuous* $t-$ *norm* $*$ *and continuous* $t-$ *conorm* $\diamond$ *defined by* $t * t \geq t$ *and* $(1-t)\diamond(1-t) \leq (1-t)$, *for all* $t \in [0, 1]$. *Let* $T_n : X \to CB(X)(n \in N)$

and mapping $I : X \to X$ *be such that* $T_n(X) \subset I(X)$*, where the pair* $\{I, T_n\}$ *is weakly compatible for every* $n \in N$ *and for every* $i, j \in N$ $(i \neq j)$ *there exists* $q \in (0, 1)$*, such that*

$$M_{\bigtriangledown}(T_i x, T_j y, qt) \geq min\{M(Ix, Iy, t), M^{\bigtriangledown}(Ix, T_i x, t), M^{\bigtriangledown}(Iy, T_j y, t), M^{\bigtriangledown}(Ix, T_j y, (2-\alpha)t), M^{\bigtriangledown}(Iy, T_i x, t)\} \quad (1)$$

and

$$N_{\Delta}(T_i x, T_j y, qt) \leq max\{N(Ix, Iy, t), N^{\Delta}(Ix, T_i x, t), N^{\Delta}(Iy, T_j y, t), N^{\Delta}(Ix, T_j y, (2-\alpha)t), N^{\Delta}(Iy, T_i x, t)\}, \quad (2)$$

for all $x, y \in X, \alpha \in (0, 2)$ *and* $t > 0$ *for every* $i, j \in N(i \neq j)$*.*

$$I(X) \text{ or } T_n(X) \text{ is a complete subspace of } X. \quad (3)$$

Then there exists a common coincidence point of T_n *and* I*, i.e.* $\exists z \in X$ *such that* $Iz \in \cap T_n z$*,* $n \in N$*.*

Proof. Let $x_0 \in X$ and $x_1 \in X$ such that $Ix_1 \in T_1 x_0$ and $y_1 = Ix_1, q \in (0, 1)$ and the inequalities hold
$M(x_0, y_1, qt) = M(x_0, Ix_1, qt) \geq M^{\bigtriangledown}(x_0, T_1 x_0, qt) - \epsilon/2$,
$N(x_0, y_1, qt) = N(x_0, Ix_1, qt) \leq N^{\Delta}(x_0, T_1 x_0, qt) + \epsilon/2$.
$x_2 \in X$ such that $Ix_2 \in T_2 x_1$ and $y_2 = Ix_2$,
$M(y_1, y_2, qt) = M(Ix_1, Ix_2, qt) \geq M^{\bigtriangledown}(y_1, T_2 x_1, qt) - \epsilon/2^2$,
$N(y_1, y_2, qt) = N(Ix_1, Ix_2, qt) \leq N^{\Delta}(y_1, T_2 x_1, qt) + \epsilon/2^2$.
Inductively, we can construct a sequence $\{y_n\}$ in X such that
$M(y_n, y_{n+1}, qt) = M(Ix_n, Ix_{n+1}, qt) \geq M^{\bigtriangledown}(y_n, T_{n+1} x_n, qt) - \epsilon/2^n$,
$N(y_n, y_{n+1}, qt) = N(Ix_n, Ix_{n+1}, qt) \leq N^{\Delta}(y_n, T_{n+1} x_n, qt) + \epsilon/2^n$.
Now, we show that $\{y_n\}$ is a Cauchy sequence. By (1) for all $t > 0$ and $\alpha = 1-k$ with $k \in (0, 1)$, we write
$M(y_n, y_{n+1}, qt) \geq M^{\bigtriangledown}(y_n, T_{n+1} x_n, qt) - \epsilon/2^n \geq M_{\bigtriangledown}(T_n x_{n-1}, T_{n+1} x_n, qt) - \epsilon/2^n \geq$
$min\{M(Ix_{n-1}, Ix_n, t), M^{\bigtriangledown}(Ix_{n-1}, T_n x_{n-1}, t), M^{\bigtriangledown}(Ix_n, T_{n+1} x_n, t)$,
$M^{\bigtriangledown}(Ix_{n-1}, T_{n+1} x_n, (2-\alpha)t), M^{\bigtriangledown}(Ix_n, T_n x_{n-1}, t)\} - \epsilon/2^n$,
$\geq min\{M(Ix_{n-1}, Ix_n, t), M(Ix_{n-1}, Ix_n, t), M(Ix_n, Ix_{n+1}, t)$,
$M(Ix_{n-1}, Ix_{n+1}, (1+k)t), M(Ix_n, Ix_n, t)\} - \epsilon/2^n$,
and by (2), we have
$N(y_n, y_{n+1}, qt) \leq N^{\Delta}(y_n, T_{n+1} x_n, qt) + \epsilon/2^n \leq N_{\Delta}(T_n x_{n-1}, T_{n+1} x_n, qt) + \epsilon/2^n$.
$\leq max\{N(Ix_{n-1}, Ix_n, t), N^{\Delta}(Ix_{n-1}, T_n x_{n-1}, t), N^{\Delta}(Ix_n, T_{n+1} x_n, t)$,

$N^{\Delta}(Ix_{n-1}, T_{n+1}x_n, (2-\alpha)t), N^{\Delta}(Ix_n, T_nx_{n-1}, t)\} + \epsilon/2^n,$
$\leq max\{N(Ix_{n-1}, Ix_n, t), N(Ix_{n-1}, Ix_n, t), N(Ix_n, Ix_{n+1}, t),$
$N(Ix_{n-1}, Ix_{n+1}, (1+k)t), N(Ix_n, Ix_n, t)\} + \epsilon/2^n.$
Now, using Definition 3 [(v) and (xi)], we have
$M(y_n, y_{n+1}, qt) \geq min\{M(y_{n-1}, y_n, t), M(y_{n-1}, y_n, t), M(y_n, y_{n+1}, t),$

$$M(y_{n-1}, y_n, t), M(y_n, y_{n+1}, kt)\} - \epsilon/2^n \tag{4}$$

and
$N(y_n, y_{n+1}, qt) \leq max\{N(y_{n-1}, y_n, t), N(y_{n-1}, y_n, t), N(y_n, y_{n+1}, t),$

$$N(y_{n-1}, y_n, t), N(y_n, y_{n+1}, kt)\} + \epsilon/2^n. \tag{5}$$

Since $t-$norm $*$,$t-$ conorm $\diamond, M(x, y, .)$ and $N(x, y, .)$ are continuous, letting $k \to 1$ in (4) and (5), we have
$M(y_n, y_{n+1}, qt) \geq min\{M(y_{n-1}, y_n, t), M(y_n, y_{n+1}, t)\} - \epsilon/2^n,$
$N(y_n, y_{n+1}, qt) \leq max\{N(y_{n-1}, y_n, t), N(y_n, y_{n+1}, t)\} + \epsilon/2^n,$
for $n = 1, 2, ...$ and so , for positive integers n and p and $\epsilon \in (0, 1)$, we have
$M(y_n, y_{n+1}, qt) \geq min\{M(y_{n-1}, y_n, t), M(y_n, y_{n+1}, t/q^p)\} - \epsilon/2^n,$
$N(y_n, y_{n+1}, qt) \leq max\{N(y_{n-1}, y_n, t), N(y_n, y_{n+1}, t/q^p)\} + \epsilon/2^n.$
Since ϵ is arbitrary, making $\epsilon \to 0, M(y_n, y_{n+1}, t/q^p) \to 1$ and
$N(y_n, y_{n+1}, t/q^p) \to 0$ as $n \to \infty$, we obtain
$M(y_n, y_{n+1}, qt) \geq M(y_{n-1}, y_n, t)$ and $N(y_n, y_{n+1}, qt) \leq N(y_{n-1}, y_n, t)$
By Lemma (12), $\{y_n\}$ is a cauchy sequence.
Since $I(X)$ is complete. Note that the subsequence $\{y_{n+1}\}$ is contained in $I(X)$ and has a limit z in $I(X)$. Let $p \in I^{-1}z$. Then $Ip = z$. By (1), we have for $\alpha = 1$,
$M^{\nabla}(T_np, Ix_{n+1}, qt) - \epsilon/2^n = M_{\nabla}(T_np, T_{n+1}x_n, qt) - \epsilon/2^n$
$\geq min\{M(Ip, Ix_n, t), M^{\nabla}(Ip, T_np, t), M^{\nabla}(Ix_n, T_{n+1}x_n, t), M^{\nabla}(Ip, T_{n+1}x_n, t),$
$M^{\nabla}(Ix_n, T_np, t)\} - \epsilon/2^n,$
$\geq min\{M(Ip, Ix_n, t), M^{\nabla}(Ip, T_np, t), M(Ix_n, Ix_{n+1}, t), M(Ip, Ix_{n+1}, t),$
$M^{\nabla}(Ix_n, T_np, t)\} - \epsilon/2^n,$
$\geq min\{M(z, Ix_n, t), M^{\nabla}(z, T_np, t), M(Ix_n, Ix_{n+1}, t), M(z, Ix_{n+1}, t),$
$M^{\nabla}(Ix_n, T_np, t)\} - \epsilon/2^n,$
Taking the limit as $n \to \infty$, we obtain
$\geq min\{M(z, z, t), M^{\nabla}(z, T_np, t), M(z, z, t), M(z, z, t), M^{\nabla}(z, T_np, t)\} - \epsilon/2^n.$
This gives $M^{\nabla}(T_np, z, qt) \geq M^{\nabla}(T_np, z, t).$
Similarly by (2), we have for $\alpha = 1$,
$N^{\Delta}(T_np, Ix_{n+1}, qt) + \epsilon/2^n \leq N_{\Delta}(T_np, T_{n+1}x_n, qt) + \epsilon/2^n$
$\leq max\{N(Ip, Ix_n, t), N^{\Delta}(Ip, T_np, t), N(Ix_n, Ix_{n+1}, t), N(Ip, Ix_{n+1}, t),$

$N^{\Delta}(Ix_n, T_np, t)\} + \epsilon/2^n$,
$\leq max\{N(z, Ix_n, t), N^{\Delta}(z, T_np, t), N(Ix_n, Ix_{n+1}, t), N(z, Ix_{n+1}, t),$
$N^{\Delta}(Ix_n, T_np, t)\} + \epsilon/2^n$,
Taking the limit as $n \to \infty$, we obtain
$\leq max\{N(z, z, t), N^{\Delta}(z, T_np, t), N(z, z, t), N(z, z, t), N^{\Delta}(z, T_np, t)\} + \epsilon/2^n$.
This gives $N^{\Delta}(T_np, z, qt) \leq N^{\Delta}(T_np, z, t)$. Therefore by Lemma (13), $z \in T_np$.
Since $Ip = z \in Tp$. i.e. p is a coincidence point of I and T_n.
Since the pair $\{I, T_n\}$ is weakly compatible , therefore, I and T_n commute at coincidence point for every $n \in N$ i.e. $IT_np = T_nIp$ or $Iz \in T_nz$. Now, we prove $T_nz = z$. By (1), we have for $\alpha = 1$
$M^{\nabla}(T_nz, Ix_{n+1}, qt) - \epsilon/2^n = M_{\nabla}(T_nz, T_{n+1}x_n, qt) - \epsilon/2^n$
$\geq min\{M(Iz, Ix_n, t), M^{\nabla}(Iz, T_nz, t), M^{\nabla}(Ix_n, T_{n+1}x_n, t), M^{\nabla}(Iz, T_{n+1}x_n, t),$
$M^{\nabla}(Ix_n, T_nz, t)\} - \epsilon/2^n$,
$\geq min\{M(Iz, Ix_n, t), M^{\nabla}(Iz, T_nz, t), M(Ix_n, Ix_{n+1}, t), M(Iz, Ix_{n+1}, t),$
$M^{\nabla}(Ix_n, T_nz, t)\} - \epsilon/2^n$,
Taking the limit as $n \to \infty$, we obtain
$\geq min\{M(z, z, t), M^{\nabla}(z, T_nz, t), M(z, z, t), M(z, z, t), M^{\nabla}(z, T_nz, t)\} - \epsilon/2^n$.
This gives $M^{\nabla}(T_nz, z, qt) \geq M^{\nabla}(z, T_nz, t)$.
Similarly by (2), we have for $\alpha = 1$,
$N^{\Delta}(T_nz, Ix_{n+1}, qt) + \epsilon/2^n = N_{\Delta}(T_nz, T_{n+1}x_n, qt) + \epsilon/2^n$
$\leq max\{N(Iz, Ix_n, t), N^{\Delta}(Iz, T_nz, t), N^{\Delta}(Ix_n, T_{n+1}x_n, t), N^{\Delta}(Iz, T_{n+1}x_n, t),$
$N^{\Delta}(Ix_n, T_nz, t)\} + \epsilon/2^n$,
$\leq max\{N(Iz, Ix_n, t), N^{\Delta}(Iz, T_nz, t), N(Ix_n, Ix_{n+1}, t), N(Iz, Ix_{n+1}, t),$
$N^{\Delta}(Ix_n, T_nz, t)\} + \epsilon/2^n$,
$\leq max\{N(z, Ix_n, t), N^{\Delta}(z, T_nz, t), N(Ix_n, Ix_{n+1}, t), N(z, Ix_{n+1}, t),$
$N^{\Delta}(Ix_n, T_nz, t)\} + \epsilon/2^n$,
Taking the limit as $n \to \infty$, we obtain
$\leq max\{N(z, z, t), N^{\Delta}(z, T_nz, t), N(z, z, t), N(z, z, t), N^{\Delta}(z, T_nz, t)\} + \epsilon/2^n$,
This gives $N^{\Delta}(T_nz, z, qt) \leq N^{\Delta}(T_nz, z, t)$. Therefore by Lemma (13), we have $z \in T_nz$. Now, we prove $Iz = z$. Using the fact that $Iz \in T_nz$, by (1), with $\alpha = 1$, we have
$M^{\nabla}(T_nz, Ix_{n+1}, qt) - \epsilon/2^n = M_{\nabla}(T_nz, T_{n+1}x_n, qt) - \epsilon/2^n$
$\geq min\{M(Iz, Ix_n, t), M^{\nabla}(Iz, T_nz, t), M^{\nabla}(Ix_n, T_{n+1}x_n, t), M^{\nabla}(Iz, T_{n+1}x_n, t),$
$M^{\nabla}(Ix_n, T_nz, t)\} - \epsilon/2^n$,
$\geq min\{M(Iz, Ix_n, t), M(Iz, Iz, t), M(Ix_n, Ix_{n+1}, t), M(Iz, Ix_{n+1}, t),$
$M(Ix_n, Iz, t)\} - \epsilon/2^n$,
Taking the limit as $n \to \infty$, we obtain
$M^{\nabla}(Iz, z, qt) - \epsilon/2^n$
$\geq min\{M(Iz, z, t), M(Iz, Iz, t), M(z, z, t), M(Iz, Iz, t), M(Iz, Iz, t)\} - \epsilon/2^n$.

This gives $M(Iz, z, qt) \geq M(Iz, z, t)$.
Using the fact that $Iz \in T_n z$, by (2), with $\alpha = 1$, we have
$N^{\Delta}(T_n z, Ix_{n+1}, qt) + \epsilon/2^n \leq N_{\Delta}(T_n z, T_{n+1}x_n, qt) + \epsilon/2^n$
$\leq max\{N(Iz, Ix_n, t), N^{\Delta}(Iz, T_n z, t), N^{\Delta}(Ix_n, T_{n+1}x_n, t), N^{\Delta}(Iz, T_{n+1}x_n, t),$
$N^{\Delta}(Ix_n, T_n z, t)\} + \epsilon/2^n,$
$\leq max\{N(Iz, Ix_n, t), N(Iz, Iz, t), N(Ix_n, Ix_{n+1}, t), N(Iz, Ix_{n+1}, t),$
$N(Ix_n, Iz, t)\} + \epsilon/2^n,$
Taking the limit as $n \to \infty$, we obtain
$\leq max\{N(Iz, z, t), N(Iz, Iz, t), N(z, z, t), N(Iz, Iz, t), N(Iz, Iz, t)\} + \epsilon/2^n.$
This gives $N(z, Iz, qt) \leq N(Iz, z, t)$.
Therefore by Lemma (13), we have $Iz = z$. By weak compatibility, since $Iz \in T_n z$. Thus $z = Iz \in \cap T_n z, n \in N$.
This completes the proof of the theorem. □

Condition (3) of the theorem (23) can be removed by taking the space complete. We establish the following.

Theorem 24. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space with continuous $t-$ norm $*$ and continuous $t-$ conorm $\diamond$ defined by $t*t \geq t$ and $(1-t) \diamond (1-t) \leq (1-t)$, for all $t \in [0, 1]$. Let $T_n : X \to CB(X)(n \in N)$ and mapping $I : X \to X$ be such that $T_n(X) \subset I(X)$ where the pair $\{I, T_n\}$ is weakly compatible for every $n \in N$ and for every $i, j \in N$ $(i \neq j)$ satisfying conditions (1) and (2) of Theorem (23). Then there exists a common coincidence point of T_n and I, i.e. there exists a point z in X such that $z = Iz \in \cap T_n z$, $n \in N$.*

Proof. Theorem 24 can be proved in the similer manner as Theorem 23. □

3. Acknowledgments

The authors would like to offer their sincere gratitude and thanks to the editors and annonyms refrees for their valuable comments/suggestions.

References

[1] Atanassov, K., Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, **20** (1986), 87-96.

[2] Alaca, C., Turkoglu and Yildiz, C., Fixed points in intuitionistic fuzzy metric spaces, *Chaos, Solit. and Fract.*, **29** (2006), 1073-1078.

[3] Banach, S., Surles operations dansles ensembles abstracts at leur applications aux equations integrals,*Fund. Math.*, **3** (1922), 133-181.

[4] Cho, Y.J., Fixed Point in fuzzy metric spaces, *J. Fuzzy Math.*, **5(4)** (1997), 949- 962.

[5] Cho, Y.J., Pathak, H.K.,Kang, S.M. and Jung, J.S., Common fixed points of compatible maps of type (α) on fuzzy metric spaces,*Fuzzy sets and systems*, **93**(1998), 99-111.

[6] Ciric, L.B., Fixed Point for generalized multivalued contractions, *Math. Vesnik*, **9(24)** (1972), 99-11.

[7] Grabiec, M., Fixed point in fuzzy metric spaces, *Fuzzy sets and Systems*, **27** (1988), 385-389.

[8] George, A. and Veermani,P., On some results in fuzzy metric spaces, *Fuzzy sets and Systems*, **64** (1994), 385-399.

[9] George, A. and Veermani,P., On some results of analysis for fuzzy metric spaces, *Fuzzy sets and Systems* , **90** (1997), 365- 368.

[10] Jungck, G., Commuting mappings and fixed points, *Amer. Math. Monthly*, **83**(1976),261-263.

[11] Jungck, G., Compatible mappings and common fixed points (2), *Internat. J. Math. and Math. Sci.*, **11**(1988),285-288.

[12] Jungck, G., Murthy,P.P. and Cho, Y.J., Compatible mappings of type (A) and common fixed points , *Math. Japon.*, **38(2)**(1993), 381-390.

[13] Jungck, G. and Rhoades, B.E., Fixed points for set valued functions without continuity, *Ind. J. Pure and Appl. Maths.*, **29 (3)**(1998), 227-238.

[14] Kramosil,I. and Michalek, J., Fuzzy metric and statistical metric spaces, *Kybernetika*, **11**, 1975, 326-334.

[15] Kubiaczyk, I. and Sharma, S., Common coincidence point in fuzzy metric space, *J. Fuzzy Math.*, **11(1)** (2003), 1-5.

[16] Kubiaczyk, I. and Sharma, S., Common fixed point , multi-maps in fuzzy metric space, *East Asian Math. J.*, **18(2)** (2002), 175-182.

[17] Mishra, S.N., Sharma N., and Singh, S.L., Common fixed points of maps in fuzzy metric spaces, *Internat. J. Math. and Math. Sci.*, **17** (1994), 253-258.

[18] Park, J.H., Intuitionistic fuzzy metric spaces, *Chaos, Solit. and Fract.*, **22** (2004),1039-1046.

[19] Sharma, S. , Common Fixed Point theorems in fuzzy metric spaces, *Fuzzy Sets and Systems*, **127** (2002), 345-352.

[20] Sessa, S., On weak commutativity condition of mappings in a fixed point considerations, *Pub. Inst. Math.*,**32** (46) (1982),149-153.

[21] Sharma, S., Servet, K. and Rathore, R.S. , Common Fixed Point for multivalued mappings in intutionistic fuzzy metric space, *Comm. Kor. Math. Soc.*, **22(3)**(2007), 391-399.

[22] Turkoglu, D., Alaca C., and Yildiz C., Compatible maps and compatible maps of type (α) and (β) in intuitionistic fuzzy metric spaces, *Demons. Math.*,**39(3)** (2006),671-684.

[23] Turkoglu, D., Alaca C , Cho, Y. J. ,and Yildiz,C., Common fixed point theorems intuitionistic fuzzy metric spaces, *J. Appl. Math. Computing* **22 (1-2)** (2006), 411-424.

[24] Schweizer, B. and Skaler, A., Statistical metric spaces, *Pac. J. Math.*,**10** (1960), 313-334.

[25] Zadeh, L.A., Fuzzy Sets, *Infor. and Control*, **8** (1965), 338-353.