International Journal of Pure and Applied Mathematics
Volume 120 No. 3 2018, 461-469
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v120i3.14

SOME COMMENTS ON THE WEIBULL–R FAMILY WITH BASELINE PARETO AND LOMAX CUMULATIVE SIGMOIDS

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Abstract: In this paper we study the one–sided Hausdorff approximation of the shifted Heaviside step function by some classes of Weibull–R Family with baseline Pareto and Lomax cumulative sigmoids. The estimates of the value of the best Hausdorff approximation obtained in this article can be used in practice as one possible additional criterion in "saturation" study.

Numerical examples, illustrating our results are presented using programming environment *CAS Mathematica*.

AMS Subject Classification: 68N30, 41A46
Key Words: Weibull–G family of cumulative distribution, Weibull–R family with baseline Pareto (cdf), Weibull–R family with baseline Lomax (cdf), Heaviside function, Hausdorff approximation, upper and lower bounds

1. Introduction

The Weibull distribution has been widely used in survival and reliability analyses.

Some modifications, properties and applications of Weibull and Weibull–R families of distributions can be found in [1]–[9].

In this paper we analyze some examples of the Weibull–R family with baseline Pareto and Lomax cumulative sigmoids.

Received: April 2, 2018
Revised: December 11, 2018
Published: January 14, 2018

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The Weibull–R family with baseline Pareto cumulative sigmoid is given by [2], [5]:

$$M(t) = 1 - e^{-\left(\beta \ln\left(\frac{t}{\theta}\right)\right)^c}, \tag{1}$$

where $t > \theta > 0$, $c > 0$, $\beta > 0$.

The Weibull–R family with baseline Lomax cumulative sigmoid is given by [4]:

$$M_1(t) = 1 - e^{-\left(\beta \ln\left(1 + \frac{t}{\theta}\right)\right)^c}, \tag{2}$$

where $t > \theta > 0$, $c > 0$, $\beta > 0$.

Definition 1. The *shifted Heaviside step function* is defined by

$$h_{t_0}(t) = \begin{cases} 0, & \text{if} \quad t < t_0, \\ [0,1], & \text{if} \quad t = t_0, \\ 1, & \text{if} \quad t > t_0 \end{cases}$$

Definition 2. The Hausdorff distance [10] (the H–distance) $\rho(f,g)$ between two interval functions f,g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,

$$\rho(f,g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B||\},$$

wherein $||.||$ is any norm in $\mathbb{R}^2$, e. g. the maximum norm $||(t,x)|| = \max\{|t|,|x|\}$; hence the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in $\mathbb{R}^2$ is $||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)$.

In this note we study the Hausdorff approximation of the *shifted Heaviside step function* by the families (1) and (2).

2. Main Results

2.1. Approximation of the $h_{t_0}(t)$ by Weibull–R–Pareto c.d.f.

We consider the following class of this family with application to the population dynamics and debugging theory:

$$M(t) = 1 - e^{-\left(\beta \ln\left(\frac{t}{\theta}\right)\right)^c}, \tag{3}$$

with

$$t_0 = \theta e^{\frac{1}{\beta}(\ln 2)^{\frac{1}{c}}}; \quad M(t_0) = \frac{1}{2}. \tag{4}$$

The one–sided Hausdorff distance d between the function $h_{t_0}(t)$ and the sigmoid - ((3)–(4)) satisfies the relation

$$M(t_0 + d) = 1 - d. \tag{5}$$

The following theorem gives upper and lower bounds for d

Theorem 1. Let

$$\begin{aligned} p &= -\tfrac{1}{2}, \\ q &= 1 + \frac{c\beta}{2\theta} \frac{(\ln 2)^{\frac{c-1}{c}}}{e^{\frac{1}{\beta}(\ln 2)^{\frac{1}{c}}}}, \\ r &= 2.1q. \end{aligned} \tag{6}$$

For the one–sided Hausdorff distance d between $h_{t_0}(t)$ and the sigmoid ((3)–(4)) the following inequalities hold for $q > \frac{e^{1.05}}{2.1}$:

$$d_l = \frac{1}{r} < d < \frac{\ln r}{r} = d_r. \tag{7}$$

Proof. Let us examine the function:

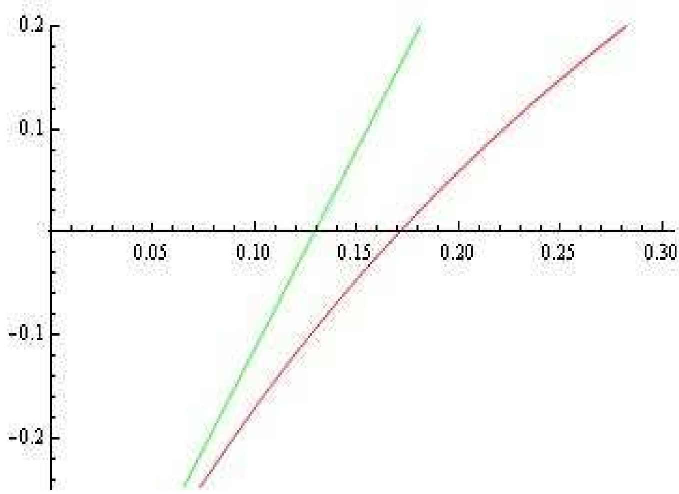

Figure 1: The functions $F(d)$ and $G(d)$ for $\beta = 1.5$; $c = 2$; $\theta = 0.25$.

$$F(d) = M(t_0 + d) - 1 + d. \tag{8}$$

From $F'(d) > 0$ we conclude that function F is increasing.

Consider the function

$$G(d) = p + qd. \tag{9}$$

From Taylor expansion we obtain $G(d) - F(d) = O(d^2)$.

Hence $G(d)$ approximates $F(d)$ with $d \to 0$ as $O(d^2)$ (see Fig. 1).
In addition $G'(d) > 0$.

Further, for $q > \dfrac{e^{1.05}}{2.1}$ we have $G(d_l) < 0$ and $G(d_r) > 0$.

This completes the proof of the theorem.

2.2. Numerical examples

The model ((3)–(4)) for $\beta = 1.5$; $c = 2$; $\theta = 0.25$, $t_0 = 0.435501$ is visualized on Fig. 2.

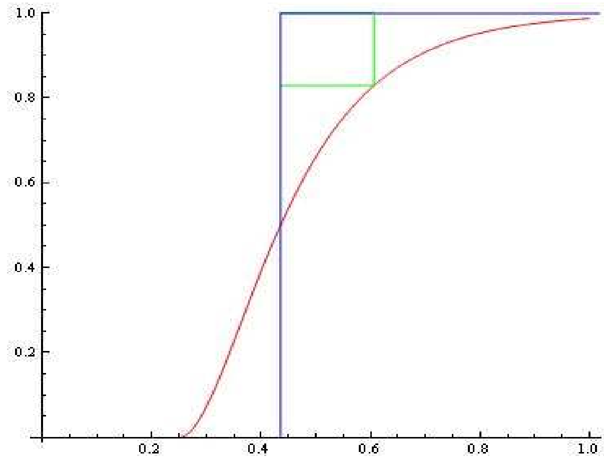

Figure 2: The model ((3)–(4)) for $\beta = 1.5$; $c = 2$; $\theta = 0.25$, $t_0 = 0.435501$; H–distance $d = 0.170909$, $d_l = 0.123124$, $d_r = 0.257891$.

From the nonlinear equation (5) and inequalities (7) we have: $d = 0.170909$, $d_l = 0.123124$, $d_r = 0.257891$.

The model ((3)–(4)) for $\beta = 1.2$; $c = 8$; $\theta = 0.15$, $t_0 = 0.332504$ is visualized on Fig. 3.

From the nonlinear equation (5) and inequalities (7) we have: $d = 0.0558694$, $d_l = 0.041497$, $d_r = 0.132049$.

The model ((3)–(4)) for $\beta = 1$; $c = 10$; $\theta = 0.1$, $t_0 = 0.26222$ is visualized on Fig. 4.

From the nonlinear equation (5) and inequalities (7) we have: $d = 0.0443688$, $d_l = 0.0323712$, $d_r = 0.111049$.

2.3. Approximation of the $h_{t_0}(t)$ by Weibull–R–Lomax c.d.f.

We consider the following class of this family:

$$M_1(t) = 1 - e^{-\left(\beta \ln\left(1 + \frac{t}{\theta}\right)\right)^c}, , \tag{10}$$

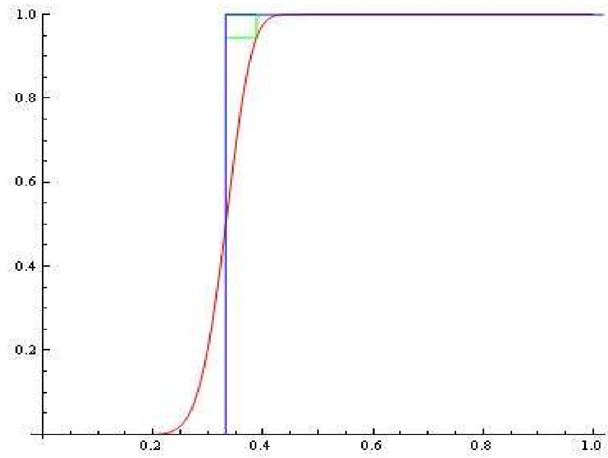

Figure 3: The model ((3)–(4)) for $\beta = 1.2$; $c = 8$; $\theta = 0.15$, $t_0 = 0.332504$; H–distance $d = 0.0558694$, $d_l = 0.041497$, $d_r = 0.132049$.

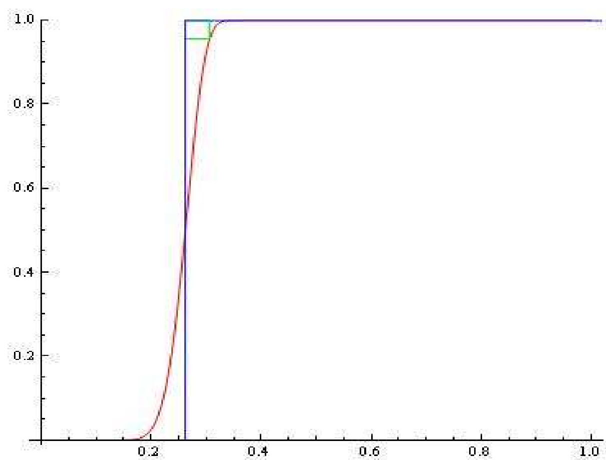

Figure 4: The model ((3)–(4)) for $\beta = 1$; $c = 10$; $\theta = 0.1$, $t_0 = 0.26222$; H–distance $d = 0.0443688$, $d_l = 0.0323712$, $d_r = 0.111049$.

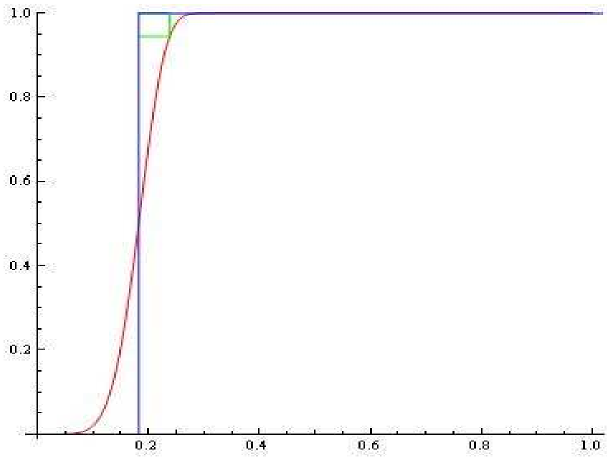

Figure 5: The model ((10)–(11)) for $\beta = 1.2$; $c = 8$; $\theta = 0.15$, $t_0^* = 0.182504$; H–distance $d = 0.0558694$.

with

$$t_0^* = \theta \left(e^{\frac{1}{\beta}(\ln 2)^{\frac{1}{c}}} - 1 \right); \qquad M_1(t_0^*) = \frac{1}{2}. \tag{11}$$

We note that for the one-sided Hausdorff approximation of the Heaviside function and the Weibull-R-Lomax cumulative sigmoid the two-sided estimations from Theorem 1 are satisfied.

Note that the Weibull-R-Lomax cdf is only a shift by θ of the W-R-Pareto cdf (see Fig. 5 and Fig. 3).

From the above examples, it can be seen that the proven estimates (see Theorem) for the value of the Hausdorff approximation is reliable when assessing the important characteristic - "saturation".

For some approximation, computational and modelling aspects, see [11]–[24].

Some software reliability models, can be found in [25]–[27].

Remark.

The Weibull–R family with baseline Cauchy cumulative sigmoid is given by [4]:

$$M_2(t) = 1 - e^{-\left(-\frac{\ln\left(\frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{t}{\theta}\right)\right)}{\gamma}\right)^c}, \tag{12}$$

where $-\infty < t < infty$, $\theta > 0$, $c > 0$, $\gamma > 0$.

Based on the methodology proposed in the present note, the reader may formulate the corresponding approximation problems for the general model $M_2(t)$ on his/her own.

Acknowledgments

This work has been supported by D01-205/23.11.2018 National Scientific Program "Information and Communication Technologies for a Single Digital Market in Science, Education and Security (ICTinSES)", financed by the Ministry of Education and Science, Bulgaria.

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