CLEAVABILITY OVER SEPARATION AXIOMS
RELATIVE SPACES

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Abstract: The propose of this paper is to discuss cleavability over relative space which studied by M. Bonanzinga in [3], we studied cleavability under some weakly and strongly forms of separations axioms relative spaces.

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1. Introduction

The notion of cleavability of one space over another space was introduced in [1], and the original word for cleavability was a Russian one (rustchepljaemostj) and it was first transported to splittability.

In (1994), M. Bonanzinga [3] defined the concept of cleavability over relative spaces. A relative topological space or simply a relative space is a pair \((Y, X)\) consisting of a topological space \(X\) and a subspace \(Y\) of \(X\). In this paper we interested for answering the following general question:

"If \((Y, X)\) is a relative space cleavable over a class \(\rho\) of relative spaces, does
(Y, X) belong to ρ?"

**Definition 1.1.** [3] A relative space (Y, X) is said to be cleavable (strongly cleavable) over a class ρ of relative spaces if for every A ⊆ Y there exist (Z, T) in ρ and a continuous mapping f : X → T such that f⁻¹(f(A)) = A and f(Y) ⊆ Z(f⁻¹(Z) = Y).

**Definition 1.2.** [3] A relative space (Y, X) is said to be pointwise cleavable (pointwise strongly cleavable) over a class ρ of relative spaces if for every y ∈ Y there exist (Z, T) in ρ and a continuous mapping f : X → T such that f⁻¹(f(y)) = y and f(Y) ⊆ Z(f⁻¹(Z) = Y).

**Definition 1.3.** A relative space (Y, X) is said to be strongly double cleavable over a class ρ of relative spaces if for every pair A, B of subset of Y, there exist (Z, T) in ρ and a continuous mapping f : X → T such that f⁻¹(f(A)) = A, f⁻¹(f(B)) = B and f⁻¹(Z) = Y.

In the previous definitions, if f is a closed continuous mapping, then (Y, X) is said to be closed cleavable, closed strongly cleavable, closed pointwise cleavable, closed strongly pointwise cleavable, closed strongly double cleavable, respectively.

Terms and notations not explained in this paper are taken from [5].

2. Cleavability over Weakly Separation Axioms Relative Spaces

**Definition 2.1.** A relative space (Y, X) is said to be $T_0$-relative space, if for every pair of distinct points $y_1, y_2 \in Y$, there exist open subset $U$ in X containing $y_1$ but not $y_2$ or containing $y_2$ but not $y_1$.

It is clear that if X is a $T_0$-space and $Y \subseteq X$, then (Y, X) is $T_0$-relative space. But the converse need not be true, consider the following example.

**Example 2.2.** Let $X = \{a, b, c\}$, $Y = \{a, c\}$ and $\tau = \{\phi, X, \{a, b\}, \{c\}\}$. Then (X, Y) is $T_0$-relative space but X is not $T_0$-space.

**Definition 2.3.** A relative space (Y, X) is said to be a strongly $T_0$-relative space, if for every $x \in X$, and $y \in Y$ with $x \neq y$, there exists an open subset $U$ in X containing $x$ but not $y$ or containing $y$ but not $x$.

It is clear that a strongly $T_0$-relative space (X, Y) is $T_0$-relative space, but the converse need not be true. consider example 2.2.

**Theorem 2.4.** Let (Y, X) be a pointwise cleavable over the class ρ of $T_0$-
relative spaces, then \((Y, X)\) is a \(T_0\)– relative space.

**Proof.** Let \(y_1, y_2 \in Y\) with \(y_1 \neq y_2\). Since \((Y, X)\) is pointwise cleavable over the class \(\rho\), then there exist \(T_0\)– relative space \((Z, T)\) and a continuous mapping \(f : X \to T\) such that \(f^{-1}(f(y_1)) = y_1\), and \(f(Y) \subseteq Z\). Now \(f(y_1), f(y_2) \in f(Y) \subseteq Z\), since \((Z, T)\) is a \(T_0\)– relative space, there exist an open subset \(U\) in \(T\) such that \(f(y_1) \in U\) and \(f(y_2) \notin U\) or \(f(y_1) \notin U\) and \(f(y_2) \in U\), without loss of generality assume that \(f(y_1) \in U\) and \(f(y_2) \notin U\), so \(y_1 \in f^{-1}(U)\) and \(y_2 \notin f^{-1}(U)\), but \(f^{-1}(U)\) is an open subset of \(X\), hence \((Y, X)\) is a \(T_0\)– relative space. \(\square\)

**Theorem 2.5.** Let \((Y, X)\) be a pointwise cleavable over the class \(\rho\) of strongly \(T_0\)–relative spaces, then \((Y, X)\) is a strongly \(-T_0\)– relative space.

**Proof.** Let \(y \in Y\), let \(x \in X\) with \(x \neq y\), since \((Y, X)\) is a pointwise cleavable over the class \(\rho\), then there exist a strongly \(-T_0\)– relative space \((Z, T)\) and a continuous mapping \(f : X \to T\) such that \(f^{-1}(f(y)) = y\), and \(f(Y) \subseteq Z\). Now, \(f(y) \in f(Y) \subseteq Z\), and \(f(x) \in T\), since \((Z, T)\) is strongly \(-T_0\)–relative space, there exists an open subset \(U\) in \(T\) such that \(f(x) \in U\) and \(f(y) \notin U\) or \(f(x) \notin U\) and \(f(y) \in U\), without loss of generality assume that \(f(x) \in U\) and \(f(y) \notin U\), so \(x \in f^{-1}(U)\) and \(y \notin f^{-1}(U)\), but \(f^{-1}(U)\) is an open subset of \(X\), hence \((Y, X)\) is a strongly \(-T_0\)– relative space. \(\square\)

**Definition 2.6.** A relative space \((Y, X)\) is said to be \(T_1\)– relative space, if for every pair of distinct points \(y_1, y_2 \in Y\), there exist an open subset \(U\) in \(X\) containing \(y_1\) but not \(y_2\) and an open subset \(V\) in \(X\) containing \(y_2\) but not \(y_1\).

Notice that for \(T_1\)– relative space \((Y, X)\), \(X\) need not be \(T_1\)– space. Consider example 2.2.

**Definition 2.7.** A relative space \((Y, X)\) is said to be strongly \(-T_1\)– relative space, if for every \(x \in X\) and \(y \in Y\) with \(x \neq y\), there exist open subsets \(U\) and \(V\) in \(X\) such that \(x \in U\), \(y \notin U\) and \(x \notin V\), \(y \in V\).

**Theorem 2.8.** Let \((Y, X)\) be a pointwise cleavable over the class \(\rho\) of \(T_1\)– relative spaces, then \((Y, X)\) is a \(T_1\)– relative space.

**Proof.** Let \(y_1 \in Y\), since \((Y, X)\) is pointwise cleavable over the class \(\rho\), then there exist a \(T_1\)– relative space \((Z, T)\) and a continuous mapping \(f : X \to T\) such that \(f^{-1}(f(y_1)) = y_1\), and \(f(Y) \subseteq Z\). So \(f(y_1) \in f(Y) \subseteq Z\). Now let \(y_2 \in Y\) with \(y_1 \neq y_2\), so \(f(y_2) \in Z\) and \(f(y_1) \neq f(y_2)\), since \((Z, T)\) is a \(T_1\)– relative space, there exist open subsets \(U\) and \(V\) in \(T\) such that \(f(y_1) \in U\), \(f(y_2) \notin U\), \(f(y_1) \notin V\) and \(f(y_2) \in V\), so \(y_1 \in f^{-1}(U)\), \(y_2 \notin f^{-1}(U)\), \(y_1 \in f^{-1}(V)\), and
y_2 \notin f^{-1}(V)$ but $f^{-1}(U)$ and $f^{-1}(V)$ are open subsets of $X$, hence $(Y, X)$ is a $T_1$-relative space. \hfill \Box

Theorem 2.9. Let $(Y, X)$ be a pointwise cleavable over the class $\rho$ of strongly $T_1$-relative spaces, then $(Y, X)$ is strongly $-T_1$-relative space.

Proof. Let $y \in Y$, since $(Y, X)$ is a pointwise cleavable over the class $\rho$ of strongly $-T_1$-relative spaces, then there exist a strongly $T_1$-relative space $(Z, T)$ and a continuous mapping $f : X \to T$ such that $f^{-1}(f(y)) = y$, and $f(Y) \subseteq Z$, so $f(y) \in f(Y) \subseteq Z$. Now let $x \in X$ with $x \neq y$, so $f(x) \in T$, since $(Z, T)$ is a strongly $-T_1$-relative space, there exist open subsets $U$ and $V$ in $T$ such that $f(x) \in U$, $f(y) \notin U$, $f(x) \notin V$ and $f(y) \in V$, so $x \in f^{-1}(U)$, $y \notin f^{-1}(U)$, $x \notin f^{-1}(V)$, and $y \in f^{-1}(V)$, but $f^{-1}(U)$ and $f^{-1}(V)$ are open subsets of $X$, hence $(Y, X)$ is strongly $-T_1$-relative space. \hfill \Box

It is clear that $T_2$-relative space implies $T_1$-relative space, and $T_1$-relative space implies $T_0$-relative space. However the converse are -in general- not true, the following examples show that.

Example 2.10. Consider the set $X = \mathbb{R}$, and topologize $X$ by the basis with $\mathfrak{B}(\tau) = \{ \{x\}|x \in \mathbb{Q}\} \cup \{\phi, \mathbb{R}\}$. Consider the subspace $Y = \{\sqrt{2}, 2\}$, then $(Y, X)$ is a $T_0$-relative space, but it is not a $T_1$-relative space, since there is no open subset containing $\sqrt{2}$ and does not contain 2.

Example 2.11. Consider the set $X = \mathbb{R}$ with $\tau = \{U \subseteq \mathbb{R}|1 \in U\} \cup \{\phi\}$. Consider the subspace $Y = \{2, 3\}$, then $(Y, X)$ is a $T_1$-relative space but it is not a $T_2$-relative space, since any two open subsets of $X$ that contain 2 and 3 have nonempty intersection. Notice that $(Y, X)$ in not strongly $-T_1$-relative space.

Example 2.12. In Example 2.10, if we take $Y = \{\sqrt{2}, \pi\}$, then $(Y, X)$ is not a $T_i$-relative space for $i = 0, 1, 2$.

Definition 2.13. A relative space $(Y, X)$ is said to be a $T_{\frac{3}{2}}$-relative space, if for every $y \in Y$, $\{y\}$ is either open or closed in $X$.

Theorem 2.14. Let $(Y, X)$ be pointwise cleavable over the class $\rho$ of $T_{\frac{3}{2}}$-relative spaces, then $(Y, X)$ is $T_{\frac{3}{2}}$-relative space.

Proof. Let $y \in Y$, since $(Y, X)$ is a pointwise cleavable over the class $\rho$ of $T_{\frac{3}{2}}$-relative spaces, then there exist a relative space $(Z, T)$ which is $T_{\frac{3}{2}}$-relative space and a continuous mapping $f : X \to T$ such that $f^{-1}(f(y)) = y$, and $f(Y) \subseteq Z$. Now $f(y) \subseteq Z$, since $(Z, T)$ is a $T_{\frac{3}{2}}$-relative space, so $f(y)$
is either open or closed in $T$, since $f$ is continuous, so $f^{-1}(f(y)) = y$ is either open or closed in $X$. Hence $(Y, X)$ is a $T_{\frac{1}{2}}$-relative space. \hfill \Box

**Definition 2.15.** A relative space $(Y, X)$ is said to be a $T_{\frac{1}{2}}$-relative space, if every convergent sequence $(x_n)$ in $Y$, has a unique limit $y \in Y$.

**Theorem 2.16.** Let $X$ be a first countable space and $(Y, X)$ is a pointwise cleavable over the class $\rho$ of first countable $T_{\frac{1}{3}}$-relative spaces, then $(Y, X)$ is a $T_{\frac{1}{3}}$-relative space.

**Proof.** Let $(x_n)$ be a sequence in $Y$ and let $y_1, y_2 \in Y$ with $y_1 \neq y_2$ such that $(x_n)$ converges to $y_1$ and $y_2$. Since $(Y, X)$ is pointwise cleavable over the class $\rho$ of $T_{\frac{1}{3}}$-relative spaces, then there exist a relative space $(Z, T)$ which is $T_{\frac{1}{3}}$-relative space and a continuous mapping $f : X \to T$ such that $f^{-1}(f(y_1)) = y_1$, with $f(Y) \subseteq Z$ and $T$ is a first countable. Now $(f(x_n)) \subseteq T$, and since $(Z, T)$ is $T_{\frac{1}{3}}$-relative space, so $(f(x_n))$ converges to $f(y_1)$ and $(f(x_n))$ converges to $f(y_2)$, by uniqueness $f(y_1) = f(y_2)$, hence $y_1 = y_2$, so $(Y, X)$ is $T_{\frac{1}{3}}$-relative space. \hfill \Box

**Definition 2.17.** A relative space $(Y, X)$ is said to be $T_4$-relative space if for a finite set $F \subseteq Y$ and $y \in X$ with $y \notin F$, there exist $A \subseteq X$ and $F \subseteq A$ and $y \notin A$ such that $A$ is either open or closed in $X$.

**Theorem 2.18.** Let $(Y, X)$ be a pointwise cleavable over the class $\rho$ of $T_{\frac{1}{4}}$-relative spaces, then $(Y, X)$ is a $T_{\frac{1}{4}}$-relative space.

**Proof.** Let $F \subseteq Y$ be a finite set and $y \notin F$, since $(Y, X)$ is a pointwise cleavable over the class $\rho$ of $T_{\frac{1}{4}}$-relative spaces, then there exist a $T_{\frac{1}{4}}$-relative spaces $(Z, T)$ and a continuous mapping $f : X \to T$ with $f^{-1}(f(y)) = y$, and $f(Y) \subseteq Z$, hence $f(y) \notin f(F)$. Since $f(F)$ is finite set in $Z$, there exist $B \subseteq T$ with $f(F) \subseteq B$ such that $B$ is either open or closed in $T$, so $F \subseteq f^{-1}(B)$, $y \notin f^{-1}(B)$ and $f^{-1}(B)$ is either open or closed in $X$, hence $(Y, X)$ is a $T_{\frac{1}{4}}$-relative space. \hfill \Box

**Definition 2.19.** A relative space $(Y, X)$ is said to be $T_D$-relative space or (locally closed) if for all $W \subseteq Y$ there exists an open set $U$ and a closed set $F$ in $X$ such that $W = U \cap F$.

**Theorem 2.20.** Let $(Y, X)$ be a cleavable relative space over the class $\rho$ of $T_D$-relative spaces, then $(Y, X)$ is a $T_D$-relative space.
Proof. Let $W \subseteq Y$. Since $(Y, X)$ is relative cleavable space over the class $\rho$ of $T_D$–relative spaces, there exist a $T_D$–relative space $(Z, T)$ and a continuous mapping $f : X \to T$ such that $f^{-1}(f(W)) = W$ and $f(Y) \subseteq Z$. Now, $f(W) \subseteq Z$, but $(Z, T)$ is a $T_D$–relative space, hence there exist an open set $U$ and a closed set $F$ in $T$ such that $f(W) = U \cap F$, so

$$W = f^{-1}(f(W)) = f^{-1}(U \cap F) = f^{-1}(U) \cap f^{-1}(F).$$

Since $f$ is continuous $f^{-1}(U)$ is open in $X$ and $f^{-1}(F)$ is closed in $X$. Therefore $(Y, X)$ is $T_D$–relative space.

\[\blacksquare\]

**Definition 2.21.** [4] A subset $A$ of a topological space $X$ is called semiopen if $A \subseteq cl(int(A))$. A subset $A$ of a topological space $X$ is called semiclosed if $X - A$ is semiopen.

**Definition 2.22.** A relative space $(Y, X)$ is said to be a semi–$T_0$–relative space, if for all $y_1, y_2 \in Y$, with $y_1 \neq y_2$, there exists a semiopen set $A$ in $X$ containing one point but not the other.

**Theorem 2.23.** Let $(Y, X)$ be pointwise open cleavable relative space over a class $\rho$ of semi–$T_0$–relative spaces, then $(Y, X)$ is semi–$T_0$–relative space.

**Proof.** Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since $(Y, X)$ is a pointwise open cleavable relative space over a class $\rho$ of semi–$T_0$–relative spaces, there exist semi–$T_0$–relative space $(Z, T)$ and an open continuous mapping $f : X \to Y$ such that $f^{-1}(f(y_1)) = y_1$, and $f(Y) \subseteq Z$. Then $f(y_1)$ and $f(y_2)$ are distinct points in $Z$, but $(Z, T) \in \rho$, so there exists a semi-open subset $A$ in $T$ such that $f(y_1) \in A$, $f(y_2) \notin A$ or $f(y_2) \in A$, $f(y_1) \notin A$. Without loss of generality, assume that $f(y_1) \in A$, $f(y_2) \notin A$, but $A$ is semi-open so

$$f(y_1) \in A \subseteq cl(int(A)),$$

and

$$y_1 \in f^{-1}(A) \subseteq f^{-1}(cl(int(A))),$$

since $f$ is open

$$y_1 \in f^{-1}(A) \subseteq f^{-1}(cl(int(A))) \subseteq cl(f^{-1}(int(A))) \subseteq cl(int(f^{-1}(A))),$$

hence $f^{-1}(A)$ is a semi-open subset of $X$ with $y_2 \notin f^{-1}(A)$, hence $(Y, X) \in \rho$. \[\blacksquare\]
Definition 2.24. [7] A subset \( A \subseteq X \) is said to be feebly open, if there exists an open subset \( U \) of \( X \) such that \( U \subseteq A \subseteq Scl(U) \), where \( Scl(U) \) is the intersection of all semi-closed subsets containing \( U \).

Definition 2.25. A relative space \((Y, X)\) is said to be a feebly \( T_0 \)-relative space, if for all \( y_1, y_2 \in Y \), with \( y_1 \neq y_2 \), there exist a feebly open set containing one point but not the other.

Theorem 2.26. [6] If \( \{A_\alpha|\alpha \in \Delta\} \) is a family of semi-open sets, then \( \bigcup_{\alpha \in \Delta} \{A_\alpha\} \) is semiopen.

Definition 2.27. [6] A mapping \( f : X \to Y \) is said to be irresolute if \( f^{-1}(A) \) is a semiopen set in \( X \) for all semi-open subset \( A \) in \( Y \).

Definition 2.28. A relative space \((Y, X)\) is called an irresolute cleavable over a class \( \rho \) if for every \( A \subseteq Y \) there exist \( (Z, T) \) in \( \rho \) and an irresolute mapping \( f : X \to T \) such that \( f^{-1}(f(A)) = A \) and \( f(Y) \subseteq Z \).

Theorem 2.29. Let \((Y, X)\) be a pointwise irresolute cleavable relative space over a class \( \rho \) of feebly \( T_0 \)-relative spaces, then \((Y, X)\) is a feebly \( T_0 \)-relative space.

Proof. Let \( y_1 \in Y \), since \((Y, X)\) is a pointwise irresolute cleavable over \( \rho \), there exist feebly \( T_0 \)-relative space \((Z, T)\) and an irresolute mapping \( f : X \to T \) such that \( f^{-1}(f(y_1)) = y_1 \), and \( f(Y) \subseteq Z \). Let \( y_2 \in Y \), with \( y_1 \neq y_2 \). Now \( f(y_1) \neq f(y_2) \) in \( Z \), since \((Z, T)\) is a feebly \( T_0 \)-relative space, there exists a feebly open set \( A \) such that \( f(y_1) \in U \subseteq A \subseteq Scl(U) \), and \( f(y_2) \in U \), where \( U \) is open in \( T \). Now

\[
y_1 \in f^{-1}(U) \subseteq f^{-1}(A) \subseteq f^{-1}(Scl(U)).
\]

To prove that \( f^{-1}(A) \) is feebly open it is enough to show that \( f^{-1}(Scl(U)) = Scl(f^{-1}(U)) \). We know that \( SclU = \bigcap_{\gamma \in \Gamma} \{B_\gamma\} \) where \( B_\gamma \) are semiclosed sets containing \( U \) for all \( \gamma \), so

\[
f^{-1}(SclU) = f^{-1}(\bigcap_{\gamma \in \Gamma} \{B_\gamma\}) = \bigcap_{\gamma \in \Gamma} f^{-1}(B_\gamma),
\]

but

\[
Y - \bigcap_{\gamma \in \Gamma} f^{-1}(B_\gamma) = \bigcup_{\gamma \in \Gamma} f^{-1}(Z - B_\gamma).
\]
Now $Z - B_\gamma$ is semiopen in $T$, since $f$ is irresolute, $f^{-1}(Z - B_\gamma)$ is semiopen in $X$, so by theorem 2.26 $\bigcup_{\gamma \in \Gamma} f^{-1}(Z - B_\gamma)$ is a semiopen subset. So

$$Y = f^{-1}\left(\bigcap_{\gamma \in \Gamma} \{B_\gamma\}\right)$$

is a semiopen subset, and hence

$$f^{-1}\left(\bigcap_{\gamma \in \Gamma} \{B_\gamma\}\right) = f^{-1}(SclU)$$

is semiclosed, thus $f^{-1}(A)$ is feebly open, and so $(Y, X)$ is a feebly-$T_{0}$-relative space.

\[
\square
\]

3. Cleavability over Strongly Separation Axioms Relative Spaces

**Definition 3.1.** A relative space $(Y, X)$ is said to be a $T_{2 \frac{1}{2}}$-relative space, if for every pair of distinct points $y_1, y_2 \in Y$, there exist open subsets $U$ and $V$ of $X$ such that $y_1 \in U$, $y_2 \in V$ and $\overline{U} \cap \overline{V} = \phi$.

**Theorem 3.2.** Let $(Y, X)$ be a pointwise cleavable over the class $\rho$ of $T_{2 \frac{1}{2}}$-relative spaces, then $(Y, X)$ is a $T_{2 \frac{1}{2}}$-relative space.

**Proof.** Let $y_1 \in Y$, since $(Y, X)$ is a pointwise cleavable over the class $\rho$ of $T_{2 \frac{1}{2}}$-relative spaces, then there exist a $T_{2 \frac{1}{2}}$-relative space $(Z, T)$ and a continuous mapping $f : X \to T$ such that $f^{-1}(f(y_1)) = y_1$, and $f(Y) \subseteq Z$. Now let $y_2 \in Y$ with $y_1 \neq y_2$, then $f(y_2) \in Z$, since $(Z, T)$ is a $T_{2 \frac{1}{2}}$-relative space, there exist open subsets $U$ and $V$ in $T$ such that $f(y_1) \in U$, $f(y_2) \in V$, with $\overline{U} \cap \overline{V} = \phi$. Since $f$ is a continuous mapping, we have $f^{-1}(U)$ and $f^{-1}(V)$ are open subsets of $X$, with $y_1 \in f^{-1}(U)$, $y_2 \in f^{-1}(V)$. Also, since $\overline{U} \cap \overline{V} = \phi$, we have $f^{-1}(\overline{U} \cap \overline{V}) = \phi$, so $f^{-1}(\overline{U}) \cap f^{-1}(\overline{V}) = \phi$, by continuity

$$f^{-1}(\overline{U}) \cap f^{-1}(\overline{V}) \subseteq f^{-1}(\overline{U}) \cap f^{-1}(\overline{V})$$

so

$$f^{-1}(U) \cap f^{-1}(V) = \phi,$$

hence $(Y, X)$ is a $T_{2 \frac{1}{2}}$-relative space. 

$\square$
Example 3.3. Let $X = \{(x, y) | y \geq 0, x, y \in \mathbb{Q}\}$ with irrational slope topology, see ([2], page 93). Consider $Y = \{(x, y) \in X | y \leq 1\}$, then $(Y, X)$ is $T_2$–relative space, since any two distinct points in $Y$ must project onto distinct pairs of irrational points on the $x$–axis, which have disjoint neighborhoods, but $(Y, X)$ is not a $T_{2\frac{1}{2}}$– relative space, since the closure of each neighborhood must intersect the closure of the other subsets.

Definition 3.4. A relative space $(Y, X)$ is said to be strongly $T_{2\frac{1}{2}}$– relative space, if for every pair of distinct points $x \in X$, and $y \in Y$ there exist open subsets $U$ and $V$ of $X$ such that $x \in U$ and $y \in V$ and $\overline{U} \cap \overline{V} = \phi$.

Theorem 3.5. Let $(Y, X)$ be a pointwise cleavable relative space over the class $\rho$ of strongly $T_{2\frac{1}{2}}$– relative spaces, then $(Y, X)$ is a strongly $T_{2\frac{1}{2}}$– relative space.

Definition 3.6. A relative space $(Y, X)$ is said to be a strongly functionally Hausdorff relative space if for all $x \in X$ and $y \in Y$ with $x \neq y$, there exists a continuous mapping $f : X \to ([0, 1], \tau_u)$ such that $f(x) = 0$ and $f(y) = 1$.

Theorem 3.7. Let $(Y, X)$ be a pointwise cleavable over the class $\rho$ of strongly functionally Hausdorff relative spaces, then $(Y, X)$ is a strongly functionally Hausdorff relative space.

Proof. Let $y \in Y$, since $(Y, X)$ is pointwise cleavable over the class $\rho$ of strongly functionally Hausdorff relative spaces, there exist strongly functionally Hausdorff relative space $(Z, T)$ and a continuous mapping $g : X \to ([0, 1], \tau_u)$ such that $g^{-1}(g(y)) = y$ and $g(Y) \subseteq Z$. Now, let $x \in X$ with $x \neq y$, so $g(x) \in T$, since $(Z, T)$ is a strongly functionally Hausdorff relative space, there exist continuous mapping $h : T \to [0, 1]$ with $h(g(x)) = 0$ and $h(g(y)) = 1$. Consider the function $f = h \circ g$, so $f$ is a continuous mapping from $X$ into $[0, 1]$ with $f(x) = 0$, $f(y) = 1$, hence $(Y, X)$ is strongly functionally Hausdorff relative space.

Definition 3.8. A relative space $(Y, X)$ is said to be an open normal relative space if for all disjoint open sets $A$, $B$ in $Y$, there exist disjoint closed sets $F, G$ in $X$ such that $A \subseteq F$ and $B \subseteq G$.

Theorem 3.9. Let $(Y, X)$ be an open double cleavable relative space over a class $\rho$ of open normal relative spaces, then $(Y, X)$ is an open normal relative space.

Proof. Let $A$ and $B$ be disjoint open sets in $Y$, since $(Y, X)$ is open double cleavable over a class $\rho$ of open normal relative spaces, then there exist open
normal relative space \((Z, T)\) and an open continuous mapping \(f : X \to T\) such that \(f^{-1}(f(A)) = A\), \(f^{-1}(f(B)) = B\), and \(f(Y) \subseteq Z\). Since \(f\) is open, so \(f(A)\) and \(f(B)\) are open in \(Z\). Since \(A \cap B = \emptyset\), so \(f^{-1}(f(A)) \cap f^{-1}(f(B)) = \emptyset\), and so \(f^{-1}(f(A) \cap f(B)) = \emptyset\), then \(f(A) \cap f(B) = \emptyset\), so there exist two closed subsets \(F, G\) in \(Z\) such that \(f(A) \subseteq F\), \(f(B) \subseteq G\), so \(A \subseteq f^{-1}(F), B \subseteq f^{-1}(G)\), it is clear that \(f^{-1}(F)\) and \(f^{-1}(G)\) are closed and disjoint subsets in \(X\), and hence \((Y, X)\) is an open normal relative space. □

**Definition 3.10.** A relative space \((Y, X)\) is said to be semi\(-T_2\) relative space, if for all \(y_1, y_2 \in Y\), with \(y_1 \neq y_2\), there exist disjoint semi-open subsets \(U\) and \(V\) in \(X\) such that \(y_1 \in U\) and \(y_2 \in V\).

**Theorem 3.11.** Let \((Y, X)\) be pointwise irresolute cleavable relative space over a class \(\rho\) of semi\(-T_2\) relative spaces, then \((Y, X)\) is semi\(-T_2\) relative space.

*Proof.* Let \(y_1 \in Y\), since \((Y, X)\) is a pointwise irresolute cleavable over the class \(\rho\) of a semi\(-T_2\) relative spaces, then there exist semi\(-T_2\) relative space \((Z, T)\) and an irresolute mapping \(f : X \to T\) such that \(f^{-1}(f(y_1)) = y_1\), and \(f(Y) \subseteq Z\). Now \(f(y_1) \in f(Y) \subseteq Z\), let \(y_2 \in Y\) with \(y_1 \neq y_2\), so \(f(y_1) \neq f(y_2)\) in \(Z\), since \((Z, T)\) is a semi\(-T_2\) relative space, there exist disjoint semi-open subsets \(U\) and \(V\) in \(T\) such that \(f(y_1) \in U\) and \(f(y_2) \in V\), then \(f^{-1}(U)\) and \(f^{-1}(V)\) are disjoint semi-open subsets of \(X\) such that \(y_1 \in f^{-1}(U)\) and \(y_2 \in f^{-1}(V)\), hence \((Y, X)\) semi\(-T_2\) relative spaces. □

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**References**


