International Journal of Pure and Applied Mathematics
Volume 120 No. 3 2018, 329-337
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: 10.12732/ijpam.v120i3.4

ijpam.eu

ORTHOGONAL BASED ZERO-STABLE NUMERICAL INTEGRATOR FOR SECOND ORDER IVPs IN ODEs

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Abstract: This paper presents a set of newly constructed polynomials valid in interval [-1, 1] with respect to weight function $w(x) = (1-x^2)^2$. For applicability sake, the polynomials shall be employed as trial function to develop a fast, efficient and reliable block algorithm for the numerical solution of ordinary differential equations with application to second order initial value problems. Collocation and interpolation techniques were adopted for the formulation of self-starting continuous hybrid schemes. Findings from the analysis of the basic properties of the method using appropriate existing theorems show that the developed schemes are consistent, zero-stable and hence convergent. On implementation, the superiority of the scheme over the existing method is established numerically.

1. Introduction

The focus here is to construct a set ofpolynomials and develop a continuous

Received: June 28, 2016
Revised: April 6, 2018
Published: December 4, 2018

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method for the solution of initial value problems of the form

$$y^{''} = f(x, y, y^{'}), y(a) = y_0, y^{'}(a) = z_0 \quad x \in [a, b] \tag{1}$$

where f is continuous in $[a, b]$. Since most of these equations are difficult to solve, efficient ODEs solvers are much needed to approximate them. The numerical solution of ODEs by collocation methods has been studied (see Lambert (1973), Zennaro (1985) and Fairweather and Meade (1989)). In the recent years, the solutions of (1) for cases m = 1, 2 and 3 have been extremely discussed (see Adeniyi et al. (2006), Adeniyi and Alabi (2006), Adesanya et al. (2012) and Adeyefa et al. (2014)). The approach of reducing the higher order of (1) to a system of first order and the consequent setback has also been discussed in Lambert (1973). Predictor-corrector method was later adopted and applied but has its setbacks which were discussed in Lambert (1973). To cater for the setback of predictor-corrector methods, the approach of block method came into being. Milne (1953) proposed a method called block method as a means of obtaining starting values which Rosser (1967) developed into algorithms for general use. Later, the modified self-starting block method was developed (see Shampine and Watts (1969), Awoyemi et al. (2014) and Kayode (2009)). The block (4) is a simultaneous producing approximations to the solution of (1) at a block of desired points. However, the effectiveness of these ODE solvers depends on the types of trial functions used in developing the schemes. Various trial functions such as, the Chebyshev polynomials which was introduced in Lanczos (1938) as basis function for the solution of linear differential equations in term of finite expansion, the Legendre polynomials, Power series and the Canonical polynomials have been used to derive continuous schemes. In what follows, we shall construct a set of polynomials valid in interval [-1, 1] with respect to weight function $w(x) = (1 - x^2)^2$.

2. Materials and Methods

2.1. Construction of the Orthogonal Polynomial

Let the function $q_r(x)$ be defined as

$$q_r(x) = \sum_{r=0}^{n} C_r^{(n)} x^r \tag{2}$$

on the interval [a, b] where $q_r(x)$ must satisfy

$$\langle q_m, q_n \rangle = \int_a^b w(x) q_m(x) q_n(x) dx = 0, \quad m = 0, 1, 2...n-1. \tag{3}$$

For the purpose of constructing the basis function, we adopt the approach extensively explained in Adeyefa and Adeniyi (2015) and use additional property (the normalization) $q_n(1) = 1$ where our non-negative weight function is defined as $w(x) = (1-x^2)^2$. With these requirements, equation (3) yields

$$q_o(x) = 1$$

$$q_1(x) = x$$

$$q_2(x) = \frac{1}{6}(7x^2 - 1)$$

$$q_3(x) = \frac{1}{2}(3x^3 - x)$$

$$q_4(x) = \frac{1}{16}(33x^4 - 18x^2 + 1)$$

$$q_5(x) = \frac{1}{48}(143x^5 - 110x^3 + 15x)$$

$$q_6(x) = \frac{1}{32}(143x^6 - 143x^4 + 33x^2 - 1)$$

$$q_7(x) = \frac{1}{32}(221x^7 - 273x^5 + 91x^3 - 7x)$$

$$q_8(x) = \frac{1}{384}(4199x^8 - 6188x^6 + 2730x^4 - 364x^2 + 7)$$

$$q_9(x) = \frac{1}{128}(2261x^9 - 3876x^7 + 2142x^5 - 420x^3 + 21x)$$

$$q_{10}(x) = \frac{1}{256}(7429x^{10} - 14535x^8 + 9690x^6 - 2550x^4 + 225x^2 - 3)$$

In the spirit of [7], these equations must satisfy three-term recurrence relation

$$c_j p(t) = (t - a_j) p_{j-1}(t) - b_j p_{j-2}(t),$$

$j = 1, 2, ...,$ $p_{-1}(t) = 0$, $p_0(t) \equiv p_0$, where b_j, $c_j > 0$ for $j \geq 1$ (b_1 is arbitrary);

$$c_j p(t) = (n+5) P_{n+1}(x), \quad (t - a_j) p_{j-1}(t) = (2n+5) x P_n(x),$$

$$b_j p_{j-2}(t) = n P_{n-1}(x), \quad n = 1, 2, ...$$

Hence, the recurrence relation of the polynomials above is

$$P_{n+1}(x) = \frac{1}{n+5}[(2n+5)xP_n - nP_{n-1}],$$

$n \geq 1$, $P_0(x) = 1$, $P_1(x) = x$.

2.2. Derivation of a Numerical Integrator

We consider here the derivation of the proposed continuous hybrid two-step block methods. This we do by approximating the analytical solution of (1) using orthogonal polynomials

$$y(x) = \sum_{j=0}^{r+s-1} a_j q_j(x) \tag{4}$$

transformed to shifted form by $x = \frac{X - x_n - h}{h}$ in [0, 1] where $a_j's$ are constants to be determined, r is the number of collocation points and s is the number of interpolation points, on the partition

$$a = x_0 < x_1 < ... < x_n < x_{n+1} < ... < x_N = b$$

of the integration interval [a,b] with a constant step size h, given by

$$h = x_{n+1} - x_n, \quad n = 0, 1, ...N - 1.$$

We interpolate (4) at $x = x_{n+s}$, $s = \frac{1}{3}, \frac{2}{3}$ and collocate the second derivative of (4) at $x = x_{n+r}$, r=0,$\frac{1}{3}$, $\frac{2}{3}$, 1 and 2 respectively to obtain a system of equations from which a_j are obtained.

Substituting a_j into (4), we have the continuous scheme

$$\begin{aligned} y(x) = \alpha_{\frac{1}{3}}(x) y_{n+\frac{1}{3}} + \alpha_{\frac{2}{3}}(x) y_{n+\frac{2}{3}} + h^2(\beta_0(x) f_n \\ + \beta_{\frac{1}{3}}(x) f_{n+\frac{1}{3}} + \beta_{\frac{2}{3}}(x) f_{n+\frac{2}{3}} + \beta_1(x) f_{n+1} + \beta_2(x) f_{n+2} \end{aligned} \tag{5}$$

where $\alpha(x)$ are $\beta(x)$ parameters to be determined. Evaluating the continuous scheme at the grid point $x = x_{n+2}$, we have

$$\begin{aligned} y_{n+2} = 5y_{n+\frac{2}{3}} - 4y_{n+\frac{1}{3}} + \frac{h^2}{3240}(-490 f_n + 2388 f_{n+\frac{1}{3}} \\ - 2715 f_{n+\frac{2}{3}} + 4220 f_{n+1} + 197 f_{n+2}) \end{aligned} \tag{6}$$

To develop the block method from the continuous scheme, we adopt the general block formula proposed in Awoyemi et al. (2014) in the normalized form given as

$$A^{(0)} Y_m = e y_m + h^{\mu-\lambda} d f(y_m) + h^{\mu-\lambda} b F(y_m) \tag{7}$$

Evaluating the first derivative of (5) at $x = x_{n+j}$, j=0,$\frac{1}{3}$, $\frac{2}{3}$, 1 and 2, substituting the resulting equations and the main method into (7) and solving simultaneously gives a block formulae represented as

$$\begin{aligned}
y_{n+\frac{1}{3}} =& y_n + \frac{h}{3}y_n^{'} + \frac{h^2}{64800}(1870f_n + 2532f_{n+\frac{1}{3}} - 1095f_{n+\frac{2}{3}} \\
&+ 300f_{n+1} - 7f_{n+2}), \\
y_{n+\frac{2}{3}} =& y_n + \frac{2h}{3}y_n^{'} + \frac{h^2}{4050}(270f_n + 696f_{n+\frac{1}{3}} - 105f_{n+\frac{2}{3}} \\
&+ 40f_{n+1} - f_{n+2}) \\
y_{n+1} =& y_n + hy_n^{'} + \frac{h^2}{2400}(250f_n + 756f_{n+\frac{1}{3}} + 135f_{n+\frac{2}{3}} + 60f_{n+1} - f_{n+2}) \\
y_{n+2} =& y_n + 2hy_n^{'} + \frac{h^2}{750}(50f_n + 1080f_{n+\frac{1}{3}} - 675f_{n+\frac{2}{3}} \\
&+ 1000f_{n+1} + 45f_{n+2}) \\
y_{n+\frac{1}{3}}^{'} =& y_n^{'} + \frac{h}{32400}(3860f_n + 9234f_{n+\frac{1}{3}} - 3105f_{n+\frac{2}{3}} + 830f_{n+1} - 19f_{n+2}) \\
y_{n+\frac{2}{3}}^{'} =& y_n^{'} + \frac{h}{4050}(440f_n + 1836f_{n+\frac{1}{3}} + 405f_{n+\frac{2}{3}} + 20f_{n+1} - f_{n+2}) \\
y_{n+1}^{'} =& y_n^{'} + \frac{h}{1200}(140f_n + 486f_{n+\frac{1}{3}} + 405f_{n+\frac{2}{3}} + 170f_{n+1} - f_{n+2}) \\
y_{n+2}^{'} =& y_n^{'} + \frac{h}{150}(-40f_n + 324f_{n+\frac{1}{3}} - 405f_{n+\frac{2}{3}}380f_{n+1} + 41f_{n+2}).
\end{aligned} \tag{8}$$

3. Analysis of the Method

The basic properties of this method such as order, error constant, zero stability and consistency are analyzed here under.
The linear operator L of the block (7) is defined as

$$L(y(x):h) = Y_m - ey_m + h^{\mu-\lambda}df(y_m) + h^{\mu-\lambda}bF(y_m) \tag{9}$$

Using Taylor series expansion to expand $y(x_n + ih)$ and $f(x_n + jh)$, equation (9) becomes

$$L(y(x):h) = C_0y(x) + C_1hy^{'}(x) + ...C_{p+2}h^{p+2}y^{p+2}(x) \tag{10}$$

The block (7) and associated linear operator are said to have order p if $C_0 = C_1 = C_2 = ... = C_p = C_{p+1} = 0$, $C_{p+2} \neq 0$

The term $C_{p+2} \neq 0$ is called the error constant and the local truncation error is given as

$$t_{n+k} = C_{p+2}h^{(p+2)}y^{(p+2)}(x_n) + 0h^{(p+3)}.$$

Thus, equations (6) is of order 5 and error constant $-\frac{73}{87480}$. The formulae in the block (8) are all of order 5 with error constants C_{p+2} given as

$$C_{p+2} = \left[\frac{829}{110224800} \frac{61}{3444525} \frac{13}{453600} \frac{-11}{14175} \frac{211}{5248800} \frac{7}{328050} \frac{1}{21600} \frac{-1}{450}\right]^T,$$

respectively.

3.1. Zero Stability of the Method

The block (7) is said to be zero-stable if the roots z $= 1, 2, ...,$ N of the characteristic polynomial $\rho(z) = det(zA - E)$, satisfy $|z| \leq 1$ and the root$|z| = 1$ has multiplicity not exceeding the order of the differential equation. Also, as $h^\mu \longrightarrow 0$, $\rho(z) = z^{r-\mu}(\lambda - 1)^(\mu)$, where μ is the order of the differential equation, $r = dim(A^{(0)})$.

The proposed method has been investigated to be zero stable.

3.2. Consistency of the Method

A numerical method is consistent if the order, $p \geq 1$.

This method is consistent owing to the fact that the order is five (5).

3.3. Convergence of the Method

According to Dahlguist's theorem, the necessary and sufficient condition for an LMM to be convergent is to be consistent and zero-stable. Since the method satisfies these two conditions, hence its convergence.

4. Numerical Experiment

We demonstrate the method with three test problems.
1. $y^{''} - y^{'} = 0$, $y(0) = 0$, $y^{'}(0) = -1$
whose analytical solution is $y(x) = 1 - e^x$.
2. $y^{''} - x(y^{'})^2 = 0$, $y(0) = 1$, $y^{'}(0) = \frac{1}{2}$, h=0.0025
Analytical Solution: $y(x) = 1 + \frac{1}{2}ln\frac{2+x}{2-x}$

3. We consider Van der Pols equation $y'' + y' + y + y^2 y' = 2cost - cos^3 t$, y(0)=0, $y'(0)$=1
whose analytical solution is $y(x) = sint$

Tables of Results

Table 1: Numerical Results for Example 1

X	Exact Solution	Proposed Method	Error	Error in Yahaya
0.1	-0.10517091807564762480	-0.10517091807239943619	-3.24818861e-12	8.79316e-05
0.2	-0.22140275816016983390	-0.22140275824581250946	8.564267556e-11	3.26718e-04
0.3	-0.34985880757600310400	-0.34985880792001473211	3.4401162811e-10	2.215564e-03
0.4	-0.49182469764127031780	-0.49182469838377994138	7.4250962358e-10	4.857093e-03
0.5	-0.64872127070012814680	-0.64872127207862860168	1.37850045488e-09	9.097734e-03
0.6	-0.82211880039050897490	-0.82211880260985294537	2.21934397047e-09	1.4391394e-02
0.7	-1.01375270747047652160	-1.01375271085798121930	3.3875046977e-09	2.1437918e-02
0.8	-1.22554092849246760460	-1.22554093333950033000	4.8470327254e-09	2.9898724e-02
0.9	-1.45960311115694966380	-1.45960311790878502900	6.7518353652e-09	4.0300719e-02
1.0	-1.71828182845904523540	-1.71828183752183259550	9.0627873601e-09	5.255213e-02

Table 2: Numerical Results for Example 2

X	Exact Solution	Proposed Method	Error
0.0025	1.00125000065104227700	1.00125000065104227700	0
0.0050	1.00250000520835286470	1.00250000520835286470	0
0.0075	1.00375001757827331690	1.00375001757827331700	1.0e-19
0.0100	1.00500004166729167780	1.00500004166729167790	1.0e-19
0.0125	1.00625008138211573520	1.00625008138211573530	1.0e-19
0.0150	1.00750014062974628450	1.00750014062974628460	1.0e-19
0.0175	1.00875022331755040640	1.00875022331755040660	2.0e-19
0.0200	1.01000033335333476200	1.01000033335333476220	2.0e-19
0.0225	1.01125047464541890790	1.01125047464541890810	2.0e-19
0.0250	1.01250065110270863570	1.01250065110270863600	3.0e-19

Table 3: Numerical Results for Example 3

X	Exact Solution	Proposed Method	Error
0.1	0.09983341664682815231	0.09983341664952788353	2.699731223e-12
0.2	0.19866933079506121546	0.19866933071823573020	7.682548526e-11
0.3	0.29552020666133957511	0.29552020638296494181	2.783746333e-10
0.4	0.38941834230865049167	0.38941834177500388267	5.33646609e-10
0.5	0.47942553860420300027	0.47942553772667219275	8.7753080752e-10
0.6	0.56464247339503535720	0.56464247215636288679	1.23867247041e-9
0.7	0.64421768723769105367	0.64421768559075318406	1.64693786961e-9
0.8	0.71735609089952276163	0.71735608886554084105	2.03398192058e-9
0.9	0.78332690962748338846	0.78332690719964676836	2.4278366201e-9
1.0	0.84147098480789650665	0.84147098203758297041	2.77031353624e-9

5. Conclusion

A successful application of a set of newly constructed polynomials has been demonstrated with the recurrence relation generated. The formulated scheme is desirable owing to the fact that it is convergent, accurate, efficient and compared favourable well with existing methods. The scheme is therefore recommended as second order initial value problems integrator.

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