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CYCLIC MODULE AMENABILITY OF BANACH ALGEBRAS

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Abstract: In this paper, we define the concept of cyclic module amenability for Banach algebras and we study the hereditary properties of cyclic module amenability on Banach algebras. For example, we investigate relationship between cyclic module amenability of I, A/I and A, where I is closed ideal and $\mathfrak{A}$-submodule of A. Also it is shown that cyclic module amenability of A and B follows from cyclic module amenability of $A \oplus_{\ell^1} B$ and cyclic module amenability of A and B implies cyclic module amenability of $A \oplus_{\ell^1} B$, if A and B are essential.

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1. Introduction

The concept of an amenable Banach algebra was defined and studied for the first time by Johnson in [7]. The concept of cyclic amenability was presented by Grønbæk in [6]. He investigated the hereditary properties of this concept, found some relations between cyclic amenability of a Banach algebra and the trace extension property of its ideals. Also he showed that Banach algebra $\mathcal{F}$, the free product of cyclic amenable Banach algebras A and B, is cyclic amenable, while this sentence is not true for weak amenability. After that, others authors rarely investigated the cyclic amenability of Banach algebras. Of course, Shojaee and

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Bodaghi generalized it in [10] to follow the results of Ghahramani and Loy [5]. Also the second author has studied cyclic amenability of triangular Banach algebras in [8].

On the other hand, The concept of module amenability for a class of Banach algebras which is Banach module over another Banach algebra with compatible actions, and is in fact a generalization of the amenability has been developed by Amini in [1]. Amini and Bagha in [2] introduced the concept of weak module amenability for Banach algebras. Then the second author along with Pourabbas in [9], introduced the concept of module cohomology group for Banach algebras and studied the second module cohomology group of semigroup algebra $\ell^1(S)$ with cofisence in $\ell^1(S)^{(2n-1)}$ $(n \in \mathbb{N})$, where S is commutative inverse semigroup with idempotent set E and $\ell^1(S)$ is a Banach $\ell^1(E)$-module with actions

$$\delta_s \cdot \delta_e = \delta_e \cdot \delta_s = \delta_e * \delta_s = \delta_{se} \qquad (s \in S, e \in E), \tag{1}$$

when δ_s and δ_e are the point mass at $s \in S$ and $e \in E$, respectively.

In this paper, first we define the concept of cyclic module amenability for Banach algebras which is weaker than the concepts cyclic amenability and weak module amenability. Indeed, we indicate there exist cyclic module amenable Banach algebras which are not cyclic amenable and weak module amenable. After that we study the hereditary properties of cyclic module amenability on Banach algebras. For example, we investigate relationship between cyclic $\mathfrak{A}$-module amenability of I, A/I and A, where I is closed ideal and $\mathfrak{A}$-submodule of A. Also it is shown that cyclic $\mathfrak{A}$-module amenability A and B follows from cyclic $\mathfrak{A}$-module amenability $A \oplus_{\ell^1} B$. ALso cyclic $\mathfrak{A}$-module amenability of A and B implies cyclic $\mathfrak{A}$-module amenability of $A \oplus_{\ell^1} B$, when A and B are essential.

2. Cyclic Module Derivations and Cyclic Module Amenability

Let $\mathfrak{A}$ and A be Banach algebras such that A is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha) \quad (\alpha \in \mathfrak{A},\ a, b \in A).$$

Let X be a Banach A-module and a Banach $\mathfrak{A}$-module with compatible actions. Then we say that X is a Banach A-$\mathfrak{A}$-module and it is called a commutative (bi-commutative) Banach A-$\mathfrak{A}$-module, if $\alpha \cdot x = x \cdot \alpha$ $(a \cdot x = x \cdot a)$ for every $\alpha \in \mathfrak{A}$ $(a \in A)$ and $x \in X$ (For more details see [1], [2], [9]). If X is a (commutative) Banach A-$\mathfrak{A}$-module, then so is X^*, where the actions of A and $\mathfrak{A}$ on X^* are defined by

$$(\alpha \cdot f)(x) = f(x \cdot \alpha) \qquad \text{and} \qquad (a \cdot f)(x) = f(x \cdot a)$$
$$(\alpha \in \mathfrak{A},\ a \in A,\ f \in X^*,\ x \in X),$$

and the same for the other side actions. In particular if A is a commutative Banach $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is a commutative Banach A-$\mathfrak{A}$-module. In this case, the dual space A^* is also a commutative Banach A-$\mathfrak{A}$-module. Let $D : A \to A^*$ be a bounded map. D is called

- **derivation** if $D(ab) = a \cdot D(b) + D(a) \cdot b$,
- **$\mathfrak{A}$-module** if $D(a \pm b) = D(a) \pm D(b)$, $D(\alpha \cdot a) = \alpha \cdot D(a)$ and $D(a \cdot \alpha) = D(a) \cdot \alpha$,
- **cyclic** if $[D(a)](b) + [D(b)](a) = 0$,
- **inner** if there exists $f \in A^*$ such that $D = \mathbf{ad}_f$ when $\mathbf{ad}_f(a) := a \cdot f - f \cdot a$,

where $\alpha \in \mathfrak{A}$ and $a, b \in A$. Now we use the notations

$$\begin{aligned} \mathcal{Z}^1(A, A^*) &:= \{D : A \to A^*; D \text{ is linear and derivation}\}, \\ \mathcal{ZC}^1(A, A^*) &:= \left\{D \in \mathcal{Z}^1(A, A^*); D \text{ is cyclic}\right\}, \\ \mathcal{B}^1(A, A^*) &:= \left\{D \in \mathcal{Z}^1(A, A^*); D \text{ is inner}\right\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{Z}^1_{\mathfrak{A}}(A, A^*) &:= \{D : A \to A^*; D \text{ is } \mathfrak{A}\text{-module and derivation}\}, \\ \mathcal{ZC}^1_{\mathfrak{A}}(A, A^*) &:= \left\{D \in \mathcal{Z}^1_{\mathfrak{A}}(A, A^*); D \text{ is cyclic}\right\}, \\ \mathcal{B}^1_{\mathfrak{A}}(A, A^*) &:= \left\{D \in \mathcal{Z}^1_{\mathfrak{A}}(A, A^*); D \text{ is inner}\right\}. \end{aligned}$$

Note that, although $D \in \mathcal{Z}^1_{\mathfrak{A}}(A, A^*)$ is not necessarily linear, but its boundedness still implies its norm continuity (since D preserves subtraction). Suppose A is a commutative Banach $\mathfrak{A}$-bimodule, for the above notations, we have

$$\begin{array}{ccccc} \mathcal{B}^1(A, A^*) & \subseteq & \mathcal{ZC}^1(A, A^*) & \subseteq & \mathcal{Z}^1(A, A^*), \\ || & & \cup & & \\ \mathcal{B}^1_{\mathfrak{A}}(A, A^*) & \subseteq & \mathcal{ZC}^1_{\mathfrak{A}}(A, A^*) & \subseteq & \mathcal{Z}^1_{\mathfrak{A}}(A, A^*). \end{array}$$

We recall that $\mathcal{B}^1(A, A^*) = \mathcal{B}^1_{\mathfrak{A}}(A, A^*)$ is a subspace of $\mathcal{ZC}^1(A, A^*)$, $\mathcal{Z}^1(A, A^*)$, $\mathcal{ZC}^1_{\mathfrak{A}}(A, A^*)$ and $\mathcal{Z}^1_{\mathfrak{A}}(A, A^*)$. The first (cyclic) cohomology group and the first (cyclic) $\mathfrak{A}$-module cohomology group is defined by:

$$\mathcal{H}^1(A, A^*) = \frac{\mathcal{Z}^1(A, A^*)}{\mathcal{B}^1(A, A^*)}, \qquad \left(\mathcal{HC}^1(A, A^*) = \frac{\mathcal{ZC}^1(A, A^*)}{\mathcal{B}^1(A, A^*)}\right),$$

and

$$\mathcal{H}^1_{\mathfrak{A}}(A, A^*) = \frac{\mathcal{Z}^1_{\mathfrak{A}}(A, A^*)}{\mathcal{B}^1_{\mathfrak{A}}(A, A^*)} \qquad \left(\mathcal{HC}^1_{\mathfrak{A}}(A, A^*) = \frac{\mathcal{ZC}^1_{\mathfrak{A}}(A, A^*)}{\mathcal{B}^1_{\mathfrak{A}}(A, A^*)}\right).$$

Definition 1. The Banach algebra A is called weak amenable (resp. cyclic amenable) if $\mathcal{H}^1(A, A^*) = 0$ (resp. $\mathcal{HC}^1(A, A^*) = 0$). Also if A is commutative Banach $\mathfrak{A}$-bimodule, A is called weak $\mathfrak{A}$-module amenable (resp. cyclic $\mathfrak{A}$-module amenable) if $\mathcal{H}^1_{\mathfrak{A}}(A, A^*) = 0$ (resp. $\mathcal{HC}^1_{\mathfrak{A}}(A, A^*) = 0$).

Remark 2. In the above, $\mathfrak{A}$-module maps are not necessarily linear. But if we consider $\mathfrak{A} = \mathbb{C}$, then all additive maps will be linear, which means $\mathcal{C}^1_{\mathfrak{A}}(A, A^*) = \mathcal{C}^1(A, A^*)$ and so in this case, the concepts weak $\mathbb{C}$-module amenability (resp. cyclic $\mathbb{C}$-module amenability) and weak amenability (resp. cyclic amenability) are coincide.

Note that

$$\text{weak amenablity} \Rightarrow \text{cyclic amenablity} \Rightarrow \text{cyclic module amenablity},$$

and

$$\text{weak module amenablity} \Rightarrow \text{cyclic module amenablity}.$$

It is shown that, there is a cyclic amenable Banach algebra which is not weak amenable. In the Example 4, we will discusses the distinction between the four above concepts and will show that the reverse of the above relations are not necessarily true.

Recall that Banach algebra A is called essential, if $\overline{A^2} = A$. We know that every weak amenable Banach algebra is essential, but this is not necessarily true for cyclic amenable Banach algebra.

Lemma 3. *Let A be an essential Banach algebra. Suppose $\mathfrak{A} = A$ where module actions of $\mathfrak{A}$ on A are algebraic multiplication, Then A is weak $\mathfrak{A}$-module amenable and so is cyclic $\mathfrak{A}$-module amenable.*

Proof. Let $D : A \to A^*$ be a bounded $\mathfrak{A}$-module derivation. For every $a, b \in A$,

$$D(ab) = a \cdot D(b) + D(a) \cdot b = D(ab) + D(ab) = 2D(ab).$$

This means that $D|_{A^2} = 0$. By continuity, D is zero, and so is inner. Therefore A is weak $\mathfrak{A}$-module amenable and so is cyclic $\mathfrak{A}$-module amenable. □

Example 4. **(i)** There is a Banach algebra that is not cyclic module amenable.

Let A be a non-zero Banach algebra with zero product, that is $ab = 0$ for all $a, b \in A$. It is shown in Example 2.5 of [6] that such Banach algebra is cyclic amenable if and only if its dimension is one. Consider $A = \mathbb{C}$ with the zero product. Then $\mathcal{A} = A \oplus A$ is not cyclic amenable since its dimension is not one. Now if $\mathfrak{A} = \mathbb{C}$, $\mathcal{A}$ is not cyclic $\mathfrak{A}$-module amenable.

(ii) There is a Banach algebra that is cyclic module amenable but not cyclic amenable.

Let A be an essential Banach algebras which be not cyclic amenable. Lemma 3 shows that A is cyclic $\mathfrak{A}$-module amenable where $\mathfrak{A} = A$.

(iii) There is a Banach algebra that is cyclic amenable but not weak amenable.

Let A and B be two cyclic amenable Banach algebras and let $\mathcal{F}$ be a Banach algebraic free product of A and B. Examples 2.6 and 2.7 of [6] show that $\mathcal{F}$ is not weak amenable while is cyclic amenable. To give another example, consider the bicyclic semigroup $S_1 = \langle e, p, q \ |qp = e\rangle = \{p^m q^n \ |m \geq 0, n \geq 0\}$. This is the inverse semigroup generated by an identity element e and two elements p and q subject to the condition $qp = e$. It is known that semigroup algebra $\ell^1(S_1)$ is cyclic amenable but is not weak amenable by Theorems 2.8 and 2.10 of [3].

(iv) There is a Banach algebra that is cyclic module amenable but not weak module amenable.

If $\mathfrak{A} = \mathbb{C}$, then by Remark 2 the Banach algebras $\mathcal{F}$ and $\ell^1(S_1)$ in (iii), are cyclic $\mathfrak{A}$-module amenable but not weak $\mathfrak{A}$-module amenable.

In the next Lemma for any Banach $\mathfrak{A}$-module A with commutative compatible actions, we investigate conditions under which cyclic module amenability of A implies it's cyclic amenability.

Lemma 5. *Let $\mathfrak{A}$ be a commutative, weak amenable Banach algebra and let A be a commutative Banach $\mathfrak{A}$-module which is cyclic $\mathfrak{A}$-module amenable. If A has a bounded approximate identity consisting of central idempotents then A is cyclic amenable.*

Proof. Let $(e_\gamma)_{\gamma\in\Gamma}$ be a bounded approximate identity such that each e_γ is an idempotent in the center of A. Let Y_γ be the closed linear span of $\{\alpha \cdot e_\gamma \ : \ \alpha \in \mathfrak{A}\}$. For each $a, b \in A$ and $\alpha, \beta \in \mathfrak{A}$, we have $(\alpha \cdot a)(\beta \cdot b) =$

$(\alpha\beta) \cdot (ab)$. Therefore, Y_γ is a closed subalgebra of A with the following multiplication:

$$(\alpha \cdot e_\gamma) \cdot (\beta \cdot e_\gamma) := (\alpha\beta) \cdot e_\gamma \qquad (\alpha, \beta \in \mathfrak{A}).$$

For each $\gamma \in \Gamma$, define $\pi_\gamma : \mathfrak{A} \to Y_\gamma$ by $\pi_\gamma(\alpha) = \alpha \cdot e_\gamma$. It is clear that, π_γ is a continuous algebra homomorphism with dense range such that $||\pi_\gamma|| \leq ||e_\gamma||$. Thus, weak amenability of Y_γ ($\gamma \in \Gamma$) follows from commutativity and weak amenability of A, by Proposition 2.8.64 of [4]. Let $D : A \to A^*$ be a cyclic derivation. Then the restriction of D vanishes on Y_γ, because $(e_\gamma)_{\gamma\in\Gamma}$ is contained in the center of A. Hence the derivation $D|_{Y_\gamma} : Y_\gamma \to A^*$ is zero. Clearly $D : A \to A^*$ is a module derivation. Thus, weak module amenability of A implies that D is inner. □

Corollary 6. *Let $\mathfrak{A}$ be a commutative, weak amenable Banach algebra. If A is a unital Banach algebra and a commutative Banach $\mathfrak{A}$-module, then cyclic $\mathfrak{A}$-module amenability of A implies its cyclic amenability.*

We note that for every commutative inverse semigroup S with idempotent set E, the semigroup algebra $\ell^1(S)$ is a Banach $\ell^1(E)$-module with actions (1). In the next corollary we give necessary condition for cyclic module amenability of $\ell^1(S)$ as a Banach $\ell^1(E_S)$-module, by any commutative compatible actions.

Corollary 7. *Let $\ell^1(S)$ be a commutative Banach $\ell^1(E_S)$-module, where S is a unital inverse semigroup. If $\ell^1(S)$ is cyclic $\ell^1(E_S)$-module amenable, then it is cyclic amenable.*

Consider the unital inverse semigroup S has infinite reversal depth. (see more details [3]). It is known that $\ell^1(S)$ is not cyclic amenable [3, Theorem 2.15]. Consequently, there is none commutative compatible action that $\ell^1(S)$ is cyclic $\ell^1(E_S)$-module amenable with it, by Corollary 6.

3. The Hereditary Properties of Cyclic Module Amenability on Banach Algebras

Let A be a Banach algebra and commutative Banach $\mathfrak{A}$-module with compatible actions, and let I be a closed ideal of A. In general, I and A/I are not necessarily Banach $\mathfrak{A}$-module with compatible actions. Throughout this section we assume that I is closed ideal and Banach $\mathfrak{A}$-submodule of A. In this case both I and A/I are commutative Banach $\mathfrak{A}$-module with the canonical actions.

Definition 8. Let I be a closed ideal in A. We say that I has the trace extension property if for each $\lambda \in I^*$ with $a.\lambda = \lambda.a\,(a \in A)$ there is $\Lambda \in A^*$ such that $\Lambda|_I = \lambda$ and $a \cdot \Lambda = \Lambda \cdot a$ for every $a \in A$.

Proposition 9. *Let A/I be cyclic $\mathfrak{A}$-module amenable. Then I has the trace extension property.*

Proof. Let $\lambda \in I^*$, such that $a \cdot \lambda = \lambda \cdot a$ for every $a \in A$. Take $\tau \in A^*$ with $\tau|_I = \lambda$. Define

$$\begin{aligned} D : A/I &\longrightarrow A^* \\ [a] &\mapsto a \cdot \tau - \tau \cdot a. \end{aligned}$$

We see immediately that D is a derivation. Furthermore for every $\alpha \in \mathfrak{A}$ and $a, b \in A$,

$$\begin{aligned} D(\alpha \cdot [a]) = D([\alpha \cdot a]) &= (\alpha \cdot a) \cdot \tau - \tau \cdot (\alpha \cdot a) \\ &= \alpha \cdot (a \cdot \tau) - (\tau \cdot \alpha) \cdot a = \alpha \cdot (a \cdot \tau) - (\alpha \cdot \tau) \cdot a \\ &= \alpha \cdot (a \cdot \tau - \tau \cdot a) = \alpha \cdot D([a]). \end{aligned}$$

Similarly we can show that $D([a] \cdot \alpha) = D([a]) \cdot \alpha$. This means that D is $\mathfrak{A}$-module map. Suppose that $i \in I$ and $a \in A$, since $i \cdot a - a \cdot i \in I$,

$$\begin{aligned} D([a])(i) &= (a \cdot \tau - \tau \cdot a)(i) = \tau(i \cdot a - a \cdot i) \\ &= \lambda(i \cdot a - a \cdot i) = (a \cdot \lambda - \lambda \cdot a)(i) \\ &= 0. \end{aligned}$$

This shows that $D([a])|_I = 0$, for every $a \in A$. So $\operatorname{Im} D \subseteq (A/I)^* = I^\perp$. Therefore $D \in \mathcal{Z}^1_{\mathfrak{A}}(A/I, (A/I)^*)$. On the other hand, for every $[a], [b] \in A/I$,

$$\begin{aligned} \langle [a], D([b]) \rangle + \langle [b], D([a] \rangle &= \langle [a], b \cdot \tau - \tau \cdot b) \rangle + \langle [b], a \cdot \tau - \tau \cdot a \rangle \\ &= \langle [a \cdot b - b \cdot a], \tau \rangle + \langle [b \cdot a - a \cdot b], \tau \rangle \\ &= \langle [a \cdot b - b \cdot a + b \cdot a - a \cdot b], \tau \rangle \\ &= 0. \end{aligned}$$

Hence D is cyclic and so $D \in \mathcal{ZC}^1_{\mathfrak{A}}(A/I, (A/I)^*)$. Since A/I is cyclic $\mathfrak{A}$-module amenable, there exists a $\lambda' \in (A/I)^* = I^\perp$ such that $D = \mathbf{ad}_{\lambda'}$. Set $\Lambda = \tau - \lambda' \in A^*$. We have

$$\Lambda|_I = (\tau - \lambda')|_I = \tau|_I - \lambda'|_I = \tau|_I - 0 = \lambda$$

and for every $a \in A$,

$$\begin{aligned} a \cdot \Lambda - \Lambda \cdot a &= (\tau - \lambda') \cdot a - a \cdot (\tau - \lambda') \\ &= (\tau \cdot a - a \cdot \tau) + (a \cdot \lambda' - \lambda' \cdot a) \\ &= (\tau \cdot a - a \cdot \tau) + ([a] \cdot \lambda' - \lambda' \cdot [a]) \\ &= -D([a]) + \mathbf{ad}_{\lambda'}([a]) \\ &= -D([a]) + D([a]) = 0. \end{aligned}$$

This shows that I has the trace extension property. □

Proposition 10. *Let A be cyclic $\mathfrak{A}$-module amenable and I has the trace extension property. Then A/I is cyclic $\mathfrak{A}$-module amenable.*

Proof. Suppose $\pi : A \longrightarrow A/I$ is the quotient map and $D \in \mathcal{ZC}^1_{\mathfrak{A}}(A/I, (A/I)^*)$. Set $\widetilde{D} = \pi^* \circ D \circ \pi : A \to A^*$. Clearly $\widetilde{D}$ is a derivation. Furthermore for every $\alpha \in \mathfrak{A}$ and $a, b \in A$,

$$\begin{aligned} \langle b, \widetilde{D}(\alpha \cdot a)\rangle &= \langle b, (\pi^* \circ D \circ \pi)(\alpha \cdot a)\rangle = \langle [b], D([\alpha \cdot a])\rangle \\ &= \langle [b], \alpha \cdot D([a])\rangle = \langle [b] \cdot \alpha, D([a])\rangle \\ &= \langle [b \cdot \alpha], D([a])\rangle = \langle b \cdot \alpha, (\pi^* \circ D \circ \pi)(a)\rangle \\ &= \langle b \cdot \alpha, \widetilde{D}(a)\rangle = \langle b, \alpha \cdot \widetilde{D}(a)\rangle. \end{aligned}$$

So $\widetilde{D}(\alpha \cdot a) = \alpha \cdot \widetilde{D}(a)$ and similarly we can show that $\widetilde{D}(a \cdot \alpha) = \widetilde{D}(a) \cdot \alpha$. This means that $\widetilde{D}$ is $\mathfrak{A}$-module map. Since D is cyclic, for every $a, b \in A$, we have

$$\begin{aligned} \langle b, \widetilde{D}(a)\rangle + \langle a, \widetilde{D}(b)\rangle &= \langle b, (\pi^* \circ D \circ \pi)(a)\rangle + \langle a, (\pi^* \circ D \circ \pi)(b)\rangle \\ &= \langle [b], D([a])\rangle + \langle [a], D([b])\rangle \\ &= 0. \end{aligned}$$

So $\widetilde{D}$ is cyclic and thus $\widetilde{D} \in \mathcal{ZC}^1_{\mathfrak{A}}(A, (A^*)$. Since A is cyclic amenable, there exists $\lambda \in A^*$ with

$$\widetilde{D}(a) = \mathbf{ad}_{\lambda}(a) = a \cdot \lambda - \lambda \cdot a \qquad (a \in A).$$

Clearly $\widetilde{D}(a)|_I = 0$. Set $\lambda' = \lambda|_I$. For every $a \in A$, we have

$$\begin{aligned} a \cdot \lambda' - \lambda' \cdot a &= a \cdot (\lambda|_I) - (\lambda|_I) \cdot a = (a \cdot \lambda)|_I - (\lambda \cdot a)|_I \\ &= (a \cdot \lambda - \lambda \cdot a)|_I = \widetilde{D}(a)|_I \\ &= 0. \end{aligned}$$

Thus by assumption there exists a $\Lambda \in A^*$ such that $\Lambda|_I = \lambda'$ and $a \cdot \Lambda - \Lambda \cdot a = 0$ for every $a \in A$. But $\lambda - \Lambda \in I^\perp$ and for every $a, b \in A$,

$$\begin{aligned}\langle [b], D([a]) \rangle &= \langle b, (\pi^* \circ D \circ \pi)(a) \rangle \\ &= \langle b, \widetilde{D}(a) \rangle \\ &= \langle b, a \cdot \lambda - \lambda \cdot a \rangle \\ &= \langle b, a \cdot (\lambda - \Lambda) - (\lambda - \Lambda) \cdot a \rangle \\ &= \langle [b], [a] \cdot (\lambda - \Lambda) - (\lambda - \Lambda) \cdot [a] \rangle.\end{aligned}$$

This shows that $D([a]) = \mathbf{ad}_\omega([a])$, where $\omega = \lambda - \Lambda$. Thus D is inner and it follows that A/I is cyclic $\mathfrak{A}$-module amenable. □

The following Corollary confirms the correctness of Corollary 3.1.2 (ii) of [8].

Corollary 11. *Let A and B be commutative Banach $\mathfrak{A}$-bimodules such that $A \oplus_{\ell^1} B$ is cyclic $\mathfrak{A}$-module amenable. Then both A and B are cyclic $\mathfrak{A}$-module amenable. In particular, the cyclic amenability of A and B follows from the cyclic amenability of $A \oplus_{\ell^1} B$.*

Proposition 12. *Let I and A/I be cyclic $\mathfrak{A}$-module amenable and $\overline{I^2} = I$. Then A is cyclic $\mathfrak{A}$-module amenable.*

Proof. Suppose $\boldsymbol{\iota} : I \longrightarrow A$ is the natural embedding and $D \in \mathcal{ZC}^1_{\mathfrak{A}}(A, A^*)$. Define $D' := \boldsymbol{\iota}^* \circ D \circ \boldsymbol{\iota}$. It is easy to prove that D' is cyclic derivation. We show that D' is $\mathfrak{A}$-module map. Let $\alpha \in \mathfrak{A}$ and $a, b \in A$,

$$\begin{aligned}\langle b, D'(\alpha \cdot a) \rangle &= \langle b, (\boldsymbol{\iota}^* \circ D \circ \boldsymbol{\iota})(\alpha \cdot a) \rangle = \langle b, D(\alpha \cdot a) \rangle \\ &= \langle b, \alpha \cdot D(a) \rangle = \langle b \cdot \alpha, D(a) \rangle \\ &= \langle b \cdot \alpha, (\boldsymbol{\iota}^* \circ D \circ \boldsymbol{\iota})(a) \rangle = \langle b \cdot \alpha, D'(a) \rangle \\ &= \langle b, \alpha \cdot D'(a) \rangle.\end{aligned}$$

So $D' \in \mathcal{ZC}^1_{\mathfrak{A}}(I, I^*)$ and since I is cyclic $\mathfrak{A}$-module amenable, there exists $\lambda \in I^*$ with

$$(\boldsymbol{\iota}^* \circ D)(i) = \mathbf{ad}_\lambda(i) \qquad (i \in I).$$

Now define $\widetilde{D} = D - \mathrm{ad}_{\widetilde{\lambda}}$, where $\widetilde{\lambda} \in A^*$ is extended of λ. Therefore $(\boldsymbol{\iota}^* \circ \widetilde{D})|_I = 0$ and for every $i, j \in I$ and $a \in A$,

$$\begin{aligned}\langle a, \widetilde{D}(ij) \rangle &= \langle a, i \cdot \widetilde{D}(j) + \widetilde{D}(i) \cdot j \rangle \\ &= \langle a, i \cdot \widetilde{D}(j) \rangle + \langle a, \widetilde{D}(i) j \rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle ai, \widetilde{D}(j)\rangle + \langle ja, \widetilde{D}(i)\rangle \\
&= \langle ai, (\boldsymbol{\iota}^* \circ \widetilde{D})(j)\rangle + \langle ja, (\boldsymbol{\iota}^* \circ \widetilde{D})(i)\rangle \\
&= 0.
\end{aligned}
$$

Hence $\widetilde{D}|_{I^2} = 0$. Since $\overline{I^2} = I$ so $\widetilde{D}|_I = 0$. If we set $F = \overline{IA + AI}$, then $F = \overline{I^2} = I$. Now for each $a \in A$ and $i \in I$, $\widetilde{D}(a) \cdot i = \widetilde{D}(ai) - a \cdot \widetilde{D}(i) = 0$, and so $\widetilde{D}(a) \cdot i = 0$. Taking $b \in A$, we get

$$\langle i \cdot b, \widetilde{D}(a)\rangle = \langle b, \widetilde{D}(a) \cdot i\rangle = 0,$$

and so $\widetilde{D}(a)|_{IA} = 0$. Similarly $\widetilde{D}(a)|_{AI} = 0$, and hence $\widetilde{D}(a)|_I = 0$. Thus $\operatorname{Im} \widetilde{D} \subseteq I^\perp$ and the map

$$
\begin{aligned}
\widehat{D} : A/I &\longrightarrow (A/I)^* = I^\perp \\
[a] &\longmapsto \widetilde{D}(a)
\end{aligned}
$$

is well define. Regard the $\widetilde{D}(a) : A/I \to \mathbb{C}$ by $\langle [b], \widetilde{D}(a)\rangle = \langle b, \widetilde{D}(a)\rangle$. Clearly $\widehat{D}$ is derivation. Let $\alpha \in \mathfrak{A}$ and $a, b \in A$,

$$
\begin{aligned}
\widehat{D}([\alpha \cdot a]) &= \widetilde{D}(\alpha \cdot a) \\
&= (D - \mathbf{ad}_{\widetilde{\lambda}})(\alpha \cdot a) \\
&= D(\alpha \cdot a) - \mathbf{ad}_{\widetilde{\lambda}}(\alpha \cdot a) \\
&= \alpha \cdot D(a) - (\alpha \cdot a) \cdot \widetilde{\lambda} + \widetilde{\lambda} \cdot (\alpha \cdot a) \\
&= \alpha \cdot D(a) - \alpha \cdot (a \cdot \widetilde{\lambda}) + (\widetilde{\lambda} \cdot \alpha) \cdot a \\
&= \alpha \cdot D(a) - \alpha \cdot (a \cdot \widetilde{\lambda}) + (\alpha \cdot \widetilde{\lambda}) \cdot a \\
&= \alpha \cdot D(a) - \alpha \cdot (a \cdot \widetilde{\lambda}) + \alpha \cdot (\widetilde{\lambda} \cdot a) \\
&= \alpha \cdot (D(a) - a \cdot \widetilde{\lambda} + \widetilde{\lambda} \cdot a) \\
&= \alpha \cdot (D - \mathbf{ad}_{\widetilde{\lambda}})(a) \\
&= \alpha \cdot \widetilde{D}(a) = \alpha \cdot \widehat{D}([a])
\end{aligned}
$$

so $\widehat{D}([\alpha \cdot a]) = \alpha \cdot \widehat{D}([a])$ and similarly $\widehat{D}([a \cdot \alpha]) = \widehat{D}([a]) \cdot \alpha$. This means that $\widehat{D}$ is $\mathfrak{A}$-module map. Since D is cyclic, for every $a, b \in A$, we have

$$
\begin{aligned}
\langle [b], \widehat{D}([a])\rangle + \langle [a], \widehat{D}([b])\rangle &= \langle [b], \widetilde{D}(a)\rangle + \langle [a], \widetilde{D}(b)\rangle \\
&= \langle b, \widetilde{D}(a)\rangle + \langle a, \widetilde{D}(b)\rangle \\
&= \langle b, (D - \mathbf{ad}_{\widetilde{\lambda}})(a)\rangle + \langle a, (D - \mathbf{ad}_{\widetilde{\lambda}})(b)\rangle
\end{aligned}
$$

$$\begin{aligned}&= \langle b, D(a)\rangle + \langle a, D(b)\rangle \\ &\quad - \langle b, \mathbf{ad}_{\widetilde{\lambda}}(a)\rangle - \langle a, \mathbf{ad}_{\widetilde{\lambda}}(b)\rangle \\ &= 0 - \langle ba - ab, \widetilde{\lambda}\rangle - \langle ab - ba, \widetilde{\lambda}\rangle \\ &= 0.\end{aligned}$$

This shows that $\widehat{D}$ is cyclic. According to previous findings, $\widehat{D} \in \mathcal{ZC}^1_{\mathfrak{A}}(A/I, (A/I)^*)$ and since by hypothesis A/I is cyclic $\mathfrak{A}$-module amenable, so there exists a $\lambda' \subseteq I^{\perp}$ such that $\widehat{D} = \mathbf{ad}_{\lambda'}$. With some simple calculations given to the reader, it can be shown that $D = \mathbf{ad}_{\lambda+\lambda'}$. It now follows that A is cyclic $\mathfrak{A}$-module amenable. □

The following Corollary not only confirms, but also it improves the Corollary 3.1.2 (i) of [8].

Corollary 13. *Let A and B be commutative Banach $\mathfrak{A}$-bimodule, cyclic $\mathfrak{A}$-module amenable and essential. Then $A \oplus_{\ell^1} B$ is cyclic $\mathfrak{A}$-module amenable. In particular, if A and B are cyclic amenable and essential Banach algebras, then $A \oplus_{\ell^1} B$ is cyclic amenable.*

Definition 14. Let I be a closed ideal in A. We say that a bounded approximate identity $\{e_\gamma\}_{\gamma\in\Gamma}$ of I is quasi central for A if $\lim_\gamma \|ae_\gamma - e_\gamma a\| = 0$ for all $a \in A$.

Proposition 15. *Let A be cyclic $\mathfrak{A}$-module amenable Banach algebra and closed ideal I of A has a quasi-central bounded approximate identity for A. Then I is cyclic $\mathfrak{A}$-module amenable.*

Proof. Let $D \in \mathcal{ZC}^1_{\mathfrak{A}}(I, I^*) \subseteq \mathcal{ZC}^1(I, I^*)$. Let $\{e_\gamma\}_{\gamma\in\Gamma} \subseteq I$ be a quasi-central bounded approximate identity for A. We define a family of bounded linear maps $\mathfrak{J}_\gamma : I^* \to A^* \quad (\gamma \in \Gamma)$ by

$$\langle a, \mathfrak{J}_\gamma(m)\rangle = \langle e_\gamma a, m\rangle \qquad (a \in A, m \in I^*).$$

By identifying $\mathcal{C}^1(I^*, A^*) \simeq (I^* \widehat{\otimes} A)^*$ and passing to a subnet if necessary, we may assume that the net $\{\mathfrak{J}_\gamma\}_{\gamma\in\Gamma}$ is convergent in the weak operator topology to a bounded linear map $\mathfrak{J} : I^* \to A^*$. Using the quasi-central property of $\{e_\gamma\}_{\gamma\in\Gamma}$ one easily checks that $\mathfrak{J}$ is a module map with the property that $\mathfrak{J}(m)$ is an extension of m for each $m \in I^*$. Now D can be lifted to a bounded derivation $\widehat{D} : A \longrightarrow A^*$ where $\widehat{D} = \mathfrak{J}(\overline{D})$ when $\overline{D}(a) := w^*\text{-}\lim_\gamma D(e_\gamma a)$. With some simple calculations given to the reader, it can be shown that $\widehat{D}$ is cyclic and $\mathfrak{A}$-module derivation and from which, the result follows immediately. □

Theorem 16. *Let A and B be Banach algebras and commutative Banach $\mathfrak{A}$-bimodule. Suppose A is cyclic $\mathfrak{A}$-module amenable. If $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow A$ are continuous $\mathfrak{A}$-module homomorphisms such that $\varphi \circ \psi = I_B$, then B is cyclic $\mathfrak{A}$-module amenable.*

Proof. Let $D : B \longrightarrow B^*$ be a cyclic $\mathfrak{A}$-module derivation. Put $\widetilde{D} = \varphi^* \circ D \circ \varphi : A \to A^*$. For every $a_1, a_2, a_3 \in A$,

$$\begin{aligned}\langle a_3, \widetilde{D}(a_1 a_2)\rangle &= \langle \varphi(a_3), D(\varphi(a_1)\varphi(a_2))\rangle \\ &= \langle \varphi(a_3), D(\varphi(a_1)) \cdot \varphi(a_2) + \varphi(a_1) \cdot D(\varphi(a_2))\rangle \\ &= \langle \varphi(a_2 a_3), D(\varphi(a_1))\rangle + \langle \varphi(a_3 a_1), D(\varphi(a_2))\rangle \\ &= \langle a_2 a_3, \widetilde{D}(a_1)\rangle + \langle a_3 a_1, \widetilde{D}(a_2)\rangle \\ &= \langle a_3, \widetilde{D}(a_1) \cdot a_2 + a_1 \cdot \widetilde{D}(a_2)\rangle.\end{aligned}$$

Hence, $\widetilde{D}$ is a continuous derivation. Moreover, $\widetilde{D}$ is cyclic, since D is cyclic. Now let $\alpha \in \mathfrak{A}$ and $a_1, a_2 \in A$,

$$\begin{aligned}\langle a_2, \widetilde{D}(\alpha \cdot a_1)\rangle &= \langle \varphi(a_2), D(\varphi(\alpha \cdot a_1))\rangle \\ &= \langle \varphi(a_2), \alpha \cdot D(\varphi(a_1))\rangle \\ &= \langle \varphi(a_2 \cdot \alpha), D(\varphi(a_1))\rangle \\ &= \langle a_2 \cdot \alpha, \widetilde{D}(a_1)\rangle \\ &= \langle a_2, \alpha \cdot \widetilde{D}(a_1)\rangle,\end{aligned}$$

so $\widetilde{D}(\alpha \cdot a_1) = \alpha \cdot \widetilde{D}(a_1)$ and similarly we can show that $\widetilde{D}(a_1 \cdot \alpha) = \widetilde{D}(a_1) \cdot \alpha$. This means that $\widetilde{D}$ is $\mathfrak{A}$-module map and $\widetilde{D} \in \mathcal{ZC}^1_{\mathfrak{A}}(A, A^*)$. By cyclic $\mathfrak{A}$-module amenability of A, there exists a $f \in A^*$ such that $\widetilde{D} = \mathbf{ad}_f$. The equality $\varphi \circ \psi = I_B$ implies $\psi^* \circ \varphi^* = I_{B^*}$. For every $b_1, b_2 \in B$, we get

$$\begin{aligned}\langle b_2, D(b_1)\rangle &= \langle b_2, \psi^* \circ \varphi^* \circ D \circ \varphi \circ \psi(b_1)\rangle \\ &= \langle b_2, \psi^*(\widetilde{D}(\psi(b_1))\rangle \\ &= \langle \psi(b_2), \mathbf{ad}_f(\psi(b_1))\rangle \\ &= \langle \psi(b_2), \psi(b_1) \cdot f - f \cdot \psi(b_1)\rangle \\ &= \langle \psi(b_2 b_1 - b_1 b_2), f\rangle \\ &= \langle b_2 b_1 - b_1 b_2, \psi^*(f)\rangle \\ &= \langle b_2, b_1 \cdot \psi^*(f) - \psi^*(f) \cdot b_1\rangle \\ &= \langle b_2, \mathbf{ad}_{\psi^*(f)}(b_1)\rangle.\end{aligned}$$

This shows that $D = \mathbf{ad}_{\psi^*(f)}$. It follows that B is cyclic $\mathfrak{A}$-module amenable. □

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