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ON EQUIVALENCE OF MODIFIED TRIGONOMETRIC SUMS

Karanvir Singh[1 §], Kanak Modi[2]

[1]Department of Applied Mathematics
GZS Campus College of Engineering and Technology
Maharaja Ranjit Singh Punjab Technical University
Bathinda, Punjab, INDIA

[2]Department of Mathematics
Amity University of Rajasthan
Jaipur, INDIA

Abstract: We establish L^1-convergence equivalence of modified sums introduced by Rees and Stanojević and Kumari and Ram. It is shown that all the results regarding integrability and L^1-convergence of cosine series (or sine series) which have been established by different authors so far by using modified cosine sums or sine sums of Rees and Stanojević can also be proved by considering the corresponding sums introduced by Kumari and Ram under same classes of coefficients. We also introduce modified cosine and sine sums and compare them with modified sums introduced by Kaur and Bhatia.

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1. Introduction

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{1.1}$$

be the cosine series.

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§Correspondence author

Let the partial sum of (1.1) be denoted by $S_n(x)$ and $f(x) = \lim_{n\to\infty} S_n(x)$. There are many classes of coefficients introduced by different authors to study the integrability and L^1-convergence of trigonometric series . Historically the work on integrability and L^1-convergence on trigonometric series was initiated by Young [30] and Kolmogorov [17] by taking the classes of convex sequences and quasi-convex sequences respectively after Riesz ([1], Vol.II, Ch.VIII§ 22) gave an example to show that the Fourier series of a function f may not converge to f in L^1- metric. Sidon [25] proved the integrability of cosine series by considering a condition which is weaker than quasi convex and is described by Telyakovskii [26] as

Definition [25, 26]. A sequence $\{a_k\}$ is said to belong to class S if $a_k = o(1)$, $k \to \infty$ and there exists a sequence $\{A_k\}$ such that (a) $A_k \downarrow 0, k \to \infty$ (b) $\sum_{k=0}^{\infty} A_k < \infty$ (c) $|\triangle a_k| \leq A_k \quad for \quad all \quad k.$

Though above mentioned authors considered different classes of coefficients but they proved the same necessary and sufficient condition for L^1-convergence of cosine series which is as follows

$a_n \log n = o(1)$**,** $n \to \infty$ **iff** $\|f - S_n\| = o(1),\ n \to \infty.$ **(*)**

Many other classes with modifications and generalisations of above mentioned classes have been introduced to study integrability and L^1-convergence of trigonometric series by other authors like Fomin [8], Moricz [19] Tomovski [28] and Tikhonov [27] . But the result (*) could not be modified under all classes of coefficients instead their findings confirmed the result.

So authors start introducing modified cosine sums for studying L^1-convergence of (1.1) as these approximate their limits better than classical partial sums. In the literature so far , many authors like Rees and Č.V. Stanojevič [21], Kumari and Ram [18], K. Kaur, Bhatia and Ram [12] , J. Kaur [16], Braha [5] and Krasniqi [13] introduced modified trigonometric sums and studied their integrability and L^1-convergence under various classes.

We will consider only modified cosine sums as results on modified sine sums can be interpreted on similar lines.

Rees and Stanojević [21] introduced following modified cosine sums

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} (\triangle a_j) \cos kx$$

and it is also studied by many authors like Garett and Stanojević [9, 10, 11], Ram [20], Bor [2], Zahid and Hasan [31] and Singh and Sharma [22, 23].

Garett and Stanojević by considering the class of bounded variation and class **C** [9] proved the following result;

Theorem A. [9] If $\{a_k\}$ belongs to class **C** and is of bounded variation, then $\|f - g_n\| = o(1),\ n \to \infty$.

Ram proved the following theorem by considering the class S.
Theorem B. [20] If $\{a_k\}$ belongs to class S, then $\|f - g_n\| = o(1),\ n \to \infty$.

Singh and Sharma proved the following two results under two different classes of coefficients.
Theorem C. [22] If $\{a_k\}$ is null generalised quasi-convex sequence, then $\|f - g_n\| = o(1),\ n \to \infty$.
Theorem D. [23] If $\{a_k\}$ belongs to class $S^{'}$, then $\|f - g_n\| = o(1),\ n \to \infty$.

From above results , it is clear that authors proved the results without considering the condition $a_n \log n = o(1), \quad n \to \infty$.

Kumari and Ram [18] introduced modified cosine sums as

$$h_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle \left(\frac{a_j}{j}\right) k \cos kx$$

and studied their L^1-convergence under the condition that the coefficient sequences $\{a_k\}$ belong to the class S . They also deduced the results about L^1-convergence of cosine and sine series as corollaries. They proved the following theorem
Theorem E. [18] If $\{a_k\}$ belongs to class S, then
$\|f - h_n\| = o(1),\ n \to \infty$, if and only if $a_n \log n = o(1), \quad n \to \infty$.

Bor by considering the class $S(\delta)$[see [3]] proved the following result.
Theorem F. [3] If $\{a_k\}$ belongs to class $S(\delta)$, then
$\|f - h_n\| = o(1),\ n \to \infty$, if and only if $a_n \log n = o(1), \quad n \to \infty$.

Many other authors like Krasniqi [13] and Kaur and Bhatia [14] also studied these modified sums. Bhatia and Ram [4] studied the complex form of these sums.

From above mentioned results, it is clear that that most of the results regarding L^1-convergence of cosine series are proved without the condition $a_n \log n = o(1), \quad n \to \infty$ if $g_n(x)$ is considered . But this condition has been assumed when modified sums $h_n(x)$ are considered as evident from Theorem E and Theorem F.

Our claim in this paper is that all the results which have been proved by considering $g_n(x)$ are also true for $h_n(x)$ as far as integrability and L^1-convergence of cosine series (or sine series) is concerned irrespective of the consideration of classes. In support of our claim we will prove the follwing theorems.

Theorem 1. If $\{a_k\}$ belongs to class S, then $\|f - h_n\| = o(1),\ n \to \infty$.

This Theorem 1 is modification of Theorem E in which we will remove the condition $a_n \log n = o(1), \quad n \to \infty$, while proving the result.

Secondly we prove the result which establishes the L^1- convergence equivalence of two modified sums $g_n(x)$ and $h_n(x)$ which means all the results which have been proved by taking $g_n(x)$ [Theorem A, Theorem B, Theorem C, Theorem D] can also be proved by taking $h_n(x)$ and vice versa.

Theorem 2. Let $\Im$ be any sub class of coefficients of (1.1) and if $\{a_k\}$ belongs to class $\Im$, then $\|f - g_n\| = o(1) \Leftrightarrow \|f - h_n\| = o(1)\ as\ n \to \infty$.

Remark. When we say that $\Im$ be any sub class of coefficients of (1.1) and if $\{a_k\}$ belongs to class $\Im$, we mean that $\{a_k\}$ is null sequence convex or quasi-convex or belongs to class **S** of Sidon or any other class of coefficients for which $\|f - g_n\| = o(1)$ or $\|f - h_n\| = o(1) \quad n \to \infty$, is true.

Thirdly we consider rth derivative of $g_n(x)$ and $h_n(x)$ and establish the relationship between them concerning L^1-convergence of r-times differentiated cosine series which in turn will modify the results in this context like [14]. Moreover all the results in which the condition $n^r a_n \log n = o(1), \quad n \to \infty$, is necessary and sufficient for $\|f^r - g_n^r\| = o(1)$ or $\|f^r - h_n^r\| = o(1)\ as\ n \to \infty$ can be modified as per Theorem 3 below.

Theorem 3. Let $\Im$ be any sub class of coefficients of (1.1) and if $\{a_k\}$ belongs to class $\Im$, then $\|f^r - g_n^r\| = O(n^r)\ or\ o(1) \Rightarrow \|f^r - h_n^r\| = o(1)\ as\ a_n n^r \to 0,\ n \to \infty$ and $\|f^r - h_n^r\| = O(n^r)\ or\ o(1) \Rightarrow \|f^r - g_n^r\| = o(1)\ as\ a_n n^r \to 0,\ n \to \infty$.

2. Lemmas

We require the following lemmas for the proof of our results.

2.1. Lemma [7]

If $|a_k| \leq 1$, then

$\int\limits_0^\pi \left| \sum\limits_{k=0}^{n} a_k D_k(x) \right| dx \leq C(n+1)$, where C is a positive absolute constant.

2.2. Lemma [1, 32]

The results mentioned in this Lemma are well known.
If $D_n(x)$ and $\tilde{D}_n(x)$ are Dirichlet and conjugate Dirichlet kernels respectively and are defined by

$$D_n(x) = \frac{\sin\left(n + \frac{1}{2}\right)x}{2\sin\frac{x}{2}}, \quad \tilde{D}_n(x) = \frac{\cos\frac{x}{2} - \cos\left(n + \frac{1}{2}\right)x}{2\sin\frac{x}{2}},$$

then as per [24]

(i) $\|D_n^r(x)\| = \frac{4}{\pi}(n^r \log n) + O(n^r)$, $r = 0, 1, 2, 3...$, where $D_n^r(x)$ represent r-th derivative of the Dirichlet kernel.

(ii) $\|\tilde{D}_n^r(x)\| = O(n^r \log n)$, $r = 0, 1, 2, 3....$

Again if $K_n(x)$ denotes Fezér kernel defined by

$K_n(x) = \dfrac{1}{n+1} \sum\limits_{j=0}^{n} D_j(x)$, then

(a) (i)$\tilde{D}_n^{'}(x) = (n+1)D_n(x) - (n+1)K_n(x)$

(ii)$\tilde{D}_n^{r+1}(x) = (n+1)D_n^r(x) - (n+1)K_n^r(x)$

(b)(i)$\|K_n(x)\| = o(1)$ (ii) $\|K_n^r(x)\| = O(n^r),\ r = 0, 1, 2, 3....$

3. Main Results

Now we shall prove our main results.
Proof of Theorem 1. We have

$$h_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle\left(\frac{a_j}{j}\right) k\cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx\left[\triangle\left(\frac{a_k}{k}\right) + \triangle\left(\frac{a_{k+1}}{k+1}\right) + \cdots + \triangle\left(\frac{a_n}{n}\right)\right]$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} k\cos kx\left(\frac{a_k}{k} - \frac{a_{n+1}}{n+1}\right)$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos kx - \frac{a_{n+1}}{n+1}\sum_{k=1}^{n} k\cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos kx - \frac{a_{n+1}}{n+1}\tilde{D}_n'(x)$$

Applying Abel's transformation, we have

$$h_n(x) = \sum_{k=0}^{n} \triangle a_k D_k(x) + a_{n+1}D_n(x) - \frac{a_{n+1}}{n+1}\tilde{D}_n'(x)$$

$$= \sum_{k=0}^{n} \triangle a_k D_k(x) + a_{n+1}K_n(x), \quad \text{by Lemma (2.2).}$$

Now making use of Abel's transformation and Lemma (2.1), we have

$$\int_0^\pi |f(x) - h_n(x)|\,dx$$

$$\le \int_0^\pi \left|\sum_{k=n+1}^{\infty} \triangle a_k D_k(x)\right| dx + |a_{n+1}|\int_0^\pi |K_n(x)|\,dx$$

$$= \int_0^\pi \left|\sum_{k=n+1}^{\infty} A_k\frac{\triangle a_k}{A_k} D_k(x)\right| dx + |a_{n+1}|\int_0^\pi |K_n(x)|\,dx$$

$$\leq \int_0^\pi \left| \sum_{k=n+1}^{\infty} \triangle A_k \sum_{j=0}^{k} \frac{\triangle a_j}{A_j} D_j(x) \right| dx + |a_{n+1}| \int_0^\pi |K_n(x)|\, dx$$

$$\leq C \sum_{k=n+1}^{\infty} (k+1) \triangle A_k + |a_{n+1}| \int_0^\pi |K_n(x)|\, dx$$

The first term converges as per hypothesis and for second term $\|K_n(x)\| = o(1)$ as per lemma 2.2
This completes the proof of the theorem.

Proof of Theorem 2. We have

$$g_n(x) = \frac{1}{2} \sum_{k=0}^{n} \triangle a_k + \sum_{k=1}^{n} \sum_{j=k}^{n} (\triangle a_j) \cos kx$$

$$= S_n(x) - a_{n+1} D_n(x)$$

and also we have

$$h_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle \left(\frac{a_j}{j} \right) k \cos kx$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}}{n+1} \tilde{D}_n'(x)$$

$$= S_n(x) - \frac{a_{n+1}}{n+1} \tilde{D}_n'(x)$$

Therefore we can write

$$g_n(x) - h_n(x) = -a_{n+1} D_n(x) + \frac{a_{n+1}}{n+1} \tilde{D}_n'(x)$$

$$= -a_{n+1} K_n(x) \text{ , by using Lemma 2.2}$$

Now Suppose $\|f - g_n\| = o(1)$, $n \to \infty$.

Then

$$\|f - h_n\| = \|f - g_n - a_{n+1} K_n(x)\|$$

$$\leq \|f - g_n\| + |a_{n+1}| \|K_n\|$$

$$= o(1)\ ,\ n \to \infty.$$

Similarly converse is obvious.

Indeed $\|g_n(x) - h_n(x)\| = |a_{n+1}|\|K_n(x)\| = o(1)\ ,\ n \to \infty.$

This completes the proof of Theorem 2.

Proof of Theorem 3. We have

$$g_n^r(x) = S_n^r(x) - a_{n+1} D_n^r(x)$$

and also

$$h_n^r(x) = S_n^r(x) - \frac{a_{n+1}}{n+1} \tilde{D}_n^{r+1}(x)$$

Clearly by using Lemma 2.2
$g_n^r(x) - h_n^r(x) = -a_{n+1} K_n^r(x)$ which implies

$$\|g_n^r(x) - h_n^r(x)\| = |a_{n+1}|\|K_n^r(x)\|$$

Now proceeding as in Theorem 2 and using Lemma 2 [b(ii)], we get the result.

4. New Modified sums

We introduce the following cosine and sine sums

$$u_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle \left(\frac{a_j \cos jx}{2^j} \right) 2^k$$

$$v_n(x) = \sum_{k=1}^{n} \sum_{j=k}^{n} \triangle \left(\frac{a_j \sin jx}{2^j} \right) 2^k$$

We also have $u_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle \left(\frac{a_j \cos jx}{2^j}\right) 2^k$

$$= \frac{a_0}{2} + \sum_{k=1}^{n}\left(\frac{a_k \cos kx}{2^k} - \frac{a_{n+1}\cos(n+1)x}{2^{n+1}}\right)2^k$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}\cos(n+1)x}{2^{n+1}}\sum_{k=1}^{n} 2^k$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx - \frac{a_{n+1}\cos(n+1)x}{2^{n+1}} 2(2^n - 1)$$

$$= S_n(x) - a_{n+1}\cos(n+1)x + \frac{a_{n+1}\cos(n+1)x}{2^n} \qquad (**)$$

Clearly $f(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} S_n(x)$ as $\{a_k\}$ is null sequence.

Now if we compare these modified sums with the modified sums introduced by Jkaur and Bhatia [16] , then it is clear that the results proved in [16] can be established in much easier way by taking our modified sums. Even under other different classes of coefficients, say, under the class **S**, we can prove the following result:

Theorem 4.1. If $\{a_k\}$ is convex null sequence (or quasi-convex or belongs to class **S** of Sidon). Then $\|f - u_n\| = o(1), \quad n \to \infty$, iff $a_n \log n = o(1), \quad n \to \infty$

Proof. From (**), result can be proved easily as the 2nd and 3rd terms in (**) simply goes to zero as $n \to \infty$.

Remark.. The above modified sums can also be written in generalised form as;

$$p_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle \left(\frac{a_j \cos jx}{c^j}\right) c^k$$

$$q_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \triangle \left(\frac{a_j \sin jx}{c^j}\right) c^k$$

where c is constant >1

Then the result (**) reduces to

$$p_n(x) = S_n(x) - a_{n+1}\cos(n+1)x + \frac{a_{n+1}\cos(n+1)x}{c^n}$$

and consequently this generalised form can also be considered in proving the results of Theorem 4.1.

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