ON COMMON FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN INTUITIONISTIC FUZZY METRIC SPACE

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Abstract: This study deals with some common fixed point theorems for multi-valued mappings in intuitionistic fuzzy metric space by relaxing the condition of continuous mapping and replacing the completeness of the space with a set of an alternative conditions. We improve some earlier results.

AMS Subject Classification: 47H10, 54H25
Key Words: common fixed point, multi-valued mappings, weak compatible maps, intuitionistic fuzzy metric space

1. Introduction and Preliminaries

There have been a number of generalizations after the notion of fuzzy sets given by Zadeh [25]. Atanassov [1] introduced the concept of intuitionistic fuzzy sets and thereby attracted other researchers involved in the field of non linear analysis to explore further. Working in the same line, Park [18] used the concept of intuitionistic fuzzy sets to generalize fuzzy metric space due to George and Veeramani (see [8], [9]) to intuitionistic fuzzy metric spaces coinciding with con-
Continuous \(t\) norms and continuous \(t\)-conorms. Thereafter, Jungck and Rhoades [13] came out with the more generalized form of compatibility called weak compatibility and established common fixed point theorems for the same.

Kubiaczyk and Sharma (see [15],[16]) defined multivalued mappings in fuzzy metric spaces and studied common fixed point theorems for it. Sharma et al. [21] obtained common fixed point for multivalued mappings in intuitionistic fuzzy metric spaces. For more details, we refer to (see [2], [3],[4], [5], [6], [7], [10],[11], [12], [14], [17], [18], [19], [20], [22], [23], [24]). In this paper, we established a common fixed point theorem for multi-valued mappings in intuitionistic fuzzy metric space by relaxing the condition of continuity and replacing the completeness of the space with an alternative set of conditions.

**Definition 1.** [24] A binary operation \(\ast : [0,1] \times [0,1] \rightarrow [0,1]\) is continuous \(t\)-norm if \(\ast\) is satisfying the following conditions:

(i) \(\ast\) is commutative and associative,

(ii) \(\ast\) is continuous,

(iii) \(a \ast 1 = a, \forall a \in [0,1]\),

(iv) \(a \ast b \leq c \ast d\) whenever \(a \leq c\) and \(b \leq d, \forall a, b, c, d \in [0,1]\).

**Definition 2.** [24] A binary operation \(\odot : [0,1] \times [0,1] \rightarrow [0,1]\) is continuous \(t\)-norm if \(\odot\) is satisfying the following conditions:

(i) \(\odot\) is commutative and associative,

(ii) \(\odot\) is continuous,

(iii) \(a \odot 0 = a, \forall a \in [0,1]\),

(iv) \(a \odot b \leq c \odot d\) whenever \(a \leq c\) and \(b \leq d, \forall a, b, c, d \in [0,1]\).

**Definition 3.** [2] A 5-tuple \((X, M, N, \ast, \odot)\) is said to be an intuitionistic fuzzy metric space if \(X\) is an arbitrary set, \(\ast\) is a continuous \(t\)-norm, \(\odot\) is a continuous \(t\)-conorm and \(M, N\) are fuzzy sets on \(X^2 \times [0,\infty)\) satisfying the following conditions (\(\forall x, y, z \in X\) and \(t, s > 0\)):

(i) \(M(x, y, t) + N(x, y, t) \leq 1\),

(ii) \(M(x, y, 0) = 0\),

(iii) \(M(x, y, t) = 1, \forall t > 0\) if and only if \(x = y\),

(iv) \(M(x, y, t) = M(y, x, t)\),

(v) \(M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)\),

(vi) \(M(x, y, \cdot) : [0,\infty) \rightarrow [0,1]\) is left continuous,

(vii) \(\lim_{t \rightarrow \infty} M(x, y, t) = 1, \forall x, y \in X\),

(viii) \(N(x, y, 0) = 1\),
(ix) \( N(x, y, t) = 0, \forall t > 0 \) if and only if \( x = y \),
(x) \( N(x, y, t) = N(y, x, t) \),
(xi) \( N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s) \),
(xii) \( N(x, y, t) : [0, \infty) \rightarrow [0, 1] \) is right continuous.

Then \((M, N)\) is called an intuitionistic fuzzy metric on \( X \). The functions \( M(x, y, t) \) and \( N(x, y, t) \) denote the degree of nearness and the degree of non-nearness between \( x \) and \( y \) with respect to \( t \), respectively.

**Remark 4.** Every fuzzy metric space \((X, M, \ast)\) is an intuitionistic fuzzy metric space of the form \((X, M, 1 - M, \ast, \diamond)\) such that \( t \)-norm \( \ast \) and \( t \)-conorm \( \diamond \) are associated, i.e., \( x \diamond y = 1 - ((1 - x) \ast (1 - y)), \forall x, y \in X \).

**Example 5.** [18] Let \((X, d)\) be a metric space. Define \( t \)-norm \( a \ast b = \min\{a, b\} \) and \( t \)-conorm \( a \diamond b = \max\{a, b\} \) and \( \forall x, y \in X \) and \( t > 0 \),

\[
M(x, y, t) = \frac{t}{t + d(x, y)}, \quad N(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.
\]

Then \((X, M, N, \ast, \diamond)\) is an intuitionistic fuzzy metric space. We call this intuitionistic fuzzy metric \((M, N)\) induced by the metric \( d \), the standard intuitionistic fuzzy metric.

**Example 6.** [18] Let \( X = N \). Define \( a \ast b = \max\{0, a + b - 1\} \) and \( a \diamond b = a + b - ab, \forall a, b \in [0, 1] \) and let \( M \) and \( N \) be the fuzzy sets on \( X^2 \times (0, \infty) \) as follows:

\[
M(x, y, t) = \begin{cases} \frac{x}{y}, & \text{if } x \leq y, \\ \frac{y}{x}, & \text{if } y \leq x, \end{cases}
\]

and

\[
N(x, y, t) = \begin{cases} \frac{y-x}{y}, & \text{if } x \leq y, \\ \frac{x-y}{x}, & \text{if } y \leq x, \end{cases}
\]

\( \forall x, y \in X \) and \( t > 0 \). Then \((X, M, N, \ast, \diamond)\) is an intuitionistic fuzzy metric space.

**Remark 7.** Note that, in the above example, \( t \)-norm \( \ast \) and \( t \)-conorm \( \diamond \) are not associated and there exists no metric \( d \) on \( X \) satisfying

\[
M(x, y, t) = \frac{t}{t + d(x, y)}, \quad N(x, y, t) = \frac{d(x, y)}{t + d(x, y)},
\]

where \( M(x, y, t) \) and \( N(x, y, t) \) are as defined in above example. Also note the above functions \((M, N)\) is not an intuitionistic fuzzy metric with the \( t \)-norm \( \ast \) and \( t \)-conorm \( \diamond \) defined as \( a \ast b = \min\{a, b\} \) and \( a \diamond b = \max\{a, b\} \).
Lemma 8. [2] In an intuitionistic fuzzy metric space \( X, M(x, y, .) \) is non-decreasing and \( N(x, y, .) \) is non-increasing, \( \forall x, y \in X \).

Definition 9. [2] Let \( (X, M, N, *, \diamond) \) be an intuitionistic fuzzy metric space. Then

(i) A sequence \( \{x_n\} \) in \( X \) is said to be convergent to a point \( x \in X \) (denoted by \( \lim_{n \to \infty} x_n = x \)) if, \( \forall t > 0 \), \( \lim_{n \to \infty} M(x_n, x, t) = 1 \), \( \lim_{n \to \infty} N(x_n, x, t) = 0 \).

(ii) A sequence \( \{x_n\} \) in \( X \) is said to be a cauchy sequence if \( \forall t > 0 \) and \( p > 0 \), \( \lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1 \), \( \lim_{n \to \infty} N(x_{n+p}, x_n, t) = 0 \).

Remark 10. Since * and \( \diamond \) are continuous, the limit is uniquely determined from (v) and (xi), respectively.

Definition 11. [1] An intuitionistic fuzzy metric space \( (X, M, N, *, \diamond) \) is said to be complete if and only if every cauchy sequence in \( X \) is convergent. It is called compact if every sequence contains a convergent subsequence.

Lemma 12. [21] Let \( (X, M, N, *, \diamond) \) be an intuitionistic fuzzy metric space and \( \{y_n\} \) be sequence in \( X \). If there exists a number \( k \in (0, 1) \) such that:

(i) \( M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t) \),

(ii) \( N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t) \), \( \forall t > 0 \) and \( n = 1, 2, ... \), then \( \{y_n\} \) is a cauchy sequence in \( X \).

Proof. By simple induction with the condition (i) and with the help of Alaca et. al. [2], we have \( \forall t > 0 \) and \( n = 1, 2, ... \),

(iii) \( M(y_{n+1}, y_{n+2}, t) \geq M(y_1, y_2, \frac{t}{k^n}) \), \( N(y_{n+1}, y_{n+2}, t) \leq N(y_1, y_2, \frac{t}{k^n}) \)

Thus by (iii) and definition (3) [(v) and (xi)], for any positive integer \( p \) and real number \( t > 0 \), we have

\[
M(y_{n+p}, y_{n+1}, t) \geq M(y_n, y_{n+1}, \frac{t}{p}) \ast ... p - times... \ast M(y_{n+p-1}, y_{n+p}, \frac{t}{p})
\]

\[
\geq M(y_1, y_2, \frac{t}{pk^{n-p-1}}) \ast ... p - times... \ast M(y_1, y_2, \frac{t}{pk^{n-p-1}})
\]

and

\[
N(y_{n+p}, y_{n+1}, t) \leq N(y_n, y_{n+1}, \frac{t}{p}) \diamond ... p - times... \diamond N(y_{n+p-1}, y_{n+p}, \frac{t}{p})
\]

\[
\leq N(y_1, y_2, \frac{t}{pk^{n-p-1}}) \diamond ... p - times... \diamond N(y_1, y_2, \frac{t}{pk^{n-p-1}}).
\]

Therefore by Definition (3) [(vii) and (xiii)], we have

\[
\lim_{n \to \infty} M(y_n, y_{n+p}, t) \geq 1 \ast ... p - times... \ast 1 \geq 1
\]
and
\[ \lim_{n \to \infty} N(y_n, y_{n+p}, t) \leq 0 \odot p \text{ times} \odot 0 \leq 0, \]
which implies that \( \{y_n\} \) is a cauchy sequence in \( X \). This completes the proof. The following Lemma establishes a relationship between \( x \) and \( y \) by virtue of Kubiaczyk and Sharma [15].

**Lemma 13.** [21] Let \((X, M, N, *, \odot)\) be an intuitionistic fuzzy metric space and \( \forall x, y \in X, t > 0 \) and if for a number \( k \in (0, 1) \), \( M(x, y, kt) \geq M(x, y, t) \) and \( N(x, y, kt) \leq N(x, y, t) \), then \( x = y \)

**Proof.** Since \( M(x, y, kt) \geq M(x, y, t) \) and \( N(x, y, t) \leq N(x, y, t) \), then using results of Kubiaczyk and Sharma[15], we have
\[ M(x, y, t) \geq M(x, y, \frac{t}{k^n}) \]

By repeated application of above inequalities , we have
\[ M(x, y, t) \geq M(x, y, \frac{t}{k^n}) \geq \ldots \geq M(x, y, \frac{t}{k^{2n}}) \geq \ldots, \]
which tend to 1 and 0, respectively as \( n \to \infty \). Thus \( M(x, y, t) = 1 \) and \( N(x, y, t) = 0 \) for all \( t > 0 \) and we get \( x = y \). \( \Box \)

**Definition 14.** [1] Let \( A \) and \( B \) be maps from an intuitionistic fuzzy metric space \((X, M, N, *, \odot)\) into itself. The maps \( A \) and \( B \) are said to be compatible if \( \forall t \geq 0, \lim_{n \to \infty} M(ABx_n, BAx_n, t) = 1 \) and \( \lim_{n \to \infty} N(ABx_n, BAx_n, t) = 0 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = z \) for some \( z \in X \).

**Definition 15.** [15] Two self maps \( A \) and \( B \) on a set \( X \) are said to be weakly compatible if they commute at coincidence points; i.e., if \( Au = Bu \) for some \( u \in X \), then \( ABu = BAu \).

**Definition 16.** [21] Let \((X, M, N, *, \odot)\) be an intuitionistic fuzzy metric space and consider \( I : X \to X \) and \( T : X \to CB(X) \). A point \( z \in X \) is called a coincidence point of \( I \) and \( T \) if and only if \( Iz \in Tz \). Denote \( CB(X) \), the set of all non-empty bounded and closed subsets of \( X \). We have \( M^\triangledown(B, y, t) = \max \{M(b, y, t); b \in B\} \),
\[ N^\Delta(B, y, t) = \min\{N(b, y, t); b \in B\} \]
\[ M_\nabla(A, B, t) \geq \min\{\min M_\nabla(a, B, t); a \in A, \min M_\nabla(A, b, t); b \in B\} \]
\[ N_\Delta(A, B, t) \leq \max\{\max N^\Delta(a, B, t); a \in A, \min N^\Delta A, b, t); b \in B\} \]
for all \( A, B \in X \) and \( t > 0 \).

**Remark 17.**  
(i) In [11],[12] and [13] we can find the equivalent formulations of definitions of compatible maps, compatible maps of type \((\alpha)\) and compatible maps of type \((\beta)\). Such maps are independent of each other and more general than commuting and weakly commuting maps ([10], [20]).

(ii) Compatible or compatible of type \((\alpha)\) or compatible of type \((\beta)\) maps are weakly compatible but converse need not true.

Alaca, Turkoglu and Yildiz [2] established the following.

**Theorem 18.** (Intuitionistic fuzzy Banach contraction theorem). Let \((X, M, N, \ast, \odot)\) be a complete intuitionistic fuzzy metric space. Let \(T : X \to X\) be a mapping satisfying

\[ M(Tx, Ty, kt) \geq M(x, y, t) \quad \text{and} \quad N(Tx, Ty, kt) \leq N(x, y, t), \]

for all \( x, y \in X, 0 < k < 1 \). Then \( T \) has a unique fixed point.

**Theorem 19.** (Intuitionistic fuzzy Edelstein contraction theorem). Let \((X, M, N, \ast, \odot)\) be a compact space. Let \(T : X \to X\) be a mapping satisfying

\[ M(Tx, Ty, .) > M(x, y, .) \quad \text{and} \quad N(Tx, Ty, .) < N(x, y, .) \]

for all \( x \neq y \), i.e. \( M(Tx, Ty, .) \geq M(x, y, .) \) and \( M(Tx, Ty, .) \neq M(x, y, .) \) and \( N(Tx, Ty, .) \leq N(x, y, .) \) and \( N(Tx, Ty, .) \neq N(x, y, .) \), for all \( x \neq y \). Then \( T \) has a unique fixed point.

Kubiaczky and Sharma [15] proved the following for fuzzy metric space.

**Theorem 20.** Let \((X, M, .)\) be a complete fuzzy metric space with \( t \ast t \geq t \) for all \( t \in [0, 1] \). Let \( P, Q : X \to CB(X) \) be continuous and there exists mappings \( S, T : X \to X \) satisfying:

(i) \( SP = PS, QT = TQ \),

(ii) \( P(X) \subseteq S(X), Q(X) \subseteq T(X) \),

(iii) there exists a number \( q \in (0, 1) \) such that

\[ M_\nabla(Px, Qy, qt) \geq \min\{M_\nabla(Sx, Tx, t), M_\nabla(Px, Sx, t), M_\nabla(Qy, Ty, t), M_\nabla(Px, Ty, (2 - \alpha)t), M_\nabla(Qy, Sx, t)\}, \]
for all \( x, y \in X, \alpha \in (0, 2), t > 0 \). Then \( P, Q, S \) and \( T \) have a common coincidence point, i.e. \( S z \in P z \) and \( T z \in Q z \).

Sharma, Servet and Rathore [21] proved the following results for intuitionistic fuzzy metric spaces.

**Theorem 21.** Let \((X, M, N, *, \diamond)\) be a complete intuitionistic fuzzy metric space with continuous \( t \)–norm \(* \) and continuous \( t \)–conorm \( \diamond \) defined by \( t * t \geq t \) and \( (1 - t) \diamond (1 - t) \leq (1 - t) \), for all \( t \in [0, 1] \). Let \( P : X \to C(X) \), such that:

(i) \( M(Px, Py, kt) \geq M(x, y, t) \)

and

(ii) \( N(Px, Py, kt) \leq N(x, y, t) \), for all \( x, y \in X \) and \( 0 < k < 1 \). Then \( P \) has a fixed point. This means that there exists a point \( u \) such that \( u \in P u \).

**Theorem 22.** Let \((X, M, N, *, \diamond)\) be a complete intuitionistic fuzzy metric space with continuous \( t \)–norm \(* \) and continuous \( t \)–conorm \( \diamond \) defined by \( t * t \geq t \) and \( (1 - t) \diamond (1 - t) \leq (1 - t) \), for all \( t \in [0, 1] \). Let \( T_n : X \to CB(X)(n \in N) \) and continuous mapping \( I : X \to X \) be such that \( T_n(X) \subset I(X) \) where \( I \) commute with \( T_n \) for every \( n \in N \) and

(i) there exists \( q \in (0, 1) \), such that

\[
M_\gamma(T_i x, T_j y, qt) \geq \min\{M(I x, I y, t), M(\gamma(I x, T_i x, t), M(\gamma(I y, T_j y, t), \]
\[
M(\gamma(I x, T_j y, (2 - \alpha)t), M(\gamma(I y, T_i x, t)\}
\]

and

\[
N_\Delta(T_i x, T_j y, qt) \leq \max\{N(I x, I y, t), N(\Delta(I x, T_i x, t), N(\Delta(I y, T_j y, t), \]
\[
N(\Delta(I x, T_j y, (2 - \alpha)t), N(\Delta(I y, T_i x, t)\}
\]

for all \( x, y \in X, \alpha \in (0, 2) \) and \( t > 0 \) for every \( i, j \in N(i \neq j) \). Then there exists a common coincidence point of \( T_n \) and \( I \), i.e. there exists a point \( z \in X \) such that \( I z \in \cap T_n z \), \( n \in N \).

2. Main Results

We improve Theorem (22) by dropping the condition of continuity and replacing the completeness of the space with a set of an alternative conditions.

**Theorem 23.** Let \((X, M, N, *, \diamond)\) be an intuitionistic fuzzy metric space with continuous \( t \)–norm \(* \) and continuous \( t \)–conorm \( \diamond \) defined by \( t * t \geq t \) and \( (1 - t) \diamond (1 - t) \leq (1 - t) \), for all \( t \in [0, 1] \). Let \( T_n : X \to CB(X)(n \in N) \)
and mapping $I : X \to X$ be such that $T_n(X) \subset I(X)$, where the pair \{I, T_n\} is weakly compatible for every $n \in N$ and for every $i, j \in N \ (i \neq j)$ there exists $q \in (0, 1)$, such that

$$M_\gamma(T_ix, T_jy, qt) \geq \min\{M(Ix, Iy, t), M_\gamma(Ix, T_ix, t), M_\gamma(Iy, T_jy, t), M_\gamma(Ix, T_jy, (2 - \alpha)t), M_\gamma(Iy, T_ix, t)\} \tag{1}$$

and

$$N_\Delta(T_ix, T_jy, qt) \leq \max\{N(Ix, Iy, t), N_\Delta(Ix, T_ix, t), N_\Delta(Iy, T_jy, t), N_\Delta(Ix, T_jy, (2 - \alpha)t), N_\Delta(Iy, T_ix, t)\}, \tag{2}$$

for all $x, y \in X, \alpha \in (0, 2)$ and $t > 0$ for every $i, j \in N (i \neq j)$.

$I(X)$ or $T_n(X)$ is a complete subspace of $X$. \tag{3}

Then there exists a common coincidence point of $T_n$ and $I$, i.e. $\exists z \in X$ such that $Iz \in \cap T_nz, \ n \in N$.

**Proof.** Let $x_0 \in X$ and $x_1 \in X$ such that $Ix_1 \in T_1x_0$ and $y_1 = Ix_1, q \in (0, 1)$ and the inequalities hold

$$M(x_0, y_1, qt) = M(x_0, Ix_1, qt) \geq M_\gamma(x_0, T_1x_0, qt) - \epsilon/2,$$

$$N(x_0, y_1, qt) = N(x_0, Ix_1, qt) \leq N_\Delta(x_0, T_1x_0, qt) + \epsilon/2, \ n \in N$$

$x_2 \in X$ such that $Ix_2 \in T_2x_1$ and $y_2 = Ix_2$,

$$M(y_1, y_2, qt) = M(Ix_1, Ix_2, qt) \geq M_\gamma(y_1, T_2x_1, qt) - \epsilon/2^2,$$

$$N(y_1, y_2, qt) = N(Ix_1, Ix_2, qt) \leq N_\Delta(y_1, T_2x_1, qt) + \epsilon/2^2.$$

Inductively, we can construct a sequence \{yn\} in $X$ such that

$$M(y_n, y_{n+1}, qt) = M(Ix_n, Ix_{n+1}, qt) \geq M_\gamma(y_n, T_{n+1}x_n, qt) - \epsilon/2^n,$$

$$N(y_n, y_{n+1}, qt) = N(Ix_n, Ix_{n+1}, qt) \leq N_\Delta(y_n, T_{n+1}x_n, qt) + \epsilon/2^n.$$

Now, we show that \{yn\} is a Cauchy sequence. By (1) for all $t > 0$ and $\alpha = 1 - k$ with $k \in (0, 1)$, we write

$$M(y_n, y_{n+1}, qt) \geq M_\gamma(y_n, T_{n+1}x_n, qt) - \epsilon/2^n \geq M_\gamma(T_nx_{n-1}, T_{n+1}x_n, qt) - \epsilon/2^n \geq$$

$$\min\{M(Ix_{n-1}, Ix_n, t), M_\gamma(Ix_{n-1}, T_nx_{n-1}, t), M_\gamma(Ix_n, T_{n+1}x_n, t), M_\gamma(Ix_{n-1}, T_nx_{n-1}, t), M_\gamma(Ix_n, T_{n+1}x_n, t)\} - \epsilon/2^n,$$

$$\geq \min\{M(Ix_{n-1}, Ix_n, t), M(Ix_{n-1}, Ix_n, t), M(Ix_n, Ix_{n-1}, t), M(Ix_{n-1}, Ix_n, t)\} - \epsilon/2^n,$$

and by (2), we have

$$N(y_n, y_{n+1}, qt) \leq N_\Delta(y_n, T_{n+1}x_n, qt) + \epsilon/2^n \leq N_\Delta(T_nx_{n-1}, T_{n+1}x_n, qt) + \epsilon/2^n.$$

$$\leq \max\{N(Ix_{n-1}, Ix_n, t), N_\Delta(Ix_{n-1}, T_nx_{n-1}, t), N_\Delta(Ix_n, T_{n+1}x_n, t), N_\Delta(T_nx_{n-1}, T_{n+1}x_n, qt) + \epsilon/2^n.$$


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\[ N^\Delta (Ix_{n-1}, Tn+1x_n, (2 - \alpha)t), N^\Delta (Ix_n, Tnx_{n-1}, t) \} + \epsilon/2^n, \]
\[ \leq \max \{ N(Ix_{n-1}, Ix_n, t), N(Ix_{n-1}, Ix_n, t), N(Ix_n, Ix_{n+1}, t), N(Ix_{n-1}, Ix_{n+1}, (1 + k)t), N(Ix_n, Ix_{n+1}, t) \} + \epsilon/2^n. \]

Now, using Definition 3 [(v) and (xi)], we have
\[ M(y_n, y_{n+1}, qt) \geq \min \{ M(y_{n-1}, y_n, t), M(y_{n-1}, y_n, t), M(y_n, y_{n+1}, t), M(y_{n-1}, y_n, t), M(y_n, y_{n+1}, t) \} - \epsilon/2^n \]
and
\[ N(y_n, y_{n+1}, qt) \leq \max \{ N(y_{n-1}, y_n, t), N(y_{n-1}, y_n, t), N(y_n, y_{n+1}, t), N(y_{n-1}, y_n, t), N(y_n, y_{n+1}, t) \} + \epsilon/2^n. \]

Since \( t \)-norm \( \ast, t \)-conorm \( \diamond, M(x, y, .) \) and \( N(x, y, .) \) are continuous, letting \( k \to 1 \) in (4) and (5), we have
\[ M(y_n, y_{n+1}, qt) \geq \min \{ M(y_{n-1}, y_n, t), M(y_n, y_{n+1}, t) \} - \epsilon/2^n, \]
\[ N(y_n, y_{n+1}, qt) \leq \max \{ N(y_{n-1}, y_n, t), N(y_n, y_{n+1}, t) \} + \epsilon/2^n, \]
for \( n = 1, 2, \ldots \) and so on, for positive integers \( n \) and \( p \) and \( \epsilon \in (0, 1) \), we have
\[ M(y_n, y_{n+1}, qt) \geq \min \{ M(y_{n-1}, y_n, t), M(y_n, y_{n+1}, t/q^p) \} - \epsilon/2^n, \]
\[ N(y_n, y_{n+1}, qt) \leq \max \{ N(y_{n-1}, y_n, t), N(y_n, y_{n+1}, t/q^p) \} + \epsilon/2^n. \]

Since \( \epsilon \) is arbitrary, making \( \epsilon \to 0, M(y_n, y_{n+1}, t/q^p) \to 1 \) and \( N(y_n, y_{n+1}, t/q^p) \to 0 \) as \( n \to \infty \), we obtain
\[ M(y_n, y_{n+1}, qt) \geq M(y_{n-1}, y_n, t) \quad \text{and} \quad N(y_n, y_{n+1}, qt) \leq N(y_{n-1}, y_n, t) \]
By Lemma (12), \( \{ y_n \} \) is a cauchy sequence.

Since \( I(X) \) is complete. Note that the subsequence \( \{ y_{n+1} \} \) is contained in \( I(X) \) and has a limit \( z \) in \( I(X) \). Then \( Ip = z \). By (1), we have for \( \alpha = 1, \)
\[ M^\nabla (Tn+1p, Ix_{n+1}, qt) - \epsilon/2^n = M^\nabla (Tn+1p, Tn+1x_n, qt) - \epsilon/2^n, \]
\[ \geq \min \{ M(Ip, Ix_n, t), M^\nabla (Ip, Tn+1p, t), M^\nabla (Ix_{n+1}, Tn+1x_n, t), M^\nabla (Ip, Tn+1x_n, t), \]
\[ M^\nabla (Ix_{n+1}, Tn+1x_n, t) \} - \epsilon/2^n, \]
\[ \geq \min \{ M(Ip, Ix_n, t), M^\nabla (Ip, Tn+1p, t), M(Ix_n, Ix_{n+1}, t), M(Ip, Ix_{n+1}, t), \]
\[ M^\nabla (Ix_{n+1}, Tn+1x_n, t) \} - \epsilon/2^n, \]
\[ \geq \min \{ M(z, Ix_{n+1}, t), M^\nabla (z, Tn+1p, t), M(Ix_n, Ix_{n+1}, t), M(z, Ix_{n+1}, t), \]
\[ M^\nabla (Ix_{n+1}, Tn+1x_n, t) \} - \epsilon/2^n, \]
Taking the limit as \( n \to \infty \), we obtain
\[ \geq \min \{ M(z, z, t), M^\nabla (z, Tn+1p, t), M(z, z, t), M(z, z, t), M^\nabla (z, Tn+1p, t) \} - \epsilon/2^n. \]
This gives \( M^\nabla (Tn+1p, z, qt) \geq M^\nabla (Tn+1p, z, t) \).

Similarly by (2), we have for \( \alpha = 1, \)
\[ N^\Delta (Tn+1p, Ix_{n+1}, qt) + \epsilon/2^n \leq N^\Delta (Tn+1p, Tn+1x_n, qt) + \epsilon/2^n \]
\[ \leq \max \{ N(Ip, Ix_n, t), N^\Delta (Ip, Tn+1p, t), N(Ix_n, Ix_{n+1}, t), N(Ip, Ix_{n+1}, t), \]
\[ N(Ix_{n+1}, Tn+1p, t) \} - \epsilon/2^n, \]
\[ \leq \max \{ N(Ip, Ix_n, t), N^\Delta (Ip, Tn+1p, t), N(Ix_n, Ix_{n+1}, t), N(Ip, Ix_{n+1}, t), \]
\[ N(Ix_{n+1}, Tn+1p, t) \} + \epsilon/2^n, \]
\[ \leq \max \{ N(Ip, Ix_n, t), N^\Delta (Ip, Tn+1p, t), N(Ix_n, Ix_{n+1}, t), N(Ip, Ix_{n+1}, t), \]
\[ N(Ix_{n+1}, Tn+1p, t) \} + \epsilon/2^n. \]
\(N^\Delta(Ix_n, T_n p, t) + \epsilon/2^n,\)
\(\leq \max\{N(z, Ix_n, t), N^\Delta(z, T_n p, t), N(Ix_n, Ix_{n+1}, t), N(z, Ix_{n+1}, t),\)
\(N^\Delta(Ix_n, T_n p, t)\} + \epsilon/2^n.\)
Taking the limit as \(n \to \infty\), we obtain
\(\leq \max\{N(z, z, t), N^\Delta(z, T_n p, t), N(z, z, t), N(z, z, t), N^\Delta(z, T_n p, t)\} + \epsilon/2^n.\)
This gives \(N^\Delta(T_n p, z, qt) \leq N^\Delta(T_n p, z, t)\). Therefore by Lemma (13), \(z \in T_n p.\)
Since \(Ip = z \in T_p\) i.e. \(p\) is a coincidence point of \(I\) and \(T_n.\)
Since the pair \(\{I, T_n\}\) is weakly compatible, therefore, \(I\) and \(T_n\) commute at coincidence point for every \(n \in N\) i.e. \(IT_n p = T_n Ip\) or \(Iz \in T_n z.\) Now, we prove \(T_n z = z.\) By (1), we have for \(\alpha = 1\)
\(M^\nabla(T_n z, Ix_{n+1}, qt) - \epsilon/2^n = M^\nabla(T_n z, T_n+1 x_n, qt) - \epsilon/2^n\)
\(\geq \min\{M(Iz, Ix_n, t), M^\nabla(Iz, T_n z, t), M^\nabla(Ix_n, T_n+1 x_n, t), M^\nabla(Iz, T_n+1 x_n, t), M^\nabla(Ix_n, T_n z, t)\} - \epsilon/2^n,\)
\(\geq \min\{M(Iz, Ix_n, t), M^\nabla(Iz, T_n z, t), M(Ix_n, Ix_{n+1}, t), M(Iz, Ix_{n+1}, t), M^\nabla(Ix_n, T_n z, t)\} - \epsilon/2^n.\)
Taking the limit as \(n \to \infty\), we obtain
\(\geq \min\{M(z, z, t), M^\nabla(z, T_n z, t), M(z, z, t), M(z, z, t), M^\nabla(z, T_n z, t)\} - \epsilon/2^n.\)
This gives \(M^\nabla(T_n z, z, qt) \geq M^\nabla(z, T_n z, t).\)
Similarly by (2), we have for \(\alpha = 1,\)
\(N^\Delta(T_n z, Ix_{n+1}, qt) + \epsilon/2^n = N^\Delta(T_n z, T_n+1 x_n, qt) + \epsilon/2^n\)
\(\leq \max\{N(Iz, Ix_n, t), N^\Delta(Iz, T_n z, t), N^\Delta(Ix_n, T_n+1 x_n, t), N^\Delta(Iz, T_n+1 x_n, t), N^\Delta(Ix_n, T_n z, t)\} + \epsilon/2^n,\)
\(\leq \max\{N(Iz, Ix_n, t), N^\Delta(Iz, T_n z, t), N(Ix_n, Ix_{n+1}, t), N(Iz, Ix_{n+1}, t), N^\Delta(Ix_n, T_n z, t)\} + \epsilon/2^n,\)
\(\leq \max\{N(z, Ix_n, t), N^\Delta(z, T_n z, t), N(Ix_n, Ix_{n+1}, t), N(z, Ix_{n+1}, t), N^\Delta(Ix_n, T_n z, t)\} + \epsilon/2^n.\)
Taking the limit as \(n \to \infty\), we obtain
\(\leq \max\{N(z, z, t), N^\Delta(z, T_n z, t), N(z, z, t), N(z, z, t), N^\Delta(z, T_n z, t)\} + \epsilon/2^n.\)
This gives \(N^\Delta(T_n z, z, qt) \leq N^\Delta(T_n z, z, t).\) Therefore by Lemma (13), we have \(z \in T_n z.\) Now, we prove \(Iz = z.\) Using the fact that \(Iz \in T_n z,\) by (1), with \(\alpha = 1,\) we have
\(M^\nabla(T_n z, Ix_{n+1}, qt) - \epsilon/2^n = M^\nabla(T_n z, T_n+1 x_n, qt) - \epsilon/2^n\)
\(\geq \min\{M(Iz, Ix_n, t), M^\nabla(Iz, T_n z, t), M^\nabla(Ix_n, T_n+1 x_n, t), M^\nabla(Iz, T_n+1 x_n, t), M^\nabla(Ix_n, T_n z, t)\} - \epsilon/2^n,\)
\(\geq \min\{M(Iz, Ix_n, t), M(Iz, Ix_n, t), M(Ix_n, Ix_{n+1}, t), M(Iz, Ix_{n+1}, t), M(Ix_n, Ix_{n+1}, t)\}, M(Ix_n, Ix_{n+1}, t)\} - \epsilon/2^n,\)
Taking the limit as \(n \to \infty\), we obtain
\(M^\nabla(Iz, z, qt) - \epsilon/2^n\)
\(\geq \min\{M(Iz, z, t), M(Iz, Ix_n, t), M(z, z, t), M(Iz, Ix_n, t), M(Iz, Ix_n, t)\} - \epsilon/2^n.\)
This gives \( M(Iz, z, qt) \geq M(Iz, z, t) \).

Using the fact that \( Iz \in T_n z \), by (2), with \( \alpha = 1 \), we have
\[
N^\Delta(T_n z, Ix_{n+1}, qt) + \epsilon/2^n \leq N^\Delta(T_n z, T_{n+1} x_n, qt) + \epsilon/2^n
\]
\[
\leq \max\{N(Iz, Ix_n, t), N^\Delta(Iz, T_n z, t), N^\Delta(Ix_n, T_{n+1} x_n, t), N^\Delta(Iz, T_{n+1} x_n, t),
N^\Delta(Ix_n, T_n z, t)\} + \epsilon/2^n,
\]
\[
\leq \max\{N(Iz, Ix_n, t), N(Iz, Ix_n, t), N(Ix_n, Ix_{n+1}, t), N(Iz, Ix_{n+1}, t),
N(Ix_n, Ix_n, t)\} + \epsilon/2^n,
\]
Taking the limit as \( n \to \infty \), we obtain
\[
\leq \max\{N(Iz, z, t), N(Iz, Ix_n, t), N(z, z, t), N(Iz, Ix_n, t), N(Iz, Ix_n, t)\} + \epsilon/2^n.
\]
This gives \( N(z, Ix_n, qt) \leq N(Iz, z, t) \).

Therefore by Lemma (13), we have \( Iz = z \). By weak compatibility, since \( Iz \in T_n z \). Thus \( z = Iz \in \bigcap T_n z, n \in N \).

This completes the proof of the theorem. \( \square \)

Condition (3) of the theorem (23) can be removed by taking the space complete. We establish the following.

**Theorem 24.** Let \((X, M, N, *, \diamond)\) be a complete intuitionistic fuzzy metric space with continuous \( t^-\) norm \(*\) and continuous \( t^-\) conorm \( \diamond \) defined by \( t* t \geq t \) and \((1 - t) \diamond (1 - t) \leq (1 - t)\), for all \( t \in [0, 1] \). Let \( T_n : X \to CB(X)(n \in N) \) and mapping \( I : X \to X \) be such that \( T_n(X) \subset I(X) \) where the pair \( \{I, T_n\} \) is weakly compatible for every \( n \in N \) and for every \( i, j \in N \) (\( i \neq j \)) satisfying conditions (1) and (2) of Theorem (23). Then there exists a common coincidence point of \( T_n \) and \( I \), i.e. there exists a point \( z \) in \( X \) such that \( z = Iz \in \bigcap T_n z, n \in N \).

**Proof.** Theorem 24 can be proved in the similer manner as Theorem 23. \( \square \)

3. Acknowledgments

The authors would like to offer their sincere gratitude and thanks to the editors and anonyms referees for their valuable comments/suggestions.

**References**


