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CLEAVABILITY OVER SEPARATION AXIOMS RELATIVE SPACES

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Abstract: The propose of this paper is to discuss cleavability over relative space which studied by M. Bonanzinga in [3], we studied cleavability under some weakly and strongly forms of separations axioms relative spaces.

AMS Subject Classification: 54C05, 54D10

Key Words: cleavability, cleavability over relative spaces, separation axioms relative spaces

1. Introduction

The notion of cleavability of one space over another space was introduced in [1], and the original word for cleavability was a Russian one (rustchepljaemostj) and it was first transported to splittability.
In (1994), M.Bonanzinga [3] defined the concept of cleavability over relative spaces. A relative topological space or simply a relative space is a pair (Y, X) consisting of a topological space X and a subspace Y of X. In this paper we interested for answering the following general question:
"If (Y, X) is a relative space cleavable over a class ρ of relative spaces, does

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(Y, X) belong to ρ ?"

Definition 1.1. [3] A relative space (Y, X) is said to be cleavable (strongly cleavable) over a class ρ of relative spaces if for every $A \subseteq Y$ there exist (Z, T) in ρ and a continuous mapping $f : X \to T$ such that $f^{-1}(f(A)) = A$ and $f(Y) \subseteq Z(f^{-1}(Z) = Y)$.

Definition 1.2. [3] A relative space (Y, X) is said to be pointwise cleavable (pointwise strongly cleavable) over a class ρ of relative spaces if for every $y \in Y$ there exist (Z, T) in ρ and a continuous mapping $f : X \to T$ such that $f^{-1}(f(y)) = y$ and $f(Y) \subseteq Z(f^{-1}(Z) = Y)$.

Definition 1.3. A relative space (Y, X) is said to be strongly double cleavable over a class ρ of relative spaces if for every pair A, B of subset of Y, there exist (Z, T) in ρ and a continuous mapping $f : X \to T$ such that $f^{-1}(f(A)) = A$, $f^{-1}(f(B)) = B$ and $f^{-1}(Z) = Y$.

In the previous definitions, if f is a closed continuous mapping, then (Y, X) is said to be closed cleavable , closed strongly cleavable, closed pointwise cleavable, closed strongly pointwise cleavable, closed strongly double cleavable, respectively.

Terms and notations not explained in this paper are taken from [5].

2. Cleavability over Weakly Separation Axioms Relative Spaces

Definition 2.1. A relative space (Y, X) is said to be T_0- relative space, if for every pair of distinct points $y_1, y_2 \in Y$, there exist open subset U in X containing y_1 but not y_2 or containing y_2 but not y_1.

It is clear that if X is a T_0-space and $Y \subseteq X$, then (Y, X) is T_0- relative space. But the converse need not be true, consider the following example.

Example 2.2. Let $X = \{a, b, c\}$, $Y = \{a, c\}$ and $\tau = \{\phi, X, \{a, b\}, \{c\}\}$. Then (X, Y) is T_0- relative space but X is not T_0-space.

Definition 2.3. A relative space (Y, X) is said to be a strongly T_0-relative space, if for every $x \in X$, and $y \in Y$ with $x \neq y$, there exists an open subset U in X containing x but not y or containing y but not x .

It is clear that a strongly $-T_0-$relative space (X, Y) is T_0-relative space, but the converse need not be true. consider example 2.2.

Theorem 2.4. *Let (Y, X) be a pointwise cleavable over the class ρ of T_0-*

relative spaces, then (Y, X) is a T_0- relative space.

Proof. Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since (Y, X) is pointwise cleavable over the class ρ ,then there exist a T_0- relative space (Z, T) and a continuous mapping $f : X \to T$ such that $f^{-1}(f(y_1)) = y_1$, and $f(Y) \subseteq Z$. Now $f(y_1), f(y_2) \in f(Y) \subseteq Z$, since (Z, T) is a T_0- relative space, there exist an open subset U in T such that $f(y_1) \in U$ and $f(y_2) \notin U$ or $f(y_1) \notin U$ and $f(y_2) \in U$, without loss of generality assume that $f(y_1) \in U$ and $f(y_2) \notin U$, so $y_1 \in f^{-1}(U)$ and $y_2 \notin f^{-1}(U)$, but $f^{-1}(U)$ is an open subset of X, hence (Y, X) is a T_0- relative space. □

Theorem 2.5. *Let (Y, X) be a pointwise cleavable over the class ρ of strongly T_0-relative spaces,then (Y, X) is a strongly $-T_0-$ relative space.*

Proof. Let $y \in Y$, let $x \in X$ with $x \neq y$, since (Y, X) is a pointwise cleavable over the class ρ, then there exist a strongly $-T_0-$ relative space (Z, T) and a continuous mapping $f : X \to T$such that $f^{-1}(f(y)) = y$,and $f(Y) \subseteq Z$. Now, $f(y) \in f(Y) \subseteq Z$, and $f(x) \in T$, since (Z, T) is strongly $-T_0-$relative space, there exists an open subset U in T such that $f(x) \in U$ and $f(y) \notin U$ or $f(x) \notin U$ and $f(y) \in U$, without loss of generality assume that $f(x) \in U$ and $f(y) \notin U$, so $x \in f^{-1}(U)$ and $y \notin f^{-1}(U)$, but $f^{-1}(U)$ is an open subset of X, hence (Y, X) is a strongly $-T_0-$ relative space. □

Definition 2.6. A relative space (Y, X) is said to be T_1- relative space, if for every pair of distinct points $y_1, y_2 \in Y$, there exist an open subset U in X containing y_1 but not y_2 and an open subset V in X containing y_2 but not y_1.

Notice that for T_1- relative space (Y, X), X need not be T_1- space. consider example 2.2.

Definition 2.7. A relative space (Y, X) is said to be strongly $-T_1-$ relative space, if for every $x \in X$ and $y \in Y$ with $x \neq y$, there exist open subsets U and V in X such that $x \in U$, $y \notin U$ and $x \notin V, y \in V$.

Theorem 2.8. *Let (Y, X) be a pointwise cleavable over the class ρ of T_1- relative spaces, then (Y, X) is a T_1- relative space.*

Proof. Let $y_1 \in Y$, since (Y, X) is pointwise cleavable over the class ρ, then there exist aT_1- relative space (Z, T) and a continuous mapping $f : X \to T$ such that $f^{-1}(f(y_1)) = y_1$, and $f(Y) \subseteq Z$. So $f(y_1) \in f(Y) \subseteq Z$. Now let $y_2 \in Y$ with $y_1 \neq y_2$, so $f(y_2) \in Z$ and $f(y_1) \neq f(y_2)$, since (Z, T) is a T_1- relative space, there exist open subsets U and V in T such that $f(y_1) \in U$, $f(y_2) \notin U$, $f(y_1) \notin V$ and $f(y_2) \in V$, so $y_1 \in f^{-1}(U)$, $y_2 \notin f^{-1}(U)$, $y_1 \in f^{-1}(V)$, and

$y_2 \notin f^{-1}(V)$ but $f^{-1}(U)$ and $f^{-1}(V)$ are open subsets of X, hence (Y, X) is a T_1- relative space. □

Theorem 2.9. *Let (Y, X) be a pointwise cleavable over the class ρ of strongly T_1- relative spaces, then (Y, X) is strongly $-T_1-$ relative space.*

Proof. Let $y \in Y$, since (Y, X) is a pointwise cleavable over the class ρ of strongly $-T_1-$ relative spaces, then there exist a strongly T_1- relative space (Z, T) and a continuous mapping $f : X \to T$ such that $f^{-1}(f(y)) = y$, and $f(Y) \subseteq Z$, so $f(y) \in f(Y) \subseteq Z$. Now let $x \in X$ with $x \neq y$, so $f(x) \in T$, since (Z, T) is a strongly $-T_1-$ relative space, there exist open subsets U and V in T such that $f(x) \in U$, $f(y) \notin U$, $f(x) \notin V$ and $f(y) \in V$, so $x \in f^{-1}(U)$, $y \notin f^{-1}(U)$, $x \notin f^{-1}(V)$, and $y \in f^{-1}(V)$, but $f^{-1}(U)$ and $f^{-1}(V)$ are open subsets of X, hence (Y, X) is strongly $-T_1-$ relative space. □

It is clear that T_2- relative space implies T_1-relative space, and T_1-relative space implies T_0- relative space. However the converse are -in general- not true, the following examples show that.

Example 2.10. Consider the set $X = \mathbb{R}$, and topologize X by the basis with $\mathfrak{B}(\tau) = \{\{x\}|x \in \mathbb{Q}\} \cup \{\phi, \mathbb{R}\}$. Consider the subspace $Y = \{\sqrt{2}, 2\}$, then (Y, X) is a T_0- relative space, but it is not a T_1- relative space, since there is no open subset containing $\sqrt{2}$ and does not contain 2.

Example 2.11. Consider the set $X = \mathbb{R}$ with $\tau = \{U \subseteq \mathbb{R}|1 \in U\} \cup \{\phi\}$. Consider the subspace $Y = \{2, 3\}$, then (Y, X) is a T_1- relative space but it is not a T_2-relative space, since any two open subsets of X that contain 2 and 3 have nonempty intersection. Notice that (Y, X) in not strongly $-T_1-$relative space.

Example 2.12. In Example 2.10, if we take $Y = \{\sqrt{2}, \pi\}$, then (Y, X) is not a T_i- relative space for $i = 0, 1, 2$.

Definition 2.13. A relative space (Y, X) is said to be a $T_{\frac{1}{2}}-$ relative space, if for every $y \in Y$, $\{y\}$ is either open or closed in X.

Theorem 2.14. *Let (Y, X) be pointwise cleavable over the class ρ of $T_{\frac{1}{2}}-$ relative spaces, then (Y, X) is $T_{\frac{1}{2}}-$ relative space*

Proof. Let $y \in Y$, since (Y, X) is a pointwise cleavable over the class ρ of $T_{\frac{1}{2}}-$ relative spaces , then there exist a relative space (Z, T) which is $T_{\frac{1}{2}}-$ relative space and a continuous mapping $f : X \to T$ such that $f^{-1}(f(y)) = y$, and $f(Y) \subseteq Z$. Now $f(y) \subseteq Z$, since (Z, T) is a $T_{\frac{1}{2}}-$ relative space, so $f(y)$

is either open or closed in T, since f is continuous, so $f^{-1}(f(y)) = y$ is either open or closed in X. Hence (Y, X) is a $T_{\frac{1}{2}}-$ relative space. □

Definition 2.15. A relative space (Y, X) is said to be a $T_{1\frac{1}{3}}-$ relative space, if every convergent sequence (x_n) in Y, has a unique limit $y \in Y$.

Theorem 2.16. *Let X be a first countable space and (Y, X) is a pointwise cleavable over the class ρ of first countable $T_{1\frac{1}{3}}-$ relative spaces, then (Y, X) is a $T_{1\frac{1}{3}}-$ relative space .*

Proof. Let (x_n) be a sequence in Y and let $y_1, y_2 \in Y$ with $y_1 \neq y_2$ such that (x_n) converges to y_1 and y_2. Since (Y, X) is pointwise cleavable over the class ρ of $T_{1\frac{1}{3}}-$ relative spaces,then there exist a relative space (Z, T) which is $T_{1\frac{1}{3}}-$ relative space and a continuous mapping $f : X \to T$ such that $f^{-1}(f(y_1)) = y_1$, with $f(Y) \subseteq Z$ and T is a first countable. Now $(f(x_n)) \subseteq T$, and since (Z, T) is $T_{1\frac{1}{3}}-$ relative space , so $(f(x_n))$ converges to $f(y_1)$ and $(f(x_n))$ converges to $f(y_2)$, by uniqueness $f(y_1) = f(y_2)$, hence $y_1 = y_2$, so (Y, X) is $T_{1\frac{1}{3}}-$ relative space. □

Definition 2.17. A relative space (Y, X) is said to be $T_{\frac{1}{4}}-$ relative space if for a finite set $F \subseteq Y$ and $y \in X$ with $y \notin F$, there exist $A \subseteq X$ and $F \subseteq A$ and $y \notin A$ such that A is either open or closed in X.

Theorem 2.18. *Let (Y, X) be a pointwise cleavable over the class ρ of $T_{\frac{1}{4}}-$ relative spaces, then (Y, X) is a $T_{\frac{1}{4}}-$ relative space.*

Proof. Let $F \subseteq Y$ be a finite set and $y \notin F$, since (Y, X) is a pointwise cleavable over the class ρ of $T_{\frac{1}{4}}-$relative spaces, then there exist a $T_{\frac{1}{4}}-$ relative spaces (Z, T) and a continuous mapping $f : X \to T$ with $f^{-1}(f(y)) = y$, and $f(Y) \subseteq Z$, hence $f(y) \notin f(F)$. Since $f(F)$ is finite set in Z, there exist $B \subseteq T$ with $f(F) \subseteq B$ such that B is either open or closed in T, so $F \subseteq f^{-1}(B)$, $y \notin f^{-1}(B)$ and $f^{-1}(B)$ is either open or closed in X, hence (Y, X) is a $T_{\frac{1}{4}}-$ relative space. □

Definition 2.19. A relative space (Y, X) is said to be T_D- relative space or (locally closed) if for all $W \subseteq Y$ there exists an open set U and a closed set F in X such that $W = U \cap F$.

Theorem 2.20. *Let (Y, X) be a cleavable relative space over the class ρ of T_D-relative spaces, then (Y, X) is a T_D-relative space*

Proof. Let $W \subseteq Y$. Since (Y, X) is relative cleavable space over the class ρ of T_D-relative spaces, there exist a T_D-relative space (Z, T) and a continuous mapping $f : X \to T$ such that $f^{-1}(f(W)) = W$ and $f(Y) \subseteq Z$. Now, $f(W) \subseteq Z$, but (Z, T) is a T_D-relative space, hence there exist an open set U and a closed set F in T such that $f(W) = U \cap F$, so

$$W = f^{-1}(f(W)) = f^{-1}(U \cap F) = f^{-1}(U) \cap f^{-1}(F).$$

Since f is continuous $f^{-1}(U)$ is open in X and $f^{-1}(F)$ is closed in X. Therefore (Y, X) is T_D-relative space. □

Definition 2.21. [4] A subset A of a topological space X is called semiopen if $A \subseteq cl(int(A))$. A subset A of a topological space X is called semiclosed if $X - A$ is semiopen.

Definition 2.22. A relative space (Y, X) is said to be a semi$-T_0-$ relative space, if for all $y_1, y_2 \in Y$, with $y_1 \neq y_2$, there exists a semiopen set A in X containing one point but not the other.

Theorem 2.23. *Let (Y, X) be pointwise open cleavable relative space over a class ρ of semi$-T_0-$ relative spaces, then (Y, X) is semi$-T_0-$ relative space.*

Proof. Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since (Y, X) is a pointwise open cleavable relative space over a class ρ of semi$-T_0-$ relative spaces, there exist semi$-T_0-$ relative space (Z, T) and an open continuous mapping $f : X \to Y$ such that $f^{-1}(f(y_1)) = y_1$, and $f(Y) \subseteq Z$. Then $f(y_1)$ and $f(y_2)$ are distinct points in Z, but $(Z, T) \in \rho$, so there exists a semi-open subset A in T such that $f(y_1) \in A$, $f(y_2) \notin A$ or $f(y_2) \in A$, $f(y_1) \notin A$. Without loss of generality, assume that $f(y_1) \in A$, $f(y_2) \notin A$, but A is semi-open so

$$f(y_1) \in A \subseteq cl(int(A)),$$

and

$$y_1 \in f^{-1}(A) \subseteq f^{-1}(cl(int(A))),$$

since f is open

$$y_1 \in f^{-1}(A) \subseteq f^{-1}(cl(int(A))) \subseteq cl(f^{-1}(int(A))) \subseteq cl(int(f^{-1}(A))),$$

hence $f^{-1}(A)$ is a semi-open subset of X with $y_2 \notin f^{-1}(A)$, hence $(Y, X) \in \rho$. □

Definition 2.24. [7] A subset $A \subseteq X$ is said to be feebly open, if there exists an open subset U of X such that $U \subseteq A \subseteq Scl(U)$, where $Scl(U)$ is the intersection of all semi-closed subsets containing U.

Definition 2.25. A relative space (Y, X) is said to be a feebly$-T_0-$ relative space, if for all $y_1, y_2 \in Y$, with $y_1 \neq y_2$, there exist a feebly open set containing one point but not the other

Theorem 2.26. [6] *If $\{A_\alpha | \alpha \in \Delta\}$ is a family of semi-open sets, then $\bigcup_{\alpha\in\Delta} \{A_\alpha\}$ is semiopen.*

Definition 2.27. [6] A mapping $f : X \to Y$ is said to be irresolute if $f^{-1}(A)$ is a semiopen set in X for all semi-open subset A in Y.

Definition 2.28. A relative space (Y, X) is called an irresolute cleavable over a class ρ if for every $A \subseteq Y$ there exist (Z, T) in ρ and an irresolute mapping $f : X \to T$ such that $f^{-1}(f(A)) = A$ and $f(Y) \subseteq Z$.

Theorem 2.29. *Let (Y, X) be a pointwise irresolute cleavable relative space over a class ρ of feebly$-T_0-$ relative spaces, then (Y, X) is a feebly$-T_0-$ relative space.*

Proof. Let $y_1 \in Y$, since (Y, X) is a pointwise irresolute cleavable over ρ , there exist feebly$-T_0-$ relative space (Z, T) and an irresolute mapping $f : X \to T$ such that $f^{-1}(f(y_1)) = y_1$, and $f(Y) \subseteq Z$. Let $y_2 \in Y$, with $y_1 \neq y_2$. Now $f(y_1) \neq f(y_2)$ in Z, since (Z, T) is a feebly$-T_0-$ relative space, there exists a feebly open set A such that $f(y_1) \in U \subseteq A \subseteq Scl(U)$, and $f(y_2) \in U$, where U is open in T. Now

$$y_1 \in f^{-1}(U) \subseteq f^{-1}(A) \subseteq f^{-1}(Scl(U)).$$

To prove that $f^{-1}(A)$ is feebly open it is enough to show that $f^{-1}(Scl(U)) = Scl(f^{-1}(U))$. We know that $SclU = \bigcap_{\gamma\in\Gamma} \{B_\gamma\}$ where B_γ are semiclosed sets containing U for all γ, so

$$f^{-1}(SclU) = f^{-1}(\bigcap_{\gamma\in\Gamma} \{B_\gamma\}) = \bigcap_{\gamma\in\Gamma} f^{-1}(B_\gamma),$$

but

$$Y - (\bigcap_{\gamma\in\Gamma} f^{-1}(B_\gamma)) = \bigcup_{\gamma\in\Gamma} f^{-1}(Z - B_\gamma).$$

Now $Z - B_\gamma$ is semiopen in T, since f is irresolute, $f^{-1}(Z - B_\gamma)$ is semiopen in X , so by theorem 2.26 $\bigcup_{\gamma\in\Gamma} f^{-1}(Z - B_\gamma)$ is a semiopen subset. So

$$Y - f^{-1}(\bigcap_{\gamma\in\Gamma}\{B_\gamma\})$$

is a semiopen subset, and hence

$$f^{-1}(\bigcap_{\gamma\in\Gamma}\{B_\gamma\}) = f^{-1}(SclU)$$

is semiclosed, thus $f^{-1}(A)$ is feebly open, and so (Y, X) is a feebly$-T_0-$ relative space. □

3. Cleavability over Strongly Separation Axioms Relative Spaces

Definition 3.1. A relative space (Y, X) is said to be a $T_{2\frac{1}{2}}-$ relative space, if for every pair of distinct points $y_1, y_2 \in Y$, there exist open subsets U and V of X such that $y_1 \in U$, $y_2 \in V$ and $\overline{U} \cap \overline{V} = \phi$.

Theorem 3.2. *Let (Y, X) be a pointwise cleavable over the class ρ of $T_{2\frac{1}{2}}-$relative spaces, then (Y, X) is a $T_{2\frac{1}{2}}-$ relative space*

Proof. Let $y_1 \in Y$, since (Y, X) is a pointwise cleavable over the class ρ of $T_{2\frac{1}{2}}-$ relative spaces, then there exist a $T_{2\frac{1}{2}}-$ relative space (Z, T) and a continuous mapping $f : X \to T$ such that $f^{-1}(f(y_1)) = y_1$, and $f(Y) \subseteq Z$. Now let $y_2 \in Y$ with $y_1 \neq y_2$, then $f(y_2) \in Z$, since (Z, T) is a $T_{2\frac{1}{2}}-$relative space, there exist open subsets U and V in T such that $f(y_1) \in U$, $f(y_2) \in V$, with $\overline{U} \cap \overline{V} = \phi$. Since f is a continuous mapping, we have $f^{-1}(U)$ and $f^{-1}(V)$ are open subsets of$X$, with $y_1 \in f^{-1}(U)$, $y_2 \in f^{-1}(V)$. Also, since $\overline{U} \cap \overline{V} = \phi$, we have $f^{-1}(\overline{U} \cap \overline{V}) = \phi$, so $f^{-1}(\overline{U}) \cap f^{-1}(\overline{V}) = \phi$, by continuity

$$\overline{f^{-1}(U)} \cap \overline{f^{-1}(V)} \subseteq f^{-1}(\overline{U}) \cap f^{-1}(\overline{V})$$

so

$$\overline{f^{-1}(U)} \cap \overline{f^{-1}(V)} = \phi,$$

hence (Y, X) is a $T_{2\frac{1}{2}}-$ relative space □

Example 3.3. Let $X = \{(x, y)|y \geq 0, x, y \in \mathbb{Q}\}$ with irrational slope topology, see ([2], page 93). Consider $Y = \{(x, y) \in X|y \leq 1\}$, then (Y, X) is T_2-relative space, since any two distinct points in Y must project onto distinct pairs of irrational points on the $x-$ axis, which have disjoint neighborhoods, but (Y, X) is not a $T_{2\frac{1}{2}}-$ relative space, since the closure of each neighborhood must intersect the closure of the anther subsets.

Definition 3.4. A relative space (Y, X) is said to be strongly $T_{2\frac{1}{2}}-$ relative space, if for every pair of distinct points $x \in X$, and $y \in Y$ there exist open subsets U and V of X such that $x \in U$ and $y \in V$ and $\overline{U} \cap \overline{V} = \phi$.

Theorem 3.5. *Let (Y, X) be a pointwise cleavable relative space over the class ρ of strongly $-T_{2\frac{1}{2}}-$ relative spaces, then (Y, X) is a strongly $T_{2\frac{1}{2}}-$ relative space.*

Definition 3.6. A relative space (Y, X) is said to be a strongly functionally Hausdorff relative space if for all $x \in X$ and $y \in Y$ with $x \neq y$, there exists a continuous mapping $f : X \to ([0, 1], \tau_u)$ such that $f(x) = 0$ and $f(y) = 1$.

Theorem 3.7. *Let (Y, X) be a pointwise cleavable over the class ρ of strongly functionally Hausdorff relative spaces, then (Y, X) is a strongly functionally Hausdorff relative space.*

Proof. Let $y \in Y$, since (Y, X) is pointwise cleavable over the class ρ of strongly functionally Hausdorff relative spaces, there exist strongly functionally Hausdorff relative space (Z, T) and a continuous mapping $g : X \to T$ such that $g^{-1}(g(y)) = y$ and $g(Y) \subseteq Z$. Now, let $x \in X$ with $x \neq y$, so $g(x) \in T$, since (Z, T) is a strongly functionally Hausdorff relative space, there exist continuous mapping $h : T \to [0, 1]$ with $h(g(x)) = 0$ and $h(g(y)) = 1$. Consider the function $f = h \circ g$, so f is a continuous mapping from X into $[0, 1]$ with $f(x) = 0, f(y) = 1$, hence (Y, X) is strongly functionally Hausdorff relative space. □

Definition 3.8. A relative space (Y, X) is said to be an open normal relative space if for all disjoint open sets A, B in Y, there exist disjoint closed sets F, G in X such that $A \subseteq F$ and $B \subseteq G$.

Theorem 3.9. *Let (Y, X) be an open double cleavable relative space over a class ρ of open normal relative spaces, then (Y, X) is an open normal relative space.*

Proof. Let A and B be disjoint open sets in Y, since (Y, X) is open double cleavable over a class ρ of open normal relative spaces, then there exist open normal relative space (Z,T) and an open continuous mapping $f : X \to T$ such that $f^{-1}(f(A)) = A$, $f^{-1}(f(B)) = B$, and $f(Y) \subseteq Z$. Since f is open, so $f(A)$ and $f(B)$ are open in Z. Since $A \cap B = \phi$, so $f^{-1}(f(A)) \cap f^{-1}(f(B)) = \phi$, and so $f^{-1}(f(A) \cap f(B)) = \phi$, then $f(A) \cap f(B) = \phi$, so there exist two closed subsets F, G in Z such that $f(A) \subseteq F$, $f(B) \subseteq G$, so $A \subseteq f^{-1}(F)$,$B \subseteq f^{-1}(G)$, it is clear that $f^{-1}(F)$ and $f^{-1}(G)$ are closed and disjoint subsets in X, and hence (Y,X) is an open normal relative space. $\square$

Definition 3.10. A relative space (Y,X) is said to be semi$-T_2-$ relative space, if for all $y_1, y_2 \in Y$, with $y_1 \neq y_2$, there exist disjoint semiopen subsets U and V in X such that $y_1 \in U$ and $y_2 \in V$.

Theorem 3.11. *Let (Y,X) be pointwise irresolute cleavable relative space over a class ρ of semi$-T_2-$ relative spaces, then (Y,X) is semi$-T_2-$ relative space.*

Proof. Let $y_1 \in Y$, since (Y,X) is a pointwise irresolute cleavable over the class ρ of a semi$-T_2-$ relative spaces, then there exist semi$-T_2-$ relative space (Z,T) and an irresolute mapping $f : X \to T$such that $f^{-1}(f(y_1)) = y_1$, and $f(Y) \subseteq Z$. Now $f(y_1) \in f(Y) \subseteq Z$, let $y_2 \in Y$ with $y_1 \neq y_2$, so $f(y_1) \neq f(y_2)$ in Z, since (Z,T) is a semi$-T_2-$ relative space, there exist disjoint semi-open subsets U and V in T such that $f(y_1) \in U$ and $f(y_2) \in V$, then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint semi-open subsets of X such that $y_1 \in f^{-1}(U)$ and $y_2 \in f^{-1}(V)$, hence (Y,X) semi$-T_2-$ relative spaces. $\square$

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References

[1] A.V. Arhangel'skii and F.Commaroto , *On different types of cleavability of topological spaces,*. J.Austral.Math. Soc. (Series A), Vol.58 (1995), 183-199.

[2] L. A. Steen, J.A. Seebach, Jr., *Counterexamples in Topology.* Springer-Verlab, New York (1970).

[3] M.Bonanzinga , *On cleavability over relative spaces.* Questions and Answers in General Topology, Vol.12, (1994), 185-192.

[4] N. Levine, , *Generalized closed sets in topology.* Rend. Circ. Mat.Palermo, Vol.19, No.2 (1970), 89-96.

[5] R. Engelking, *General Topology*. revised and completed edition, Heldermann Verlag, Berlin, 1989.

[6] S.G. Crossely and S.K. Hilderbrand, *Semiclosure*. Texas J. Sci., Vol.22 (1971), 99-112.

[7] S.N. Masheshwari and P.C. Jain, *Some new mappings*. Mathematica, Vol. 24 (47) (1-2) (1982), 53-55.