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A NOTE ON BICOMPLEX FIBONACCI AND LUCAS NUMBERS

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Abstract: In this study, we define a new type of Fibonacci and Lucas numbers which are called bicomplex Fibonacci and bicomplex Lucas numbers. We obtain the well-known properties e.g. D'ocagnes, Cassini, Catalan for these new types. We also give the identities of negabicomplex Fibonacci and negabicomplex Lucas numbers, Binet formulas and relations of them.

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1. Introduction

Fibonacci numbers are invented by Italian mathematician Leonardo Fibonacci when he wrote his first book Liber Abaci in 1202 contains many elementary problems, including famous rabbit problem.

The Fibonacci numbers are the numbers of the following integer sequence;

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...$$

The sequence of Fibonacci numbers which is denoted by F_n is defined as the

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linear reccurence relation

$$F_n = F_{n-1} + F_{n-2}$$

with $F_1 = F_2 = 1$ and $n \in \mathbb{Z}$. Also as a result of this relation we can define $F_0 = 0$. Fibonacci numbers are connected with the golden ratio as; the ratio of two consecutive Fibonacci numbers approximates the golden ratio $1,61803399....$

Fibonacci numbers are closely related to Lucas numbers which are named after the mathematician Francois Edouard Anatole Lucas who worked on both Fibonacci and Lucas numbers. The integer sequence of Lucas numbers denoted by L_n is given by

$$2, 1, 3, 4, 7, 11, 18, 29, 47, ...$$

with the same reccurence relation

$$L_n = L_{n-1} + L_{n-2}.$$

where $L_0 = 2$, $L_1 = 1$, $L_2 = 3$ and $n \in \mathbb{Z}$.

There are many works on Fibonacci and Lucas numbers in literature. The properties, relations, results between Fibonacci and Lucas numbers can be found in, Dunlap[3], Güven and Nurkan[5], Koshy[14], Vajda[21], Verner and Hoggatt[22]. Also Fibonacci and Lucas quaternions were described by Horadam in [10], then many studies related with these quaternions were done in Akyiğit [1], Halici [6], Iyer [11], Nurkan[16], Swamy [20].

Corrado Segre introduced bicomplex numbers in 1892 [19]. The bicomplex numbers are a type of complex Clifford algebra, one of several possible generalizations of the ordinary complex numbers. The complex numbers denoted as $\mathbb{C}$ are defined by

$$\mathbb{C} = \left\{ a + bi \;\mid\; a, b \in \mathbb{R} \text{ and } i^2 = -1 \right\}.$$

The bicomplex numbers are defined by

$$\mathbb{C}_2 = \left\{ z_1 + z_2 j \;\mid\; z_1, z_2 \in \mathbb{C} \text{ and } j^2 = -1 \right\}.$$

Since z_1 and z_2 are complex numbers, writing $z_1 = a + bi$ and $z_2 = c + di$ gives us another way to represent the bicomplex numbers as;

$$z_1 + z_2 j = a + bi + cj + dij.$$

where $a, b, c, d \in \mathbb{R}$. Thus the set of bicomplex numbers can be expressed by

$$\mathbb{C}_2 = \left\{ \begin{array}{c} a + bi + cj + dij \mid\; a, b, c, d \in \mathbb{R} \text{ and} \\ i^2 = -1, j^2 = -1,\; ij = ji = k \end{array} \right\}.$$

For any bicomplex numbers $x = a_1 + b_1 i + c_1 j + d_1 ij$ and $y = a_2 + b_2 i + c_2 j + d_2 ij$, the addition and multiplication of these bicomplex numbers are given respectively by

$$x + y = (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)ij \tag{1.1}$$

and

$$\begin{aligned} x \times y = {} & (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 - c_1d_2 - d_1c_2)i \\ & + (a_1c_2 + c_1a_2 - b_1d_2 - d_1b_2)j + (a_1d_2 + d_1a_2 + b_1c_2 + c_1b_2)ij. \end{aligned} \tag{1.2}$$

The multiplication of a bicomplex number by a real scalar λ is given by

$$\lambda x = \lambda a_1 + \lambda b_1 i + \lambda c_1 j + \lambda d_1 ij.$$

The set $\mathbb{C}_2$ forms a commutative ring with addition and multiplication and it is a real vector space with addition and scalar multiplication [17].

In $\mathbb{C}$, the complex conjugate of $z = a + bi$ is $\overline{z} = a - bi$. In $\mathbb{C}_2$, for a bicomplex number $x = (a_1 + b_1 i) + (c_1 + d_1 i)j$, there are different conjugations which are;

$$\begin{aligned} x^\star &= [(a_1 + b_1 i) + (c_1 + d_1 i)j]^\star = (a_1 - b_1 i) + (c_1 - d_1 i)j \\ x^\circ &= [(a_1 + b_1 i) + (c_1 + d_1 i)j]^\circ = (a_1 + b_1 i) - (c_1 + d_1 i)j \\ x^\dagger &= [(a_1 + b_1 i) + (c_1 + d_1 i)j]^\dagger = (a_1 - b_1 i) - (c_1 - d_1 i)j \end{aligned} \tag{1.3}$$

[18]. Then the following equation are written in [12];

$$\begin{aligned} x \times x^\star &= (a_1^2 + b_1^2 - c_1^2 - d_1^2) + 2(a_1c_1 + b_1d_1)j \\ x \times x^\circ &= (a_1^2 - b_1^2 + c_1^2 - d_1^2) + 2(a_1b_1 + c_1d_1)i \\ x \times x^\dagger &= (a_1^2 + b_1^2 + c_1^2 + d_1^2) + 2(a_1d_1 - b_1c_1)ij. \end{aligned} \tag{1.4}$$

Also in [18], the modulus of a bicomplex number x is defined by

$$\begin{aligned} |x|_i &= \sqrt{x \times x^\star} \\ |x|_j &= \sqrt{x \times x^\circ} \\ |x|_k &= \sqrt{x \times x^\dagger} \\ |x| &= \sqrt{\mathrm{Re}(x \times x^\dagger)} \end{aligned} \tag{1.5}$$

where the names are $i-$modulus, $j-$modulus, $k-$modulus and real modulus, respectively.

Rochon and Shapiro gave the detailed algebraic properties of bicomplex and hyperbolic numbers in [18]. The generalized bicomplex numbers were defined by Karakuş and Aksoyak in [12], they gave some algebratic properties of them and used generalized bicomplex number product to show that $\mathbb{R}^4$ and $\mathbb{R}^4_2$ were Lie groups. Also Luna-Elizarraras *et al* [15], introduced the algebra of bicomplex numbers and described how to define elementary functions and their inverse functions in that algebra.

In this paper, we define bicomplex Fibonacci and bicomplex Lucas numbers by combining bicomplex numbers and Fibonacci, Lucas numbers. We give some identities and Binet formulas of these new numbers.

2. Bicomplex Fibonacci Numbers

Definition 1: The *bicomplex Fibonacci* and *bicomplex Lucas numbers* are defined respectively by

$$BF_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k \tag{2.1}$$

and

$$BL_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k \tag{2.2}$$

where F_n is the n^{th} Fibonacci number, L_n is the n^{th} Lucas number and i, j, k are bicomplex units which satisfy the commutative multiplication rules:

$$\begin{aligned} &i^2 = -1,\ j^2 = -1,\ k^2 = 1 \\ &ij = ji = k,\ jk = kj = -i,\ ik = ki = -j. \end{aligned}$$

Starting from $n = 0$, the bicomplex Fibonacci and bicomplex Lucas numbers can be written respectively as;

$$BF_0 = 1i + 1j + 2k,\quad BF_1 = 1 + 1i + 2j + 3k\ ,\ \ BF_2 = 1 + 2i + 3j + 5k, ...$$

$$BL_0 = 2 + 1i + 3j + 4k\ ,\ BL_1 = 1 + 3i + 4j + 7k\ ,\ BL_2 = 3 + 4i + 7j + 11k, ...$$

Let $BF_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$ and $BF_m = F_m + F_{m+1}i + F_{m+2}j + F_{m+3}k$ be two bicomplex Fibonacci numbers. By taking into account the equations (1.1) and (1.2), the addition, subtraction and multiplication of these numbers are given by

$$BF_n \pm BF_m \;=\; (F_n \pm F_m) + (F_{n+1} \pm F_{m+1})\, i + (F_{n+2} \pm F_{m+2})\, j$$

$$+ (F_{n+3} \pm F_{m+3})\, k$$

and

$$\begin{aligned} BF_n \times BF_m = & (F_n F_m - F_{n+1} F_{m+1} - F_{n+2} F_{m+2} + F_{n+3} F_{m+3}) \\ & + (F_n F_{m+1} + F_{n+1} F_m - F_{n+2} F_{m+3} + F_{n+3} F_{m+2})\, i \\ & + (F_n F_{m+2} + F_{n+2} F_m - F_{n+1} F_{m+3} - F_{n+3} F_{m+1})\, j \\ & + (F_n F_{m+3} + F_{n+3} F_m + F_{n+1} F_{m+2} + F_{n+2} F_{m+1})\, k. \end{aligned}$$

Definition 2: A bicomplex Fibonacci number can also be expressed as $BF_n = (F_n + F_{n+1} i) + (F_{n+2} + F_{n+3} i)\, j$. In that case,there is three different *conjugations* with respect to i, j and k; for bicomplex Fibonacci numbers as follows:

$$\begin{aligned} \overline{BF_n}^{i} &= \left[(F_n + F_{n+1} i) + (F_{n+2} + F_{n+3} i)\, j\right]^{i} \\ &= (F_n - F_{n+1} i) + (F_{n+2} - F_{n+3} i)\, j \\ \overline{BF_n}^{j} &= \left[(F_n + F_{n+1} i) + (F_{n+2} + F_{n+3} i)\, j\right]^{j} \\ &= (F_n + F_{n+1} i) - (F_{n+2} + F_{n+3} i)\, j \\ \overline{BF_n}^{k} &= \left[(F_n + F_{n+1} i) + (F_{n+2} + F_{n+3} i)\, j\right]^{k} \\ &= (F_n - F_{n+1} i) - (F_{n+2} - F_{n+3} i)\, j. \end{aligned}$$

By the Definition 2 and the equations (1.4) and (2.1), we can write

$$\begin{aligned} BF_n \times \overline{BF_n}^{i} &= (F_{2n+1} - F_{2n+7}) + 2\,(F_{2n+3})\, j \\ BF_n \times \overline{BF_n}^{j} &= \left(F_n^2 - F_{n+1}^2 + F_{n+3}^2 - F_{n+4}^2\right) + 2\,(F_n F_{n+1} + F_{n+2} F_{n+3})\, i \\ BF_n \times \overline{BF_n}^{k} &= (F_{2n+1} + F_{2n+7}) + 2\,(-1)^{n+1}\, k. \end{aligned}$$

Definition 3: Let a bicomplex Fibonacci number be $BF_n = F_n + F_{n+1} i + F_{n+2} j + F_{n+3} k$. The $i-$*modulus*, $j-$*modulus*, $k-$*modulus* and *real modulus* of BF_n are given respectively as;

$$\begin{aligned} |BF_n|_i &= \sqrt{BF_n \times \overline{BF_n}^{i}} \\ |BF_n|_j &= \sqrt{BF_n \times \overline{BF_n}^{j}} \\ |BF_n|_k &= \sqrt{BF_n \times \overline{BF_n}^{k}} \\ |BF_n| &= \sqrt{F_{2n+1} + F_{2n+7}} \end{aligned}$$

In [16], we proved some summation equations by supposing, for $i \geq 0$

$$\sum_{m=0}^{M} \alpha_m F_{m+i} + \sum_{m=0}^{M} \beta_m L_{m+i} = 0 \tag{2.3}$$

where α_m and β_m are fixed numbers.

By using the equations (2.1), (2.2), (2.3) and taking $i = 0, 1, 2, 3,$ we can write the following equation clearly;

$$\sum_{m=0}^{M} \alpha_m BF_m + \sum_{m=0}^{M} \beta_m BL_m = 0 \tag{2.4}$$

Now let us give theorems which give some properties.

Theorem 1. *Let BF_n and BL_n be a bicomplex Fibonacci and a bicomplex Lucas number, respectively. For $n \geq 0$, the following relations hold:*

1) $BF_n + BF_{n+1} = BF_{n+2}$
2) $BL_n + BL_{n+1} = BL_{n+2}$
3) $BL_n = BF_{n-1} + BF_{n+1}$
4) $BL_n = BF_{n+2} - BF_{n-2}$
5) $BF_n^2 + BF_{n+1}^2 = BF_{2n+1} + F_{2n+2} + F_{2n+5} - 3iF_{2n+5} - jF_{2n+6} + 3kF_{2n+4}$
6) $BF_{n+1}^2 - BF_{n-1}^2 = 2BF_{2n} + F_{2n+4} + F_{2n-1} + 2\left(-F_{2n+5}i - F_{2n+4}j + F_{2n+3}k\right)$
7) $BF_nBF_m + BF_{n+1}BF_{m+1} = 2BF_{n+m+1} + 2F_{n+m+4} - F_{n+m+1} - 2F_{n+m+6}i - 2F_{n+m+5}j + 2F_{n+m+4}k$
8) $BF_n - BF_{n+1}i + BF_{n+2}j - BF_{n+3}k = -5F_{n+3}$

Proof. By using the equations of (2.1), (2.2), (2.4) and taking appropriate fixed numbers in the equation (2.4), the proofs of 1), 2), 3), 4) are clear.

From the definition of Fibonacci number , the bicomplex Fibonacci number in (2.1) , the equations $F_n^2 + F_{n+1}^2 = F_{2n+1}$, $F_{n+1}^2 - F_{n-1}^2 = F_{2n}$ and $F_nF_m + F_{n+1}F_{m+1} = F_{n+m+1}$ (see Vajda [21]), we get

$$\begin{aligned} BF_n^2 + BF_{n+1}^2 &= \left(F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k\right)^2 \\ &\quad + \left(F_{n+1} + F_{n+2}i + F_{n+3}j + F_{n+4}k\right)^2 \\ &= \begin{bmatrix} F_{2n+3} + 2\left(F_nF_{n+1} - F_{n+2}F_{n+3}\right)i \\ +2\left(F_nF_{n+2} - F_{n+1}F_{n+3}\right)j \\ +2\left(F_nF_{n+3} + F_{n+1}F_{n+2}\right)k \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&+\left[\begin{array}{c} F_{2n+5}+2\left(F_{n+1}F_{n+2}-F_{n+3}F_{n+4}\right)i \\ +2\left(F_{n+1}F_{n+3}-F_{n+2}F_{n+4}\right)j \\ +2\left(F_{n+1}F_{n+4}+F_{n+2}F_{n+3}\right)k \end{array}\right] \\
&= F_{2n+1}+F_{2n+2}i+F_{2n+3}j+F_{2n+4}k+F_{2n+2} \\
&+ F_{2n+5}-3F_{2n+5}i-F_{2n+6}j+3F_{2n+4}k \\
&= BF_{2n+1}+F_{2n+2}+F_{2n+5}-3iF_{2n+5} \\
&- jF_{2n+6}+3kF_{2n+4}.
\end{aligned}
$$

$$
\begin{aligned}
BF_{n+1}^2-BF_{n-1}^2 &= \left(F_{n+1}+F_{n+2}i+F_{n+3}j+F_{n+4}k\right)^2 \\
&- \left(F_{n-1}+F_{n}i+F_{n+1}j+F_{n+2}k\right)^2 \\
&= \left[\begin{array}{c} F_{2n+5}+2\left(F_{n+1}F_{n+2}-F_{n+3}F_{n+4}\right)i \\ +2\left(F_{n+1}F_{n+3}-F_{n+2}F_{n+4}\right)j \\ +2\left(F_{n+1}F_{n+4}+F_{n+2}F_{n+3}\right)k \end{array}\right] \\
&- \left[\begin{array}{c} F_{2n+1}+2\left(F_{n-1}F_{n}-F_{n+1}F_{n+2}\right)i \\ +2\left(F_{n-1}F_{n+1}-F_{n}F_{n+2}\right)j \\ +2\left(F_{n-1}F_{n+2}+F_{n}F_{n+1}\right)k \end{array}\right] \\
&= 2BF_{2n}+F_{2n+4}+F_{2n-1} \\
&+ 2\left(-F_{2n+5}i-F_{2n+4}j+F_{2n+3}k\right)
\end{aligned}
$$

$$
\begin{aligned}
BF_nBF_m+BF_{n+1}BF_{m+1} &= \left(F_n+F_{n+1}i+F_{n+2}j+F_{n+3}k\right) \\
&.\left(F_m+F_{m+1}i+F_{m+2}j+F_{m+3}k\right) \\
&+ \left(F_{n+1}+F_{n+2}i+F_{n+3}j+F_{n+4}k\right) \\
&.\left(F_{m+1}+F_{m+2}i+F_{m+3}j+F_{m+4}k\right) \\
&= 2\left(F_{n+m+1}+F_{n+m+2}i\right) \\
&+ 2\left(F_{n+m+3}j+F_{n+m+4}k\right) \\
&+ 2F_{n+m+4}-F_{n+m+1}-2F_{n+m+6}i \\
&- 2F_{n+m+5}j+2F_{n+m+4}k \\
&= 2BF_{n+m+1}+2F_{n+m+4}-F_{n+m+1} \\
&- 2F_{n+m+6}i-2F_{n+m+5}j+2F_{n+m+4}k
\end{aligned}
$$

and

$$
BF_n-BF_{n+1}i+BF_{n+2}j-BF_{n+3}k = \left(F_n+F_{n+1}i+F_{n+2}j+F_{n+3}k\right)
$$

$$
\begin{aligned}
&-\left(F_{n+1}+F_{n+2}i+F_{n+3}j+F_{n+4}k\right)i \\
&+\left(F_{n+2}+F_{n+3}i+F_{n+4}j+F_{n+5}k\right)j \\
&-\left(F_{n+3}+F_{n+4}i+F_{n+5}j+F_{n+6}k\right)k \\
&=F_n+F_{n+2}-F_{n+4}-F_{n+6} \\
&=-5F_{n+3}.
\end{aligned}
$$

□

Theorem 2. *For $n, m \geq 0$ the D'ocagnes identity for bicomplex Fibonacci numbers BF_n and BF_m is given by*

$$
\begin{aligned}
BF_mBF_{n+1}-BF_{m+1}BF_n &= (-1)^n BF_{m-n} \\
&+(-1)^{n+1}\begin{pmatrix} F_{m-n}+F_{m-n+1}i \\ -\left(F_{m-n-2}+2F_{m-n+2}\right)j \\ +2F_{m-n-1}k \end{pmatrix}.
\end{aligned}
$$

Proof. If we decide the equation (2.1) and the D'ocagnes identity for Fibonacci numbers $F_mF_{n+1}-F_{m+1}F_n=(-1)^n F_{m-n}$ (see Weisstein [23]), we obtain the following calculations as;

$$
\begin{aligned}
BF_mBF_{n+1}-BF_{m+1}BF_n &= \left(F_m+F_{m+1}i+F_{m+2}j+F_{m+3}k\right) \\
&\left(F_{n+1}+F_{n+2}i+F_{n+3}j+F_{n+4}k\right) \\
&-\left(F_{m+1}+F_{m+2}i+F_{m+3}j+F_{m+4}k\right) \\
&\left(F_n+F_{n+1}i+F_{n+2}j+F_{n+3}k\right) \\
&=\left[F_{n+2}\left(F_m+F_{m+4}\right)-F_{m+2}\left(F_n+F_{n+4}\right)\right]i \\
&+\left[2(-1)^n\left(F_{m-n-2}+F_{m-n+2}\right)\right]j \\
&+\left[(-1)^n\left(F_{m-n-2}+F_{m-n+2}\right)\right]k \\
&+(-1)^n BF_{m-n}-(-1)^n BF_{m-n} \\
&=(-1)^n BF_{m-n} \\
&+(-1)^{n+1}\begin{pmatrix} F_{m-n}+F_{m-n+1}i \\ -\left(F_{m-n-2}+2F_{m-n+2}\right)j \\ +2F_{m-n-1}k \end{pmatrix}.
\end{aligned}
$$

□

Theorem 3. *If BF_n and BL_n are bicomplex Fibonacci and bicomplex Lucas numbers respectively, then for $n \geq 0$, the identities of negabicomplex Fibonacci and negabicomplex Lucas numbers are*

$$
BF_{-n}=(-1)^{n+1} BF_n+(-1)^n L_n(i+j+2k)
$$

and

$$BL_{-n} = (-1)^n BL_n + (-1)^{n+1} 5F_n (i + j + 2k)$$

Proof. From the equations (2.1) and the identity of negafibonacci numbers which is $F_{-n} = (-1)^{n+1} F_n$ (see Knuth [13] , Dunlap [3]), we have

$$\begin{aligned} BF_{-n} &= F_{-n} + F_{-n+1}i + F_{-n+2}j + F_{-n+3}k \\ &= F_{-n} + F_{-(n-1)}i + F_{-(n-2)}j + F_{-(n-3)}k \\ &= (-1)^{n+1} F_n + (-1)^n F_{n-1}i + (-1)^{n+1} F_{n-2}j + (-1)^n F_{n-3}k \\ &= (-1)^{n+1} (F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k) - (-1)^{n+1} F_{n+1}i \\ &- (-1)^{n+1} F_{n+2}j - (-1)^{n+1} F_{n+3}k + (-1)^n F_{n-1}i \\ &+ (-1)^{n+1} F_{n-2}j + (-1)^n F_{n-3}k \\ &= (-1)^{n+1} BF_n + (-1)^n (F_{n+1} + F_{n-1})\, i \\ &+ (-1)^n (F_{n+2} - F_{n-2})\, j + (-1)^n (F_{n+3} + F_{n-3})\, k \end{aligned}$$

In this equation , we take into account that $F_{n-1} + F_{n+1} = L_n$, $F_{n+2} - F_{n-2} = L_n$, $F_{n+3} + F_{n-3} = 2L_n$(see Vajda [21]) and the identity of negalucas numbers $L_{-n} = (-1)^n L_n$ (see Knuth [13] ,Dunlap [3]), thus we obtain;

$$\begin{aligned} BF_{-n} &= (-1)^{n+1} BF_n + (-1)^n L_n i + (-1)^n L_n j + (-1)^n 2L_n \\ &= (-1)^{n+1} BF_n + (-1)^n L_n (i + j + 2k) \end{aligned}$$

Now by using the equation (2.2) and the identity of negalucas numbers, we get

$$\begin{aligned} BL_{-n} &= L_{-n} + L_{-n+1}i + L_{-n+2}j + L_{-n+3}k \\ &= L_{-n} + L_{-(n-1)}i + L_{-(n-2)}j + L_{-(n-3)}k \\ &= (-1)^n L_n + (-1)^{n-1} L_{n-1}i + (-1)^n L_{n-2}j + (-1)^{n-1} L_{n-3}k \\ &= (-1)^n (L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k) - (-1)^n L_{n+1}i \\ &- (-1)^n L_{n+2}j - (-1)^n L_{n+3}k + (-1)^{n-1} L_{n-1}i + (-1)^n L_{n-2}j \\ &+ (-1)^{n-1} L_{n-3}k \\ &= (-1)^n BL_n \\ &+ (-1)^{n+1} \begin{pmatrix} (L_{n+1} + L_{n-1})\, i \\ + (L_{n+2} - L_{n-2})\, j + (L_{n+3} + L_{n-3})\, k \end{pmatrix} \end{aligned}$$

Here if we use the identity $L_{m+n} + L_{m-n} = \begin{cases} 5F_m F_n, & \text{if } n \text{ is odd,} \\ L_m L_n, & \text{otherwise} \end{cases}$ (see Koshy, [14]), namely $L_{n-1} + L_{n+1} = 5F_n$, the definition of dual Lucas number and the identity of negafibonacci number in last equation, we complete the proof as;

$$\begin{aligned} BL_{-n} &= (-1)^n BL_n + (-1)^{n+1} 5F_n i + (-1)^{n+1} 5F_n j + (-1)^{n+1} 10F_n k \\ &= (-1)^n BL_n + (-1)^{n+1} 5F_n (i + j + 2k) \end{aligned}$$

□

The Binet formulas for Fibonacci and Lucas numbers are given by (see Koshy, [14])

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Theorem 4. *Let BF_n and BL_n be bicomplex Fibonacci and bicomplex Lucas numbers, respectively. For $n \geq 0$, the Binet formulas for these numbers are given as;*

$$BF_n = \frac{\overline{\alpha}\alpha^n - \overline{\beta}\beta^n}{\alpha - \beta}$$

and

$$BL_n = \overline{\alpha}\alpha^n + \overline{\beta}\beta^n$$

where $\overline{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3$ and $\overline{\beta} = 1 + i\beta + j\beta^2 + k\beta^3$.

Proof. By using the Binet formulas for Fibonacci and Lucas numbers, taking $\overline{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3$ and $\overline{\beta} = 1 + i\beta + j\beta^2 + k\beta^3$, we find the result clearly as;

$$\begin{aligned} BF_n &= F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k \\ &= \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}i + \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}j + \frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta} \\ &= \frac{\alpha^n \left(1 + i\alpha + j\alpha^2 + k\alpha^3\right) - \beta^n \left(1 + i\beta + j\beta^2 + k\beta^3\right)}{\alpha - \beta} \\ &= \frac{\overline{\alpha}\alpha^n - \overline{\beta}\beta^n}{\alpha - \beta} \end{aligned}$$

and

$$BL_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k$$

$$= \alpha^n + \beta^n + (\alpha^{n+1} + \beta^{n+1})i + (\alpha^{n+2} + \beta^{n+2})j$$
$$+ (\alpha^{n+3} + \beta^{n+3})k$$
$$= \alpha^n(1 + i\alpha + j\alpha^2 + k\alpha^3) + \beta^n(1 + i\beta + j\beta^2 + k\beta^3)$$
$$= \overline{\alpha}\alpha^n + \overline{\beta}\beta^n.$$

□

Theorem 5. *Let BF_n and BL_n be bicomplex Fibonacci and bicomplex Lucas numbers. For $n \geq 1$,the Cassini identities for BF_n and BL_n are given by;*

$$BF_{n+1}BF_{n-1} - BF_n^2 = 3\left(-1\right)^n\left(2j + k\right)$$

and

$$BL_{n+1}BL_{n-1} - BL_n^2 = 5(-1)^{n-1}\left(2j + k\right)$$

Proof. From the equation (2.1), we have

$$BF_{n+1}BF_{n-1} - BF_n^2 = (F_{n+1} + F_{n+2}i + F_{n+3}j + F_{n+4}k)$$
$$(F_{n-1} + F_n i + F_{n+1}j + F_{n+2}k)$$
$$- \left(F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k\right)^2.$$

If we use the identity $F_mF_{n+1} - F_{m+1}F_n = \left(-1\right)^n F_{m-n}$ (see Weisstein [23]) and $F_{-n} = \left(-1\right)^{n+1} F_n$ (see Knuth [13]) in the above equation , we get

$$BF_{n+1}BF_{n-1} - BF_n^2 = \left[2\left(-1\right)^{n+2}\right]3j + \left[3\left(-1\right)^{n+2}\right]k$$
$$= 3\left(-1\right)^n\left(2j + k\right)$$

Similarly for bicomplex Lucas numbers we can simply get;

$$BL_{n+1}BL_{n-1} - BL_n^2 = (L_{n+1} + L_{n+2}i + L_{n+3}j + L_{n+4}k)\,(L_{n-1}$$
$$+ L_n i + L_{n+1}j + L_{n+2}k)$$
$$- \left(L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k\right)^2.$$

By using the identity of Lucas numbers which are $L_{n-1}L_{n+1} - L_n^2 = 5(-1)^{n-1}$ (see Koshy [14]) and $L_{n+2} = L_{n+1} + L_n$ (see Dunlap [3]) in the above equation, we compute the following expression;

$$BL_{n+1}BL_{n-1} - BL_n^2 == 5(-1)^{n-1}(1 + \varepsilon)$$
$$= 5(-1)^{n-1}\left(2j + k\right).$$

□

Theorem 6. *Let BF_n be a bicomplex Fibonacci number. The Catalan's identity for BF_n is given by*

$$BF_n^2 - BF_{n+r}BF_{n-r} = (-1)^{n-r}\left[\begin{array}{c} 2\left(F_{r-2}^2 + F_r^2\right)j \\ +\left(F_{r+1}^2 + F_{r-2}^2 - F_r^2 - F_{r-3}^2\right)k \end{array}\right].$$

Proof. From the equation (1.4), we have

$$\begin{aligned} BF_n^2 - BF_{n+r}BF_{n-r} = & \left[\begin{array}{c} F_n^2 - F_{n+1}^2 - F_{n+2}^2 + F_{n+3}^2 \\ -F_{n+r}F_{n-r} + F_{n+r+1}F_{n-r-1} \\ +F_{n+r+2}F_{n-r+2} - F_{n+r+3}F_{n-r+3} \end{array}\right] \\ & + \left[\begin{array}{c} 2F_nF_{n+1} - 2F_{n+2}F_{n+3} \\ -F_{n+r}F_{n-r+1} - F_{n+r+1}F_{n-r} \\ +F_{n+r+2}F_{n-r+3} + F_{n+r+3}F_{n-r+2} \end{array}\right] i \\ & + \left[\begin{array}{c} 2F_nF_{n+2} - 2F_{n+1}F_{n+3} \\ -F_{n+r}F_{n-r+2} - F_{n+r+2}F_{n-r} \\ +F_{n+r+1}F_{n-r+3} + F_{n+r+3}F_{n-r+1} \end{array}\right] j \\ & + \left[\begin{array}{c} 2F_nF_{n+3} + 2F_{n+1}F_{n+2} \\ -F_{n+r}F_{n-r+3} - F_{n+r+3}F_{n-r} \\ -F_{n+r+1}F_{n-r+2} - F_{n+r+2}F_{n-r+1} \end{array}\right] k \end{aligned}$$

Here using the Catalan's identity for Fibonacci numbers $F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2$ (see Weisstein [23]) , $F_{n+1}^2 - F_{n-1}^2 = F_{2n}$ (see Vajda [21]) and doing necessary calculations, we obtain that;

$$BF_n^2 - BF_{n+r}BF_{n-r} = (-1)^{n-r}\left[\begin{array}{c} 2\left(F_{r-2}^2 + F_r^2\right)j \\ +\left(F_{r+1}^2 + F_{r-2}^2 - F_r^2 - F_{r-3}^2\right)k \end{array}\right]$$

□

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